$\mathfrak{so}(4)$ Symmetry of the Kepler Problem

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1 The Classical Perspective on the Kepler Problem

The dynamics of the Kepler problem are most clear from the Lagrangian perspective. We can introduce the Lagrangian:

$$\mathcal{L} = K - V = \frac{1}{2}m\dot{\vec{x}}^2 - \left(-\frac{k}{r}\right) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{r}$$

and now use the Euler-Lagrange equations (or equivalently the minimization of the variation of the action $- \delta S = 0$ with $S = \int \mathcal{L} dt = 0$).

$$\frac{\partial \mathcal{L}}{\partial x^i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \implies mr\dot{\theta}^2 - \frac{k}{r^2} = m\ddot{r}$$

for r, and

$$\implies 0 = \frac{d}{dt}(mr^2\dot{\theta})$$

for θ.

The θ equation tells us that the angular momentum $L = mr^2\dot{\theta}$ is conserved. So we can write $\dot{\theta} = \frac{L}{mr^2}$ and simplify the r equation from above to get

$$\ddot{r} = \frac{L^2}{m^2r^3} - \frac{k}{mr^2}$$

then make a change of variables to $u(\theta) = \frac{1}{r}$ and use that $\frac{d}{dt} = \frac{L}{mr^2}\frac{d}{d\theta}$ to rewrite the equation of motion as

$$\frac{d^2u}{d\theta^2} = -u + \frac{km}{L^2}$$

which is of course solved by

$$u(\theta) = \frac{1}{r(\theta)} = \frac{km}{L^2}\left(1 + e\cos(\theta + \theta_0)\right)$$

which can be fully solved by finding $\theta(t)$ to calculate $r(t)$. This concludes the brief review of how elliptic orbits are derived from the Lagrangian.

Now we can briefly review the relative simplicity of the approach that uses the Lenz vector, a 'hidden' conserved quantity.

Conserved quantities:

$$\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} \\
\vec{A} &= \vec{p} \times \vec{L} - mk\hat{r}
\end{aligned}$$

Dotting $\vec{A}$ with $\vec{r}$:

$$\begin{aligned}
\vec{A} \cdot \vec{r} &= (\vec{p} \times \vec{L}) \cdot \vec{r} - mkr \\
&= (\vec{L} \times \vec{r}) \cdot \vec{p} - mkr \\
&= L^2 - mkr
\end{aligned}$$

We also of course can write this dot product as

$$\vec{A} \cdot \vec{r} = Ar\cos\theta$$

So we equate these to find

$$\begin{aligned}
Ar\cos\theta &= L^2 - mkr \\
r(A\cos\theta + mk) &= L^2 \\
r(\theta) &= \frac{L^2}{mk + A\cos\theta}
\end{aligned}$$

Now again define eccentricity $e = \frac{A}{mk}$, then:

$$r(\theta) = \frac{L^2/(mk)}{1 + e\cos\theta}$$

Conserved Quantities and Noether's Theorem.

Theorem 1.1. *Suppose your Lagrangian is invariant under the action of some continuous symmetry. Then there exists a corresponding conserved quantity.*

In this problem, we saw conservation of angular momentum, and conservation of energy is also present, although was not explicitly used. These are standard conserved quantities resulting from rotational and time translation symmetries, respectively.

For our purposes it's best to see that we can express conservation of angular momentum L by computing the Poisson bracket:

$$\{L_i, H\} = 0$$

In this problem, and occasionally elsewhere, there exist hidden symmetries, and therefore conserved quantities that can simplify our work. In the case of the Kepler problem, this is the *Runge-Lenz vector*, defined by

$$\vec{A} = \frac{1}{m}(L \times p) - k\frac{q}{|q|}$$

As we'll see later, this Lenz vector corresponds to an $\mathfrak{so}(4)$ symmetry algebra, which in fact can be enlarged to $\mathfrak{o}(4)$. One way of seeing this is looking at the phase space, and choosing a hypersurface at a fixed energy. It naturally becomes a 3-sphere, which has a full symmetry group of $O(4)$.

2 Mathematical Perspective

This section primarily uses Marsden's *Foundations of Mechanics* as a reference.

Theorem 2.1. *Let $\mathcal{M}$ be a pseudo-Riemannian manifold and let a group G act on $\mathcal{M}$ by isometries. Lift the action to a symplectic action on $T\mathcal{M}$. Then its momentum mapping is given by*

$$\hat{J}(\xi)(v_q) = \langle v_q, \xi_{\mathcal{M}}(q)\rangle$$

Now we'll consider the case of angular momentum from this framework. Let $\mathcal{M} = \mathbb{R}^n$ and let G be a Lie subgroup of $GL(n, \mathbb{R})$. Let G act on $\mathcal{M}$ by $\Phi : G \times \mathcal{M} \to \mathcal{M}$ by $(T, p) \mapsto Tp$. The infinitesimal generator corresponding to the Lie algebra $\mathfrak{g} \subset L(\mathbb{R}^n, \mathbb{R}^n)$ is $B_{\mathcal{M}}(p) = Bp$. Then the momentum mapping on $T^*\mathcal{M}$ is given by

$$\hat{J}(B)(p, q) = q(Bp)$$

2.1 Poisson Brackets

It's worth taking a moment to note that the Poisson bracket can be regarded as a closed 2-form on the phase space. In canonical coordinates $(q^1, \ldots q^n, p_1, \ldots p_n)$, we write

$$\omega = dq^i \wedge dp_i$$

Then we can write the Poisson bracket as

$$\{f, g\} = \omega(X_f, X_g)$$

Where X_f is the Hamiltonian vector field associated to f. Whether a vector field is Hamiltonian is determined by

$$i_{X_f}\omega = df$$

and we will now show this. We'll write X_f as a vector field in canonical coordinates on the tangent bundle as

$$X_f = \sum_{i=1}^{n} \left(A_i \frac{\partial}{\partial q_i} + B_i \frac{\partial}{\partial p_i} \right)$$

Now since we have the property

$$\iota_V(\alpha \wedge \beta) = (\iota_V \alpha)\beta - \alpha(\iota_V \beta)$$

This implies that

$$\iota_{X_f}\omega = \sum_{i=1}^{n} \big((\iota_{X_f} dq_i) dp_i - dq_i (\iota_{X_f} dp_i) \big)$$

So we find the following and then equate it to df to find

$$\iota_{X_f}\omega = \sum_{i=1}^{n} (A_i dp_i - B_i dq_i) = df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial p_i} dp_i \right)$$

So we have

$$A_i = \frac{\partial f}{\partial p_i}$$

$$B_i = -\frac{\partial f}{\partial q_i}$$

which we can then use to conclude that X_f takes the form

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

Which, if you'd like to check, of course reproduces the standard Poisson bracket:

$$\{f, g\} = \omega(X_f, X_g) = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$$

It's also useful to know that we can write $X_f = J \cdot df$, where $J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$.

2.2 New Perspectives

This section is primarily based on Guillemin and Sternberg's *Variations on a Theme by Kepler*. Iwai [Iwa81] and Mladenov [Mla89] presented a new an quite interesting way of viewing the Kepler from 'above' – from a Hamiltonian on $\mathbb{R}^4 \setminus \{0\} \times \mathbb{R}$ with Lorentzian metric

$$ds^2 = 4||y||^2(dy^2 - dw^2)$$

with $y \in \mathbb{R}^4 \setminus \{0\}$ and $w \in \mathbb{R}$. Then we define an energy function on the cotangent bundle $T^*(\mathbb{R}^4 \setminus \{0\} \times \mathbb{R})$ by

$$E(y, w; \eta, \tau) = \frac{1}{8||y||^2}(||\eta||^2 - \tau^2)$$

And now we consider the action of the group

$$G = U(1) \times \mathbb{R}$$

$\mathbb{R}$ acts by translations on the time coordinate, while the action of $U(1)$ on $\mathbb{R}^4 \setminus \{0\}$ is generated by

$$M_0 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We can continue this by defining three more similar matrices, and then define functions on $\mathbb{R}^4 \setminus \{0\}$ by

$$x_i = M_i y \cdot y$$

where $i = 1, 2, 3$. This defines a projection map π onto $\mathbb{R}^3$. What we can see from this is that $\mathbb{R}^4 \setminus \{0\} \times \mathbb{R}$ can be viewed as a principal G bundle over $\mathbb{R}^3$ for the G in consideration. Now we consider the moment map of the $\mathbb{R}$ component of the action, which is trivial; it is the function τ. So the reduced energy function becomes

$$E(y, \eta) = \frac{1}{8}\frac{||\eta||^2}{||y||^2} - \frac{\alpha}{||y||^2}$$

Which is interesting for the fact that it allows us to study the Kepler problem as this Hamiltonian on $T^*\mathbb{R}^4 \subset \mathbb{R}^8$. This can be shown by reducing the Hamiltonian under $U(1)$ now that we've done the same for $\mathbb{R}$. Say we have some $\xi = \xi_i dx^i \in T^*\mathbb{R}^3$. Then we can define a map $d\pi_y^*$ at all $y \in T^*\mathbb{R}^4$ by $d\pi_y^*(dx_i) = 2M_i y \cdot dy$. This lets us write that for $\eta = d\pi_y^*\xi$, say $||\eta||^2 = 4||y||^2||\xi||^2$. This is the final step to reduce the Hamiltonian (at $E = 0$) to

$$0 = \frac{1}{8}\frac{4||y||^2||\xi||^2}{||y||^2} - \frac{\alpha}{||y||^2} = \frac{1}{2}||\xi||^2 - \frac{\alpha}{||x||}$$

where in the last step we used $||x|| = ||y||^2$, which can be quickly derived from the definition of the x_i maps.

To give some motivation for why this is remotely useful, we should first construct the reduced Hamiltonian at arbitrary energy level E. So we derive a new Hamiltonian:

$$4||y||^2\left(H+\frac{k}{8}\right)=\frac{1}{2}(||\eta||^2+k||y||^2)-4\alpha \tag{1}$$

$$4||y||^2\left(H+\frac{k}{8}\right)=U_k-4\alpha \tag{2}$$

$$\implies H^{-1}\left(-\frac{k}{8}\right)=U_k^{-1}(4\alpha) \tag{3}$$

The use here is that U_k is now simply the Hamiltonian of a harmonic oscillator in $\mathbb{R}^8$. Now we may use these equations – somewhat surprisingly if the general picture is still unclear – *to derive the energy levels of the Hydrogen atom* [Mla89].

Start by rewriting $a = \lambda^2$ and noting that U_k has the standard result of having eigenvalues

$$\lambda\left(n+\frac{1}{2}\right)$$

And so performing this reduction at $U_k = 4\alpha$, and choosing $E = -\frac{\lambda^2}{8}$. So we find

$$4\alpha = \lambda\,(n+1)$$

and thus, after using this constraint to derive λ, we can write

$$E_n = -\frac{2\alpha^2}{(n+1)^2}$$

with $n = 0, 1, \ldots$. These are precisely the energy levels of the hydrogen atom. From my own perspective, this derivation is physically a little loose, and doesn't exactly seem novel, but this could be due to a lack of understanding. Regardless, these reduction arguments gained interest for good enough reason, especially in the 1990's.

3 Quantizing the System

We can somewhat easily extend this discussion by promoting the Hamiltonian to the operator Hamiltonian $\hat{H}$.

First, it'll be useful to briefly discuss the form of the $\mathfrak{o}(n)$ algebra. To quickly derive the form of elements of this algebra, consider an element of the tangent space at the identity

$$X = \frac{d}{dt}A\Big|_{t=0}$$

We know that any $A \in O(n)$ satisfies $A^T A = \mathbb{I}$, so we differentiate this and evaluate at $t = 0$ to find:

$$\begin{aligned} \left(\frac{d}{dt}A^T\right)\Big|_{t=0} A(0) + A^T(0)\left(\frac{d}{dt}A\right)\Big|_{t=0} &= 0 \\ X^T + X &= 0 \end{aligned}$$

So we define $\mathfrak{o}(n)$ as

$$\mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) : A^T = -A\}$$

Clearly $\dim(\mathfrak{o}(n)) = \frac{n(n-1)}{2}$ by considering the number of independent components of any $n \times n$ matrix after imposing the skew-symmetric condition.

For $\mathfrak{o}(3)$, for instance, we have the basis

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and for $\mathfrak{o}(4)$ something interesting can be observed. We can either be stubborn and enforce a similar basis of six E_{ij} skew-symmetric basis vectors for this space, with

$$E_{42} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

, for instance. We also have the more insightful approach, where we write any element of $\mathfrak{o}(4)$ as

$$\begin{pmatrix} A & b \\ -b^T & 0 \end{pmatrix}, A \in \mathfrak{o}(3), b \in \mathbb{R}^3$$

where we have the new bracketing defined by

$$[(0, b_1), (0, b_2)] = ([b_1, b_2]_{\mathfrak{o}(3)}, 0)$$

These two mentioned approaches are of course equivalent, and hopefully it's clear enough why. The second is just a bit more explicit in separating out two copies of $\mathfrak{o}(3)$, which is ultimately the goal. So we can easily see that

$$\mathfrak{o}(4) \cong \mathfrak{o}(3) \oplus \mathfrak{o}(3)$$

3.1 Back to the Quantized Model

We'll rewrite $\vec{A}$ from before as an operator, and we end up with the following commutation relations:

$$[A_i, H] = 0$$
$$[L_i, A_j] = i\hbar\epsilon_{ijk}A_k$$
$$[A_i, A_j] = -i\hbar\epsilon_{ijk}L_k\left(\frac{2H}{m}\right)$$

First two aren't surprising; they tell us that A_i is conserved, and that A_i transforms under $SO(3)$ as would any regular vector. But this last relation is very unusual; we'll address this later. What's most important to note here is that since we can see that A_i preserves our energy eigenstates, but doesn't commute with the L_i, this implies that our symmetry algebra must actually be larger than $\mathfrak{so}(3)$, unexpectedly.

In addition to these commutation relations, we can also find the following properties of A_i:

$$\vec{L}\cdot\vec{A} = \vec{A}\cdot\vec{L} = 0$$
$$A^2 = k^2\mathbb{I} + \frac{2H}{m}(L^2 + \hbar^2\mathbb{I})$$

We'll now rescale in order to get a more sensible commutation relation between the A_i-like rescaled operators. After writing

$$K = \sqrt{-\frac{m}{2E}}A$$

We can immediately rewrite all of the previous equations:

$$\begin{aligned} 2H(K^2 + L^2 + \hbar^2\mathbb{I}) &= -mk^2\mathbb{I} \\ [L_i, L_j] &= i\hbar\epsilon_{ijk}L_k \\ [L_i, K_j] &= i\hbar\epsilon_{ijk}K_k \\ [K_i, K_j] &= i\hbar\epsilon_{ijk}L_k \end{aligned}$$

Now, much like in the standard procedure of analyzing the spin representations of the Lorentz algebra (very much worth reading about if this topic interests you), we'll define new operators out of L and K in order to derive the true structure of the algebra we're seeing.

$$M = \frac{1}{2}(L + K)$$
$$N = \frac{1}{2}(L - K)$$

$$\begin{aligned}
[M_i, M_j] &= i\hbar\epsilon_{ijk}M_k \\
[N_i, N_j] &= i\hbar\epsilon_{ijk}N_k \\
[N_i, M_j] &= 0
\end{aligned}$$

and *finally* we can see that we succeeded in splitting our algebra up into two isomorphic pieces. In particular, this is in fact

$$\mathfrak{spin}(4) = \mathfrak{so}(4) = \mathfrak{so}(3) \times \mathfrak{so}(3)$$

Finally, we'll show another cleaner derivation of the Hydrogen energy levels from this. Note also that $M^2 = N^2$. To relate H to these N, M, write

$$2H(K^2 + L^2 + \hbar^2\mathbb{I}) = 2H(2M^2 + 2N^2 + \hbar^2\mathbb{I}) = 2H(4M^2 + \hbar^2\mathbb{I}) = -me^4\mathbb{I}$$

And finally note that we always have the fact that on some irrep corresponding to half-integral spin s, we have $M^2 = \hbar^2 s(s+1)$ which, after writing $n = 2s+1$, we can write

$$E_n = -\frac{me^4}{2\hbar^2(4s(s+1)+1)} = -\frac{me^4}{2\hbar^2(2s+1)^2} = -\frac{me^4}{2\hbar^2 n^2}$$

This is a nice result, especially for the fact that it shows that the n^2 degeneracy in the energy levels (there are n^2 possible states at energy level E_n) corresponds to this fact that these special operators M and N (called Casimirs) maintain equality. You might be tempted to ask what happens when we want to consider states that aren't bound, since all of this analysis requires $E < 0$. In that case we get the Lie algebra $\mathfrak{so}(3,1)$ and corresponding scattering states. (See Woit p.243 for more)

References

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