1. $\mathfrak{g}_2$

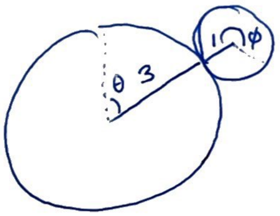
2. $\mathfrak{e}_8$

3. Why are these useful?

Lie sought to classify all Lie algebras, which results from classifying the simple Lie algebras. Killing classified the root systems $\Lambda \subset \mathbb{R}^r$, 1887-1890. $\underset{SU(N+1)}{A_n}, \underset{SO(2n+1)}{B_n}, \underset{Sp(2n)}{C_n}, \underset{SO(2n)}{D_n} \,\|\, G_2, F_4, E_6, \dots E_8$

$\underline{\mathfrak{g}_2}$: Cartan found the corresponding G_2 by consider vector fields on $\mathbb{C}^5$. Much later, 2005-2008, a new idea emerged: a ball rolling on another.

<u>Why a 1:3 ratio?</u>



Quick result: $3\theta = \phi$

$\phi + \theta = (3+1)\theta$ R+1 turns

$L = \{(\cos\theta\, u + \sin\theta\, v,\ R(u \times v, (R+1)\theta\, g) \mid \theta \in \mathbb{R}\} \subset S^2 \times SO(3)$

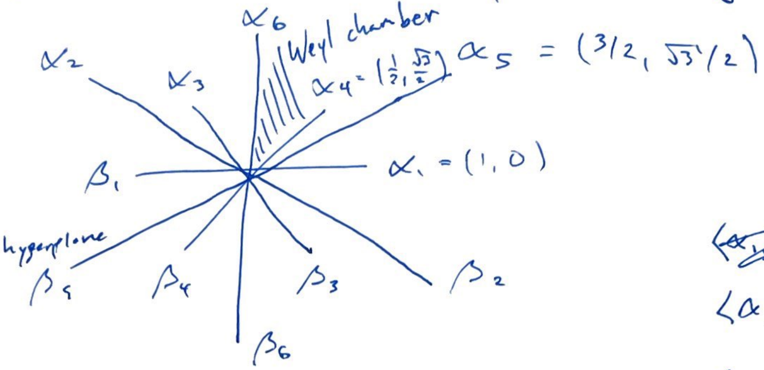
is a <u>line</u>. $\dim(\text{space of } L\text{'s}) = 5 = 2+1+3-1$.

The rolling spinor on $\mathbb{RP}^2$

Fulton-Harris

G_2 :  Dynkin diagram

Root system:



1. Spans $\mathbb{R}^n$
2. $\alpha \in R$, $\lambda\alpha \in R \Rightarrow \lambda = \pm 1$
3. R closed under reflection wrt hyperplane
4. $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

$\cancel{\langle \alpha_1, \alpha_2 \rangle^2} \geq \frac{3}{2}$

$\langle \alpha, \beta \rangle = \frac{2(\alpha, b)}{(\beta, \beta)}$

$\langle \alpha_1, \alpha_2 \rangle = \cancel{\ldots}$

$\langle \alpha_2, \alpha_1 \rangle = \frac{4 \cdot \cancel{(\ldots)}}{\cos^2\Theta}$

Weyl group is clearly D_6.

Let $H_1 \equiv [X_1, Y_1]$, $H_2 \equiv [X_2, Y_2]$

X_1 eigenvector of $\mathfrak{h}$ with eig.-value α_1 etc.

Y_1 . . . β_1

Choose Y_1, Y_2 s.t. $\alpha_1(H_1) = \alpha_2(H_2) = 2$,

so $[H_1, X_1] = [[X_1, Y_1], X_1] = 2X_1$, $[H_2, X_2] = 2X_2$

And so

$[H_1, Y_1] = -2Y_1$, $[H_2, Y_2] = -2Y_2$

Which is just $\mathfrak{sl}_2\mathbb{C}$!

Extend this by taking some path through the lattice where $\alpha_{i_1} + \cdots + \alpha_{i_l}$ are roots in the larger sum expressing any α_i.

Take $\alpha_1 \to \alpha_2 \to \cdots \alpha_6$.

$\alpha_3 = \alpha_1 + \alpha_2$ and set $X_3 = [X_1, X_2]$

$\alpha_4 = \alpha_1 + \alpha_2$ and so on. (Y follow)

$\vdots$

$\alpha_6 = \alpha_2 + \alpha_5$ We can see structure again!

H_1, H_2 are the $\mathfrak{h}$ basis, and collectively we have a 14-dim basis for $\mathfrak{g}_2$.

$$[H_1, Y_1] = H_1 Y_1 - Y_1 H_1 = \cancel{X_1 Y_1}$$

$$H_1 X_1 = [X_1, Y_1] X_1 = \alpha_1(H_1) X_1 = \cancel{2X_1} = 2X_1$$

$$H_1 Y_1 = [X_1, Y_1] Y_1 = \beta_1(H_1) \cancel{X_1} Y_1$$

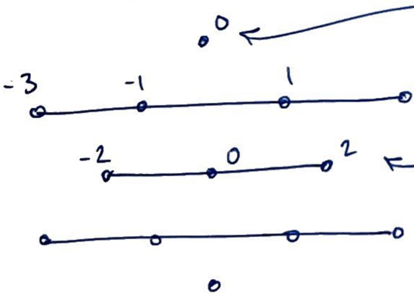
What else? Clearly $[X_i, X_j]$ lies in $\mathfrak{g}_{\alpha_i + \alpha_j} \Longrightarrow [X_i, X_j] = 0$ when $\alpha_i + \alpha_j$ is not a root.

$[X_1, X_5]$, $[X_4, X_6]$, $[Y_2, Y_3]$, for instance.

And $[X_i, Y_j] = 0$ when $\alpha_i + \beta_j = \alpha_i - \alpha_j = 0$

$[X_5, Y_2]$ for example.

How does $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{\beta_i})$ decompose under the action of $\mathfrak{s}_{\alpha_1} = \mathbb{C}\{H_1, X_1, \cancel{X_3} Y\}$?



triv rep. (span X_6, Y_6)

2 copies of $\mathrm{Sym}^3 V$ for X_2, X_3, X_4, X_5 etc.

adj. rep. as before

Computation:

$\mathrm{ad}(X_1)$ sends $X_2 \mapsto X_3$. $\mathrm{ad}(Y_1)(X_3) = \mathrm{ad}(Y_1)\mathrm{ad}(Y_1)(X_2) - \mathrm{ad}([X_1, Y_1])(X_2)$

$= 0 - \mathrm{ad}(H_1)(\cancel{X} X_2) = 3X_2$

since X_2 is an e-vec. for $\cancel{X} H_1$ with -3.

These computations build up a complete mult. table for these elements.

Connection to $\mathfrak{sl}_3$:

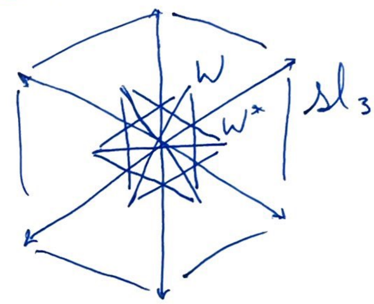
Look at the root diagram! Looks a lot like $\mathfrak{sl}_3$.

<u>def.</u> $\mathfrak{g}_0 = \mathbb{C}\{H_2, H_5, X_2, X_5, X_6, Y \ldots\}$

where $H_i = [X_i, Y_i]$ etc.

Computing the table for $\mathfrak{g}_0$

$\Longrightarrow \mathfrak{g}_0 \cong \mathfrak{sl}_3\mathbb{C}$ as expected.



The rest must be a rep. of $\mathfrak{g}_0$.

Write $\mathfrak{g}_2 = \mathfrak{sl}_3\mathbb{C} \oplus W \oplus W^*$

$W = \mathbb{C}\{X_4, Y_1, Y_3\}$

and $\mathfrak{g}_0 \times W \to W$ correspond to $\mathfrak{sl}_3 \curvearrowright \mathbb{C}^3$ etc.

$W \times W \to W^* = \Lambda^2 W$, $v \times w \mapsto -2 \cdot v \wedge w$.

Some bracket $[,] : W \times W^* \to \mathfrak{sl}_3\mathbb{C}$ can be characterized by $[v, \varphi](w) = 3\varphi(w)v - \varphi(v)w$.

<u>To check Jacobi id.</u>, we can now just use these rules on $\mathfrak{g}_0 \oplus W \oplus W^*$.

Need $Z \cdot (v \wedge w) = (Z \cdot v) \wedge w + v \wedge (Z \cdot w)$.

True from action of Lie alg. on ext. prod.

What about $u, v, w \in W$?

$(u \wedge v)(Z \cdot w) + (v \wedge w)(Z \cdot u) + (w \wedge u)(Z \cdot v) = 0$ in $\Lambda^3 W = \mathbb{C}$.

$= Z \cdot (u \wedge v \wedge w) = 0$

Shown easily for all other combinations. $\square$

Reps. of $\mathfrak{g}_2$

$\Gamma_W \subset \mathfrak{h}$ generated by H_1, H_2.

Cartan $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$\alpha_1(H_1) = 2 \qquad \alpha_1(H_2) = -1$

$\alpha_2(H_1) = -3 \qquad \alpha_2(H_2) = 2$

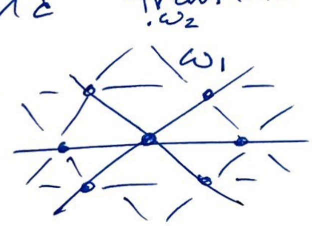
def. Fundamental weights

$\omega_1 = 2\alpha_1 + \alpha_2, \quad \omega_2 = 3\alpha_1 + 2\alpha_2$

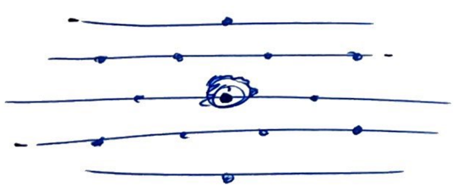
def. Γ_{ab} denotes irrep with highest weight $a\omega_1 + b\omega_2$.

ex. $\Gamma_{1,0}$: take ω_1 and translate by Weyl on ω_1.

This is the standard rep.



$\Gamma_{0,1}$ is adj.



~~A~~ $\Lambda^2 V \cong \Gamma_{0,1} \oplus V$ by weight diag.

Every irrep of $\mathfrak{g}_2$ appears in some tensor power $V^{\otimes m}$ of the standard rep.

proof. ~~dd~~ $\Gamma_{0,1}$ is contained in $\Lambda^2 V$, and

$\Gamma_{ab} \subset \mathrm{Sym}^a V \otimes \mathrm{Sym}^b \Gamma_{0,1}$.