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Deligne - Lustzig Theory

Motivation: (by V. Drinfeld)

The discrete series rep of $Sl_2(\mathbb{F}_q)$ can be obtained through ℓ-adic cohomology of the curve over $\mathbb{F}_q$, s.t. $xy^q - x^q y = 1$

Rmk: $\mathbb{F}_q$ is a finite field (i.e. Galois field) with q elts.

A finite field of order q exists $\iff$ $q = p^k$ for p be a prime, $k \in \mathbb{Z}_{>0}$

Rmk: ℓ-adic cohomology: ℓ is a prime number different from char p of $\mathbb{F}_q$.

To understand objects (varieties, groups) over $\mathbb{F}_q$, we employ ℓ-adic coh'y gps, which allows us to carry natural action of the Frobenius map.

Strategy: Rep of $Sl_2(\mathbb{F}_q) \longrightarrow$ Drinfeld curve $\longrightarrow$ Delign-Lustzig variety

I) Def'n $Sl_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{F}_q) : ad - bc = 1 \right\}$

Property: 1) $Sl_2(\mathbb{F}_q)$ is finite, non-abelian, and simple for $q > 3$ }

2) $|Sl_2(\mathbb{F}_q)| = q(q^2-1)$ (choose q^2-1 first column ; q choices for second)

Rmk. A gp is 'simple' if it has no non-trivial subgps other than its identity.

So, for $q>3$, the center $Z = \{\pm Id\}$, and the quot't gp $Sl_2(\mathbb{F}_q)/Z$ is $PSl_2(\mathbb{F}_q)$ (projective special linear gp)

For non-abelian, for any $q>2$, take matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then $AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq BA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Also, since $Sl_2(\mathbb{F}_q)$ is simple for $q>3$, it has no nontrivial 1-dimen'l rep.

Now, let us delve into the rep theory of $Sl_2(\mathbb{F}_q)$

We have $G = Sl_2(\mathbb{F}_q)$, which is finite. We want to show all irred'le complex reprzn of G.

Upshot: # irreducible rep = # conjugacy class of $Sl_2(\mathbb{F}_q) = q+4$

Center Z	$\pm Id$:	2	
semisimple	split $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$; $a \neq Id$	$\frac{q-3}{2}$	$2 + \frac{q-3}{2} + \frac{q-1}{2} + 4 = q+4$
	non-split $\begin{pmatrix} 0 & -\zeta^{-1} \\ \zeta & 0 \end{pmatrix}$	$\frac{q-1}{2}$	
unipotent	$\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$ $b \neq 1$	4	

For the semi-simple part, $Sl_2(\mathbb{F}_q)$ has two types of max'l tori

i) Split torous $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_q^\times \right\}$; diag'le over $\mathbb{F}_q$

Characters of this torus give the 'principal series' via induction

ii) Non-split torus : $T' = \left\{ \begin{pmatrix} 0 & -\zeta^{-1} \\ \zeta & 0 \end{pmatrix} : \zeta \in \mu_{q+1} \right\}$, where μ_{q+1} are roots of unity of order $q+1$.

So, $\mu_{q+1} = \mathrm{Ker}(N : \mathbb{F}_{q^2}^\times \to \mathbb{F}_q) \subseteq Sl_2(q)$; its matrices cannot be diag'n'le over $\mathbb{F}_q$, but only after passing $\mathbb{F}_{q^2}$.

'Discrete series' rep (cuspidal) follows.

We first work on Principal series rep of $Sl_2(\mathbb{F}_q)$

Rmk. In alg'c gp thery, a Borel subgp G (here $Sl_2(\mathbb{F}_q)$) is a max'l connected solvable subgp.

We know that solvable means, a gp can be break down into finitely many abelian gps.

In $SL_2(\mathbb{F}_q)$, let $B = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \}$ i.e. upper-triangular 2×2 matrices with $\det = 1$.

Inside B, a special subgp called unipotent gp $U = \{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{F}_q \}$, where all eigenvalues are 1. (We can think of U as almost identity or 2×2 upper-triangular matrices with 1s on its diagonal).

Then, we find the Borel subgp $B = T \ltimes U$ (semi-direct product)

Now, we chose a charc $\theta : T \longrightarrow \mathbb{C}^\times$, (gp hom'sm)

Fact $\begin{cases} \text{sends each } \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ to } \theta(a) \in \mathbb{C}^\times \\ \text{only depends on } a \end{cases}$

Hence, there are $q-1$ characters since $\mathbb{F}_q^\times$ is cycle of order $q-1$.

But we want θ into B not just T. i.e. θ extends trivially through U.

$$\theta\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \theta\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = \theta(a).$$

Here, the unipotent U acts trivially. We define induction from B to G, s.t. $\pi_\theta = \mathrm{Ind}_B^G(\theta)$. This means if we consider a fnc

$$\sigma : G \to \mathbb{C}^\times, \text{ st. } \sigma(bg) = \theta(b)\sigma(g) \ \forall\, b \in B, g \in SL_2(\mathbb{F}_q)$$

so $G \curvearrowright \sigma$ by right-trans'n $(g \cdot \sigma)(x) = \sigma(xg)$.

(We upgrade a small character θ into large & rich rep of G)

We have 3 cases to consider. Call it Parabolic induction

1) $\theta \neq \theta^{-1}$ (generic) so each character are distinct. Recall that $\mathbb{F}_q^\times$ has $q-1$ characters so we take off trivial & $\theta^2=1$ $\Rightarrow$ $q-3$ characters & θ and θ^{-1} in same gp, we get $\frac{q-3}{2}$

2) $\theta^2=1$ but $\theta \neq 1$ (quadratic) : Observe π_θ is not irreducible so each splits into two irred'le reps with same dimension. (2 reps)

Recall $\dim(\mathrm{Ind}_B^G(\theta)) = [G:B] = q+1 \Rightarrow \frac{q+1}{2}$

3) $\theta=1$ (Trivial) $\begin{cases} \dim \theta = 1 \\ \text{steinberg} = q \end{cases}$ 2 reps

Sum all : $\frac{q-3}{2} + 2 + 2 = \frac{q+5}{2}$ total reps.

II) Discrete series : we need to find $\frac{q+3}{2}$ irreducible reps.

Progress further. we introduce Drinfeld curve to Deligne-Lusztig variety.

Re-visit $Y = xy^q - x^q y = 1$ over $\mathbb{F}_q$: the curve Y carries large gp of automorphisms. $\begin{cases} \text{The gp } \mu_{q+1} \circlearrowright Y \text{ by multiplying coordinates} \\ SL_2(\mathbb{F}_q) \circlearrowright Y \text{ by linear transformation} \end{cases}$

In fact μ_{q+1} & $SL_2(\mathbb{F}_q)$ commute each other $\Rightarrow$ $\begin{cases} \text{symmetric} \\ \mu_{q+1} \times SL_2(\mathbb{F}_q) \circlearrowright Y \end{cases}$

What is this mean? Since μ_{q+1} acts on it by $\zeta \cdot (x,y) = (\zeta x, \zeta y)$, it admits to say $\begin{pmatrix} ax+by & (ax+by)^q \\ cx+dy & (cx+dy)^q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & x^q \\ y & y^q \end{pmatrix}$ by char p

, and so fixes $\det\begin{pmatrix} x & x^q \\ y & y^q \end{pmatrix} = xy^q - x^q y = 1$

By compactif'n the genus of $\overline{Y}$ of Y is $g = \frac{q(q-1)}{2}$.

Since $\mu_{q+1} \times SL_2(\mathbb{F}_q) \circlearrowright Y$, we study ℓ-adic ètale cohomology gp of Y : $H_c^i(Y, \overline{\mathbb{Q}_\ell})$ and this cohomology becomes rep of $SL_2(\mathbb{F}_q)$

<u>Defn</u> : Introduce Deligne - Lustzig induction

Recall $\theta : \mu_{q+1} \longrightarrow \mathbb{C}^\times$ be a character, then we define

$R(\theta) = H_c^*(Y)[\theta] = \sum_i (-1)^i H_c^i(Y)[\theta]$. For each θ, we get a virtual representa'n of $SL_2(\mathbb{F}_q)$. So when $\theta \neq 1$, it yields to irreducible discrete reps.

<u>Thm</u> Every irreducible rep of G^F (finite gp of Lie type) appears in $R_w(\theta)$ for some $w \in W = N_G(T)/T$ and $\theta : T^{wF} \longrightarrow \mathbb{C}^\times$.

Moreover, the inner product on virtual characters:

$$\langle R_w(\theta), R_w(\theta) \rangle = \# \{ x \in W^F : x \cdot \theta = \theta \}$$

T^{wF} = max'l torus assoc'd w/ w, having Frobenius structure.

W^F = Weyl group of elets fixed by F.

So in $SL_2(\mathbb{F}_q)$, $W \cong \mathbb{Z}/2\mathbb{Z}$; $\begin{cases} \theta \neq \theta^{-1} & \text{then } R_w(\theta) \text{ is irreducible} \\ \theta = \theta^{-1} & \text{" \quad splits into two irreducible} \end{cases}$

Rmk. 'Variety' means set of sol'n to a poly'l eq'n

w/ structrures (smooth, quasi- projective)

Roughly, geometric object that is cut out by poly'l eq'ns.

So, over a field $\mathbb{F}$, it is a space locally described by sol'n to poly'l eq'ns
(e.g. elliptic curve $y^2 = x^3 + x$)