# A Stroll Through Differential Geometry

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## 1 Introduction

This presentation (and these notes) will consists of a breezy stroll through the very interesting field of Differential Geometry, which has important connections to Algebraic Geometry. Seeing as this is a stroll (not a hike, which most mathematics texts are), we will gloss over, as it were, many of the details and the technical mathematics, primarily focusing on the bigger picture. I hope you find some sights to enjoy.

### 1.1 Some History

Geometry has obviously been studied extensively since antiquity (consider Euclid's seminal Elements). One might hope (correctly) that differential geometry is somehow related to differential calculus, so Newton, Liebniz, and other thinkers of the time obviously helped lay the groundwork for the subject.

The real breakthrough came from two names you may have heard before: Gauss, particularly in his 1827 paper the "Disquisitiones generales circa superficies curvas" (General Discussions About Curved Surfaces), and his student, Riemann. Gauss primarily studied curvature and fundamental forms, while Riemann considered metric tensors (all of which will be discussed in more detail below). (It's worth noting that many, many other people obviously considered to this devleopment, particularly as non-Euclidean geometry was being discovered by others at the time, and to ascribe mathematical discoveries just to "great men" is a distortion).

Differential Geometry got a big boost at the turn of the twentieth century through one Albert Einstein, who was looking for a suitable way to describe spacetime, and found it through Riemann's study of geometry. To this day it is the basis for the physical theory of relativity.

Differential Geometry was further developed/reformulated by Elie Cartan, and continue to evolve to this day where it remains a thriving area of mathematics. [1]

# 2 Manifolds

Differential Geometry is concerned with studying manifolds, special types of spaces. They are essentially curved spaces that are still reasonable enough that one can perform calculus on them.

Ever so slightly more formally, a manifold is any surface that is "locally Euclidean." That is, if you zoom in close enough, it looks like Euclidean space $\mathbb{R}^n$. As an example, I encourage you to stamp your feet and think about what you've just stamped your foot upon. Indeed, the Earth itself is not flat, it is (roughly) a sphere (most of us agree), but to our eyes, when we look around, it looks flat. This is because we are so small compared to the curvature of the surface, that is, our viewpoint is zoomed in so closely, that to us the Earth looks like $\mathbb{R}^2$. So we live on a manifold! Hopefully this provides some justification as to why we would want to study them.

**Definition** Let $d > 0$. $M \subseteq \mathbb{R}^n$ is called a smooth d-dimensional manifold if $\forall p \in M \exists \alpha : U \to V, \subseteq \mathbb{R}^d$ open, $V \subseteq M$ open st:

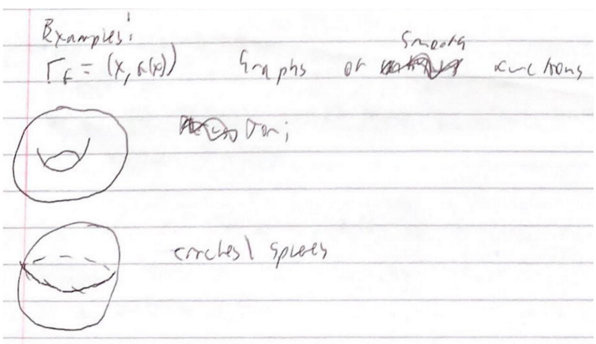
1. Alpha is a smooth function
2. Alpha is a homeomorphism (a bijective function with a continuous inverse)
3. $D\alpha(x)$ has rank d $\forall x \in U$

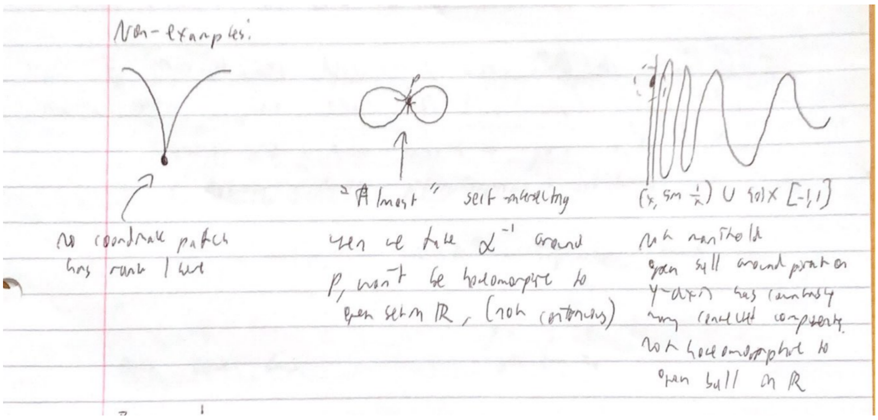
Alpha is called a coordinate patch/coordinate chart. (Note here and throughout the rest of the notes, if f is a function $\mathbb{R}^m \to \mathbb{R}^n$, then Df is that function's multivariable derivative, also called the Jacobian Matrix).

Hopefully this shows how we consider each point to be "locally Euclidean." The full rank condition makes sure that we stay in the same dimension, as, for our purposes, we always want to consider manifolds of a fixed dimension.

The collection of $\{\alpha\}$ that cover the manifold M are called a coordinate atlas. I find this terminology motivating, because you can think of the Earth as a manifold, and a bunch of maps for each part in the earth written down in an atlas as the charts for the Earth.

## 2.1 Examples






## 2.2 Locality and Globalization

An important recurring theme in these notes is that of locality and globalization. That is, as you may have noticed above, we have a bunch of local patches, but it's not necessarily clear how these different pieces fit together. If we want to do something globally, for example like differentiating across the manifold, this will be important. One important way to do this is through Partitions of Unity (In fact, this is a recurring theme in general. Sheaves are another manifestation of this idea).

A partition of unity is, in essence, a collection of smooth "bump" functions that are designed such that for any open cover there's a countable number of them whose values sum to 1 at any point. The whole point of them is to globalize local smoothness. So basically, if we have a smooth functions defined on 2 open sets that overlap, they let us combine those open sets and the 2 smooth functions into 1 "unified" smooth function.

**Definition/Theorem:** Let $\mathbb{R}^n \supseteq V$ be open, and $\{U_i\}$ an open cover of V. Then there exist $\phi_1, \phi_2, ... : [0, 1] \to V$ such that:

- For each $j \in \mathbb{N} \exists i \in I$ st $s_j :=\text{supp}\phi_j = \overline{\{x \in V | \phi_j(x) > 0\}} \subseteq U_i$
- Each $p \in V$ has a neighborhood intersecting finitely many $s_j$'s.
- $\sum_{i=j}^{\infty} \phi_j(x) = 1 \forall x \in V$
- $s_j$ is compact $\forall j$
- $\phi_j$ is smooth $\forall j$

The $\{\phi_j\}$ are called a partition of unity.

This is called a theorem since it is not immediately clear that such a construction exists, although it does. However, we will skip the proof of it, which is rather technical.

## 2.3 Boundaries

Sometimes we have things that feel like they should be manifolds, they're "mostly" manifolds, but they don't quite fit the description. Here's an example: Consider the closed unit disk $D = \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Any point on rim of the disk won't locally look like $\mathbb{R}^2$, since we can't move in 2 linearly independent directions (we can move into the disk but not out of it). However, note that it basically looks like $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}$, what's called the "upper-half plane."

We still want this thing to be a manifold, even though it doesn't quite fit the bill, so we call it a manifold with boundary, where the rim of the disk is the boundary of the manifold. We can simply adjust our definition of a manifold to use $\mathbb{H}^d$ wherever we had previously used $\mathbb{R}^d$, and this gives a definition for a manifold with boundary.

**Theorem**: If M is a d-dimensional manifold with boundary, then $M = M^o \sqcup \partial M$ where $M^o$ is the interior of the manifold and $\partial M$ is the boundary of the manifold and $\sqcup$ stands for a disjoint union. Moreover, $M^o$ is a d-dimensional manifold and $\partial M$ is a d-1 dimensional manifold, both of which are without boundary.

So this theorem tells us manifolds with boundary are actually a d-dimensional manifold and a d-1 dimensional manifold "glued together." (This may slightly violate what I said earlier about always wanting to consider things of the same dimension, but this is the only exception).

In the example of the unit disk above, we have $D^o = \{x \in \mathbb{R}^2 : |x| < 1\} = B(0, 1)$ and $\partial M = \{x \in \mathbb{R}^2 : |x| = 1\} = S^1$. That is, the interior of the disk is the open unit ball (a 2-d manifold) and the boundary of the disk is the unit circle (a 1-d manifold).

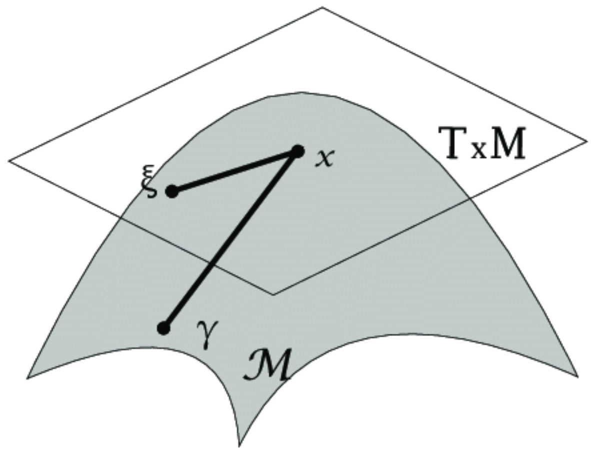
**Caution:** Manifold interiors/boundaries are not the same thing as topological interiors/boundaries, although they use the same notation.

## 2.4 Tangent Spaces

Each smooth manifold comes with a notion of a tangent space at each point, which is essentially the "linear approximation" to the manifold at that point. For a curve, this is the tangent vector at that point; for a surface, the tangent plane. This generalizes to higher dimensions.

**Definition:** Let$p \in M$ for M a d-dimensional manifold and $\alpha : U \to V$ the chart around p st $\alpha(x_0) = p$. Then the tangent space at p is $T_pM := Im(D\alpha(x_0)$, which is a d-dimensional linear space.

This is another realization of our "locally Euclidean" idea.




[2]

It's not immediately clear that this definition is independent of our choice of $\alpha$, but it is!

Moreover,

**Proposition:** If M and N are two manifolds and $f : M \to N$ a smooth function, then this defines a map $Df_p : T_pM \to T_{f(p)}N \forall p \in M$.

So the derivative of a smooth map gives a map between tangent spaces!

We finally have the notion of a tangent bundle:

**Definition:** For M a manifold with boundary in $\mathbb{R}^n$, $TM := \{(x, v) \in M \times \mathbb{R}^n | v \in T_xM\}$, basically the collection of tangent spaces of the manifold.

# 3 Tensors

**Definition:** Let V be a vectorspace. A (k-)tensor (for our purposes) is a map $T : V^k \to \mathbb{R}$ satisfying multilinearity: $T(v_1, ..., v_i + w_i, ...v_k) = T(v_1, ..., v_i, ...v_k) + T(v_1, ..., w_i, ...v_k)$ and $T(v_1, ..., c \cdot v_i, ...v_k) = cT(v_1, ..., v_i, ...v_k)$.

We want to study tensors as they will encode the properties of our manifolds. For example, we can use them to calculate distances and angles (via the metric tensor) or to tell us how to integrate functions on the manifold (via differential forms).

**Definition:**$L^kV := \{$k-tensors on V$\}$. This is a vectorspace with addition and scalar multiplication defined in the natural way.

**Definition:** The tensor product $\otimes$ is an operation that combines tensors in the "natural" way: If $G \in L^kV, H \in L^lV, G \otimes H \in L^{k+l}V, G + H(v_1, ..., v_{k+l}) = G(v_1, ..., v_k) \cdot H(v_{k+1}, ..., v_{k+l})$

**Theorem:** For $I = (i_1, ..., i_k)$ a multiindex with $1 \leq i_j \leq \dim V \forall j$, $e^I = e^{i_1} \otimes ... \otimes e^{i_k}$ and all of the possible $e^I$ form the basis for $L^kV$

Example: $V = \mathbb{R}^2$, $\mathcal{L}^2(V)$.

$I_1 = (1, 2)$ $I_2 = (2, 1)$ $I_3 = (1, 1)$ $I_4 = (2, 2)$

$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$

Then

$$e^{I_1}(v_1, v_2) = e^1(v_1) \cdot e^2(v_2) = (1, 0)\begin{pmatrix} a \\ b \end{pmatrix} \cdot (0, 1)\begin{pmatrix} c \\ d \end{pmatrix} = ad$$

$$e^{I_2}(v_1, v_2) = e^2(v_1) \cdot e^1(v_2) = bc$$

$$e^{I_3}(v_1, v_2) = e^1(v_1) \cdot e^1(v_2) = ac$$

$$e^{I_4}(v_1, v_2) = e^2(v_1) \cdot e^2(v_2) = bd$$

Ex: Consider again $V = \mathbb{R}^2$, $\mathcal{L}^2(V)$

Consider $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$

$$\phi(v_1, v_2) = a(d + 3c)$$

Then $\phi = ad + 3ca = e^{I_1} + 3e^{I_3}$ (see prev page) ✓

Each of the $e^{i_j}$ just takes out some component of the jth vector, so this tells us all k-tensors are just linear combinations of k of these 1-tensors which have been tensor-producted together.

**Definition:** If $A : V \to W$ is linear, we can define the pullback function $A^* : L^k W \to L^k V$ by $(A^* f)(v_1, ..., v_k) = f(Av_1, ..., Av_k)$ for $f \in L^k W$

The pullback tells us how the tensor should act under a linear transformation, but "in reverse," in a way.

Example: Consider $A : \mathbb{R}^3 \to \mathbb{R}^2$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$A^* : \mathcal{L}(\mathbb{R}^2) \to \mathcal{L}(\mathbb{R}^3)$

$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$ $w_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $w_2 = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$f(v_1, v_2) = ac + 3bd$$

$$A^* f(w_1, w_2) = A^* f\left(\begin{pmatrix} w \\ x \\ y \end{pmatrix}, \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right) = f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} w \\ x \\ y \end{pmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}\right)$$

$$= f\left(\begin{pmatrix} w \\ x \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix}\right) = wp + xq$$

## 3.1 Symmetric and Alternating Tensors

Consider the action of the symmetric group of k elements on $L^kV : S_k \times L^kV \to L^kV, (\sigma, f(v_1, ..., v_k)) \mapsto f(v_{\sigma(1)}, ..., v_{\sigma(k)})$. We use the notation $f^\sigma := (f, \sigma)$. All we're doing is permuting the order of the vectors in the tensor.

**Definition:** We say $f \in L^kV$ is

- symmetric if $f^\sigma = f \forall \sigma \in S_k$
- alternating if $f^\sigma = \text{sgn}(\sigma) f \forall \sigma \in S_k$

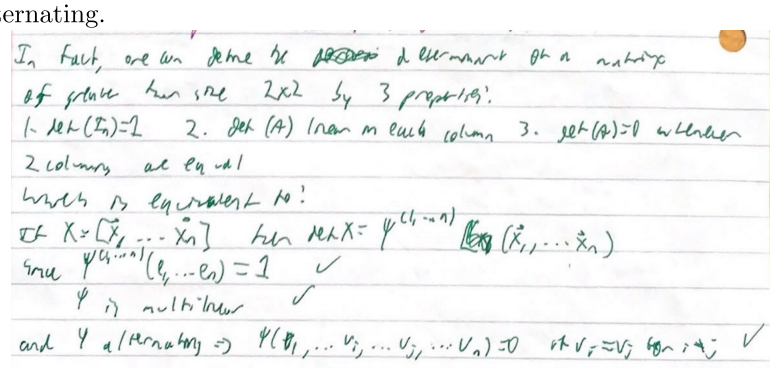
We call $S^kV$ and $A^kV$ the vector subspaces of symmetric and alternating tensors respectively.

So a k-tensor is symmetric if we get the same value when we apply all permutations of k-elements to it, and alternating if we get the same value multiplied by the sign of the permutation. These are actually quite special properties, and there are many tensors that are neither of these. We will use these special types of tensors in various applications. (In particular, metrics are symmetric tensors whereas differential forms are alternating tensors).

**Definition/Proposition:** We have a canonical operation $\wedge : A^kV \times A^lV \to A^{k+l}V, (f, g) \mapsto f \wedge g$ satisfying:

1. Associativity
2. Multilinearity if k=l
3. $f \wedge g = (-1)^{kl} g \wedge f$
4. If $e^i$ are the basis for 1-tensors, then the collection$\Psi_I = e^{i1} \wedge ... \wedge e^{ik}$ are a basis for $A^kV$ where $i1 < ... < ik$

The normal tensor product of 2 alternating tensors won't usually be alternating, so we have this wedge product, which tells us how to "product" alternating tensors into higher order tensors that remain alternating.


In fact, one can define the determinant of a matrix of greater than size 2x2 by 3 properties:
1. $\det(I_n)=1$ 2. $\det(A)$ linear in each column 3. $\det(A)=0$ whenever 2 columns are equal
which is equivalent to:
If $X=[\vec{x}_1, ... \vec{x}_n]$ then $\det X = \psi^{(1,...,n)}(\vec{x}_1, ... \vec{x}_n)$
Since $\psi^{(1,...,n)}(e_1, ... e_n) = 1$ ✓
$\psi$ is multilinear ✓
and $\psi$ alternating $\Rightarrow \psi(v_1, ... v_i, ... v_j, ... v_n) = 0$ if $v_i = v_j$ for $i \neq j$ ✓

The determinant is actually an example of an alternating tensor. Recall from Linear Algebra that a determinant is the "signed volume" of the parallelepiped that is unit square under the linear transformation (it measures the "size/orientation" of the linear transformation). This hints at how alternating tensors can be used to calculate signed volumes, and why we want to use them for differential forms.

# 4 Differential Forms

"A differential form is precisely a linear function which eats vectors, spits out numbers, and is used in integration. The strength of differential forms lies in the fact that their integrals do not depend on a choice of parametrization." [4]

**Definition:** A differential form of order k is a smooth function $\omega : \{(p, v_1, ..., v_k) \in M \times (\mathbb{R}^n)^k : v_i \in T_pM, i \in \{1, ..., k\}\} \to \mathbb{R}$ such that $\forall p \in M : \omega_p : T_pM^k \to \mathbb{R}$ is an alternating k-tensor.

$\Omega^k(M) := \{\omega | \omega$ is a smooth differential form of order k$\}$. This is a vectorspace with pointwise addition and scalar multiplication.

So it's just a smooth function that assigns an alternating tensor at every point. Thus $\Omega^0(M) = C^\infty(M)$

**Definition/Proposition:** If $f : M \to N$ is smooth we can define $f^* : \Omega^k(N) \to \Omega^k(M), (f^*\omega)(p, v_1, ..., v_k) = \omega(f(p), Df(p)v_1, ..., Df(p)v_k)$

**Proposition:**For $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, we define $\omega \wedge \eta \in \Omega^{k+l}(M)$ by simply saying we take the wedge product of the alternating tensor at each point, and we have $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$ for $f : M \to N$ smooth

## 4.1 The Exterior Derivative

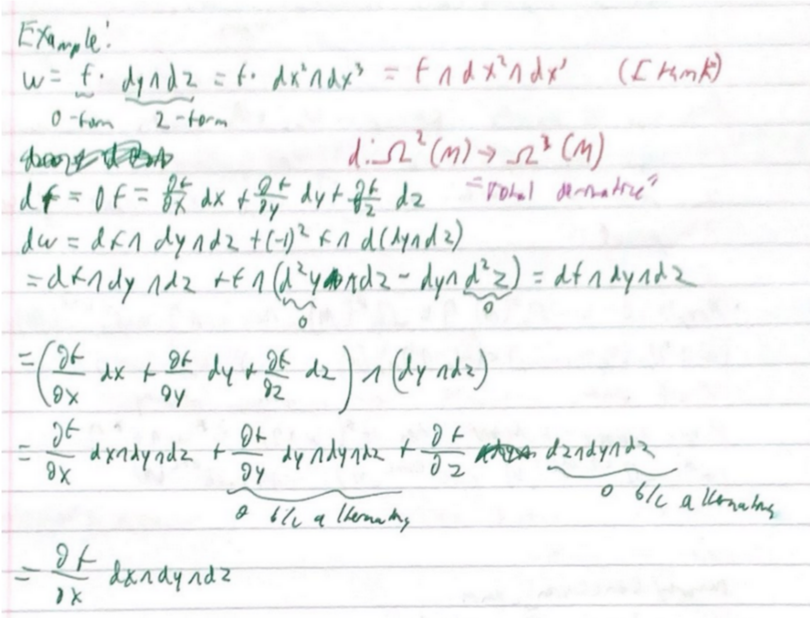
**Theorem:** Let M be a manifold with boundary. Then there exists a unique linear transformation $d : \Omega^k(M) \to \Omega^{k+1}(M) \forall k \geq 0$ called the exterior derivative satisfying:

1. For $f \in C^\infty(M), df_p(v) = Df(p)v$
2. If $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$ then $d(\omega \wedge \eta = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3. $d(d\omega) = 0$

Moreover, if $F : M \to N$ is a smooth map between manifolds then $d(F^*\omega) = F^*d\omega$ for $\omega \in \Omega^k(N)$

Let's break down this theorem. This "exterior derivative" gives a generalization of our normal derivative. This means that it should be a linear transformation, which is our "condition 0" above. Condition 1 ensures that we get out the normal derivative when we apply this to a real-valued function. Condition 2 is a generalization of the product rule, simly adding in some negative term. Condition 3 tells us that the derivative of a derivative is 0. This seems somewhat confusing, especially since, for example, $(x^2)'' = 2 \neq 0$. That is because, in the previous case, we are not really applying the "d" operator more than once, as we consider $(x^2)' = 2x$ still as a smooth function, whereas if we were to apply $d(x^2) = 2x\ dx$, then $d(2x\ dx) = 2dx \wedge dx + 2x \wedge d(dx) = 0 + 0 = 0$.

Note that for condition 1, we often write $Df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i$ where $dx^i$ is essentially the projection function onto the ith coordinate. This is analogous to our matrix definition.


Example:

$\omega = f \cdot dy \wedge dz = f \cdot dx^2 \wedge dx^3 = f \wedge dx^2 \wedge dx^3$ (Σ Hmk)

0-form, 2-form

$d: \Omega^2(M) \to \Omega^3(M)$

$df = Df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$ "total derivative"

$d\omega = df \wedge dy \wedge dz + (-1)^2 f \wedge d(dy \wedge dz)$

$= df \wedge dy \wedge dz + f \wedge (d^2y \wedge dz - dy \wedge d^2z) = df \wedge dy \wedge dz$ (with $d^2y = 0$, $d^2z = 0$)

$= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right) \wedge (dy \wedge dz)$

$= \frac{\partial f}{\partial x} dx \wedge dy \wedge dz + \frac{\partial f}{\partial y} dy \wedge dy \wedge dz + \frac{\partial f}{\partial z} dz \wedge dy \wedge dz$

0 b/c alternating; 0 b/c alternating

$= \frac{\partial f}{\partial x} dx \wedge dy \wedge dz$

Another way to think about why $d(d\omega) = 0$ is to consider some standard identities from vector calculus, which this encapsulates:

- div(curl F)=0
- curl(grad f)=0

In fact this property will be important when it comes to discussing De Rham Cohomology.

## 4.2 Integration and Stokes' Theorem

In fact, it is necessary to define orientations on Manifolds before one can talk about integrating over them. However, we will skip this for the sake of brevity. We shall simply assume we have an oriented manifold, which can be thought of most easily in the case of 2d surfaces as having 2 well-defined "faces" of the surface (e.g. an outside and an inside). This fails in some examples, like the Mobius strip, and also this motivation doesn't extend well to higher-dimensions, but is good enough for our purposes.

**Definition:** Let M be an oriented (compact) k-dimensional manifold (with boundary) and $\omega$ a k-form on M. Then $\int_M \omega$ is defined by choosing (positive) charts $\{\alpha_i, U_i\}$ and a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$. Then $\int_M \omega := \sum_i \int_{U_i} \alpha_i^*(\phi_i \cdot \omega)$.

This definition may look circular, but the integral on the right is simply an integral over Euclidean space, which we already have theories to describe (e.g. we can simply use the Lebesgue Integral). Thus any integral over a manifold simply comes from pulling the manifold back to Euclidean space and performing the integral there.

It's not immediately clear that this is well-defined for any choice of $\alpha_i, \phi_i$, but it turns out to be. We also get that this integral is still a linear operator and respects orientation (i.e. by integrating over an oppositely oriented manifold we get the negative of the integral, analogous to saying $\int_a^b f(x)dx = -\int_b^a f(x)dx$).

**Stokes' Theorem:** Let M be a compact oriented k-dimensional manifold with boundary. Let $\omega$ be a k-1 form defined on an open set containing M. Then

$$\int_M d\omega = \int_{\partial\omega} \omega$$

This theorem encapsulates all of the important theorems in vector calculus (Green's Theorem, Divergence Theorem, etc.). Let us show how with one example.

**Green's Theorem:** If C is a positively oriented piecewise smooth simple closed curve and D is the region bounded by C, and L and M are functions of (x, y) with continuous partial derivatives on D then we get $\oint_C (Ldx + Mdy) = \iint_D \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} dA$

**Proof:** (Via Stokes)

We have D as a manifold and $\partial D = C$. So we get that $\int_D df = \int_{\partial D} f = \int_C f$. So now, if $f = Ldx + Mdy$ then $df = d(Ldx + Mdy) = dL \wedge dx + dM \wedge dy = (\frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial y}dy) \wedge dx + (\frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy) \wedge dy = \frac{\partial L}{\partial x}dx \wedge dx + \frac{\partial L}{\partial y}dy \wedge dx + \frac{\partial M}{\partial x}dx \wedge dy + \frac{\partial M}{\partial y}dy \wedge dy = \frac{\partial M}{\partial x}dx \wedge dy - \frac{\partial L}{\partial y}dx \wedge dy = (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y})dA$. This chain of equalities simply comes by applying the properties of the exterior derivative and the wedge product.

Thus the result follows.

## 4.3 de Rahm Cohomology

To get a bit closer to algebraic geometry, we can consider de Rham Cohomology, which is a cohomology theory based on differential forms.

**Definition:** $\omega \in \Omega^k(M)$ is called

- Closed if $d\omega = 0$
- Exact if $\exists \eta \in \Omega^{k-1}(M)$ st $d\eta = \omega$

We call $Z^p(M) := \ker(d : \Omega^p(M) \to \Omega^{p+1}(M)) = \{\text{closed p-forms on M}\}$ and

$B^p(M) := Im(d : \Omega^{p-1}(M) \to \Omega^p(M)) = \{\text{exact p-forms on M}\}$

Because $d^2 = 0$ we see that every exact form is closed, but the converse is not necessarily true. In fact, de Rham Cohomology measures this difference, and moreover this difference can actually tell us about the topological properties of the manifold.

**Recall:**[9]

A cochain complex is a sequence of abelian groups $\to C^{n-1} \to C^n \to ...$ with a maps $d^{n-1} : C^{n-1} \to C^n$ satisfying $d^n \circ d^{n-1} = 0$

The nth cohomology group is $H^n(C^*, d^*) := \ker d^n / Im(d^{n-1})$

**Definition:** De Rham Cohomology is the cohomology theory associated to a manifold M where the cochain complex is given by $\to \Omega^k(M) \to \Omega^{k+1}(M) \to ...$ where the map is the exterior derivative d.

The nth cohomology group now is just $H^n_{dR}(M) := Z^p(M)/B^p(M)$, that is, the closed forms modded out by the exact forms.

While we won't go deeper on the technicalities of the theory of de Rham cohomology, some main ideas are:

- Every closed form is locally exact, so whether a closed form is exact in full generality depends on the global shape
- de Rham groups are homotopy invariant and thus topological invariants
- de Rham's Theorem gives an explicit isomorphism from de Rham cohomology to singular cohomology (through integration)

# 5 Metrics

Metrics, more formally called metric tensors, are a principal tool in differential geometry. They allow us to define distances and angles on manifolds, which are necessary prerequisites for geometry.

**Definition:** The metric tensor $g_p$ is a 2-tensor defined on each tangent space that is symmetric and nondegenerate ($\forall v \in T_pM$, the function $w \mapsto g_p(v, w)$ is not identically zero, that is, $\forall v \in T_pM \exists w \in T_pM$ st $g_p(v, w) \neq 0$.

A metric tensor field is a smooth function that assigns a metric tensor to each point on the manifold.

**Definition:** A Riemannian metric is a positive definite metric tensor. That is $\forall p \in M, \forall v \in T_pM, g_p(v, v) \geq 0$ and $g_p(v, v = 0 \iff v = 0$.

A manifold that admits a Riemannian metric is called a Riemannian manifold.

The most obvious (and indeed motivating) example of a (Riemannian) metric tensor is that of the standard Euclidean inner product (i.e. the dot product). An inner product space is one where lengths and angles can be calculated via the formulas $||v|| =< v, v >$ and $\cos\theta = \frac{<x,y>}{||x|| \cdot ||y||}$.

Since the metric is analogous to a dot product, we would expect (and because it is a 2-tensor, it is true) it can be represented by a matrix that acts like $(v_1, v_2) \mapsto v_1^T g v_2$. Since the tensor is symmetric, so is this matrix.

**Examples:** The metric is constant in Cartesian coordinates and simply represented by $I_n$ for $\mathbb{R}^n$.

For the unit sphere, we have the metric being $g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$ where $\theta$ refers to the angle in spherical coordinates.

The metric tensor can be used in the calculation of, for example, the length of a curve on a manifold. The length of a curve $\gamma : [a, b] \to M$ is given by $\int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{\frac{1}{2}} dt$. Compare this to the normal arclength in Euclidean Coordinates: $\int_a^b |\gamma'(t)|dt$ and recall we say $|\gamma'(t)| =< \gamma'(t), \gamma'(t) >^{\frac{1}{2}}$, where the inner product is our Riemannian metric on Euclidean space, so this definition is analogous.

# 6 Kahler Manifolds and Hodge Theory

Finally we turn to Kahler Manifolds, which have the clearest connection to Algebraic Geometry. Here we move from real manifolds to complex ones. We will not redefine things, but there are important differences in the definitions (e.g. requiring holomorphicity, etc.).

**Definition:** A symplectic form is a closed non-degenerate 2-form $\omega$.

So recall closed means $d\omega = 0$ and nondegenerate means $\forall v \in T_pM \exists w \in T_pM$ st $g_p(v, w) \neq 0$.

**Definition:** A Kahler Manifold is a complex manifold that simultaneously admits both a Riemannian metric and a symplectic form, which we call a Kahler Metric. .

## 6.1 Connection to Algebraic Geometry

**Definition:** A complex projective variety is a solution set to homogenous polynomials in complex projective space.

If X is a complex projective variety it is embedded in $P^n_{\mathbb{C}}$ (complex projective space). Then $P^n_{\mathbb{C}}$ is a Kahler manifold that has a Kahler Metric, and by restriction becomes a Kahler metric on X, which thus becomes a Kahler Manifold.

It can be shown by formal arguments that the structure on its cohomology groups can be broken down like this: $H^i(X, \mathbb{C}) = \oplus_{a+b=i} H^{a,b}(X)$, where $H^{a,b}(X)$ is the set of classes of closed forms of type (a, b) which is isomorphic to $\mathcal{H}^{a,b}$, the harmonic forms of type a, b.

Type (p, q) refers to a differential form that is of the form $dz_{i_1} \wedge ... \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q}$ where $dz_j = dx_j + idy + j$ and $d\bar{z}_j = dx_j - idy_j$

So in contrast to de Rham cohomology, which uses differential forms, the Hodge Decomposition uses Harmonic forms, which are those forms that vanish under the Laplacian operator (whose definition we have avoided here as, in full generality necessary for manifolds, it is rather complicated). That is to say, every real cohomology class on the manifold has a distinguished harmonic form as a representative.

# 7 References

## References


[1] URL: wikipedia.org.

[2] URL: https://www.researchgate.net/publication/333979280/figure/fig4/AS:773300930224131@1561380717614/llustration-of-a-tangent-space-T-x-M-is-the-tangent-space-of-the-manifold-M-at-point-x.png.

[3] Ben Andrews. *Lecture 9. Riemannian metrics*. URL: https://maths-people.anu.edu.au/~andrews/DG/DG_chap9.pdf.

[4] Daivd Bachman. *A Geometric Approach to Differential Forms*. Springer, 2012.

[5] John Lee. *Introduction to Smooth Manifolds*. Springer, 2013.

[6] James Munkeres. *Analysis on Manifolds*. Addison-Wesley, 1991.

[7] Florian Naef. *Course: Calculus on Manifolds*. Spring 2024.

[8] *The Metric Tensor: A Complete Guide With Examples*. URL: https://profoundphysics.com/metric-tensor-a-complete-guide-with-examples/#Metric_Tensor_Examples_A_Complete_List.

[9] Tony. *Cohomology in Algebraic Geometry*. 2024. URL: https://www.math.columbia.edu/~calebji/teaching/10-4-sheaf-coh.pdf.

[10] Claire Voisin. *HODGE THEORY AND COMPLEX ALGEBRAIC GEOMETRY I*. Cambridge University Press, 2002.