Fact 1 F and G be two projective curves without common componet, then $F \cap G$ is finite

Problem: the number of these intersection points ie. $\sum_{P \in F \cap G} \mu_P(F, G)$

Local rings of $\mathbb{P}^2$ For $P \in \mathbb{R}^2$, define local ring of $\mathbb{P}^2$ at P

as $\mathcal{O}_P = \mathcal{O}_{\mathbb{P}^2, P} := \{\frac{f}{g} : f, g \in K[x, y]$ homogeneous of the same degree with $g(P) \neq 0\} \cup \{0\}$

with kernel $I_P := I_{\mathbb{P}^2, P} := \{\frac{f}{g} \in \mathcal{O}_P : f(P) = 0$ and $g(P) \neq 0\} \subset \mathcal{O}_P$

describes local functions that have value 0 at P

$K[x, y]$ is the ring of global functions on the plane

$\mathcal{O}_P = K[x, y]$ $\{(x - x_0, y - y_0)\}$ consists of all reduced rational functions whose

denominator is not 0 at P $\Rightarrow$ rational functions which are definable near P

$\langle F, G \rangle$ ideal generated by F and G (i.e. anything you can get from combinations

$aF + bG$ where a, b are in $K[x, y]$)

Taking Quotient means anything in $\langle F, G \rangle$ is 0

claim **Exercise 2.7** (Finiteness of the intersection multiplicity). Let F and G be two curves without a common component that passes through the origin. Show:

(a) There is a number $n \in \mathbb{N}$ such that $x^n = y^n = 0$ in $\mathcal{O}_0 / \langle F, G \rangle$.

(b) Every element of $\mathcal{O}_0 / \langle F, G \rangle$ has a polynomial representative.

(c) $\mu_0(F, G) < \infty$.

Define intersection multiplicity of 2 curves F, G at a point $P \in \mathbb{P}^2$ $\mu_P(F, G) := \dim \mathcal{O}_P / \langle F, G \rangle \in \mathbb{N} \cup \{\infty\}$

Lemma 1 (Summing up intersection multiplicities)

Let F and G be 2 affine curves over K with no common component ($F \cap G$ finite)

consider natural ring homomorphism $\varphi: K[x, y] / \langle F, G \rangle \to \prod_{P \in F \cap G} \mathcal{O}_P / \langle F, G \rangle$

that sends the class of a polynomial $f \in K[x, y]$ to class of $f \in \mathcal{O}_P$ in each factor $\mathcal{O}_P / \langle F, G \rangle$

(a) φ is surjective

(b) if K is algebraically closed then φ is an isomorphism

Pf (a) Let $F \cap G = \{P_0 \cdots P_m\}$ with $P_i = (x_i, y_i)$ for $i = 0 \cdots m$ since $F \cap G$ is finite

There is a number $n \in \mathbb{N}$ s.t $(x - x_i)^n = (y - y_i)^n = 0 \in \mathcal{O}_{P_i} / \langle F, G \rangle$ for all i by Claim (a)

Define $f := \prod_{i = x_i \neq x_0} (x - x_i)^n \cdot \prod_{i: y_i \neq y_0} (y - y_i)^n \in K[x, y]$

$g(P_0) \neq 0$ by definition of $\mathcal{O}_P$, then there is a polynomial representative $g \in K[x, y]$

for $\frac{1}{f} \in \mathcal{O}_{P_0} / \langle F, G \rangle$. The polynomial fg is then mapped by φ

① in component $\mathcal{O}_{P_0} / \langle F, G \rangle$ to $fg = f \cdot \frac{1}{f} = 1$ since g is representative

② in all other components $\mathcal{O}_{P_i} / \langle F, G \rangle$ for $i > 0$ since $f = 0 \in \mathcal{O}_{P_i} / \langle F, G \rangle$

By symmetry, we can find in the same way for all $i = 1, \cdots m$ a polynomial that is mapped by φ to 1 in the p_i component ($\mathcal{O}_{P_i}$) and to 0 in all others

Since image of φ is a subring it follows that φ is surjective

Pf (b) It suffices to show φ is injective.

Let $f \in K[x, y]$ with $\varphi(f) = 0$ and consider the set $I := \{g \in K[x, y] : gf \in \langle F, G \rangle\}$

This is clearly an ideal containing $\langle F, G \rangle$

Recall: Any ideal I in $K[x_1, \cdots x_n]$ with $V(I) = \emptyset$ over an algebraically closed field K is the unit ideal $I = \langle 1 \rangle$

want to prove $V(I) = \emptyset$ since then $I = K[x, y]$. hence $1 \in I$ i.e. $f \in \langle F, G \rangle$ and thus $f = 0 \in K[x, y] / \langle F, G \rangle$

claim $V(I) = \emptyset$ $\quad$ $V(I)$: $p(x) = 0$ for all x in $V(I)$

Assume there is a point $P \in V(I)$. As $F, G \in I$ we know $P \in F \cap G$. Hence P is one of the points in the product in the targest space of φ so $f = 0 \in \mathcal{O}_P / \langle F, G \rangle$ as $f \in \ker \varphi$. This means that $f = \frac{a}{g} F + \frac{b}{g} G$ for $a, b, g \in K[x, y]$ with $g(P) \neq 0$

$\Rightarrow gf = aF + bG$ hence $g \in I$ and $P \in V(I) \Rightarrow g(P) = 0$, contradicts!

Lemma 2 Let F, G be two affine curves of degrees $m := \deg F$ and $n := \deg G$ such that their leading parts F_m and G_n have no common component. Then $\forall f \in \langle F, G\rangle \subset K[x,y]$ of degree $d := \deg f$ can be written as

$f = aF + bG$ for 2 polynomials a and b with $\deg a \le d-m$ and $\deg b \le d-n$

pf: As $f \in \langle F, G\rangle$ we can write $f = aF + bG$ for some $a, b \in K[x,y]$

choose such a representation with $\deg a$ minimal

Assume for a contradiction that $\deg a > d-m$ or $\deg b > d-n$. Then aF or bG contains a term of degree bigger than d $\Rightarrow$ leading terms of aF and bG must cancel in f. Hence if a_* and b_* denote the leading terms of a and b respectively, we have $a_* F_m = -b_* G_n$. but F_m and G_n have no common component by assumption, so we must have $a_* = c\,G_n$ and $b_* = -c\,F_m$ for some homogeneous polynomial c. This gives us a new representation $f = (a - cG)F + (b + cF)G$

in which leading term a_* of a cancels the leading term cG_n of cG in the first bracket

Hence $\deg(a - cG) < \deg a$, contradicts $\deg a$ is the minimal

Lemma 3 Let F and G be affine curves with no common component of degrees $m := \deg F$ and $n := \deg G$

(a) $\dim K[x,y]/\langle F, G\rangle \le mn$

$F = x^2$ $G = x$ $\Rightarrow$ have common component

(b) If the leading parts F_m and G_n have no common component then equality holds in (a).

pf: For all $d \ge m+n$ consider sequence of vector space homomorphisms

$$K[x,y]_{\le d-m} \times K[x,y]_{\le d-n} \xrightarrow{\alpha} K[x,y]_{\le d} \xrightarrow{\pi} K[x,y]/\langle F, G\rangle$$

$(a,b) \mapsto aF + bG$

all polynomials of deg at most d

which has $\dim \binom{d+2}{2}$

quotient map

kernel of α consists of all pairs (a,b) of polynomials of degrees at most $d-m$ and $d-n$, with $aF=-bG$. As F and G have no common component, this is equivalent to $a=cG$ and $b=-cF$ for some $c \in K[x,y]_{\le d-m-n}$

So that $\ker\alpha = K[x,y]_{\le d-m-n} \cdot (G,-F)$

Obvious that $\operatorname{im}\alpha \subset \ker\pi$

$d \ge m+n$

$$\dim \operatorname{im}\pi = \binom{d+2}{2} - \dim\ker\pi \le \binom{d+2}{2} - \dim\operatorname{im}\alpha$$

$$= \binom{d+2}{2} - \binom{d-m+2}{2} - \binom{d-n+2}{2} + \dim\ker\alpha$$

$$= \binom{d+2}{2} - \binom{d-m+2}{2} - \binom{d-n+2}{2} + \binom{d-m-n+2}{2}$$

$$= mn$$

$\pi: K[x,y] \to K[x,y]/\langle F,G\rangle$ surjective $\Rightarrow \dim K[x,y]/\langle F,G\rangle \le mn$

(b) if we have $\underbrace{\ker\pi \subset \operatorname{im}\alpha}_{\text{by lemma 2}} \Rightarrow \ker\pi = \operatorname{im}\alpha$ we have equality

① affine curve is the zero set of a polynomial in 2 variables.

② projective curve is zero set in a projective plane of a homogeneous polynomial in 3 variables

③ affine curve can be completed in projective curve by homogenizing its defining polynomial

④ projective curve of homogeneous equation $h(x,y,t)=0$ can be restricted to the affine curve of equation $h(x,y,1)=0$

Lemma 1.11. *Let F be an affine curve.*

(a) *If K is algebraically closed then $V(F)$ is infinite.*

(b) *If K is infinite then $\mathbb{A}^2_K \backslash V(F)$ is infinite.*

Bezout's Theorem Let F and G be projective curves without common component over an infinite field K. Then $\sum_{P\in F\cap G} \mu_P(F,G) \le \deg F \cdot \deg G$ (equality holds if K is algebraically closed

Pf Since K is infinite, then $\mathbb{A}^2_K \setminus (F^i \cap G^i)$ is infinite. $\exists$ a point Q in the affine part of $\mathbb{P}^2$? which does not lie on $F^i \cup G^i$. Moreover $F\cap G$ finite, we can pick a line L through Q which does not intersect $F \cap G$. Now we make a projective coordinate transformation ? so that L becomes the line at infinity

Then neither F nor G contains the line at infinity as a component i.e. $\deg F^i = \deg F$ and $\deg G^i = \deg G$, and all intersection points of F and G lie in the affine part. (they are also intersection points of the affine curves F^i and G^i)

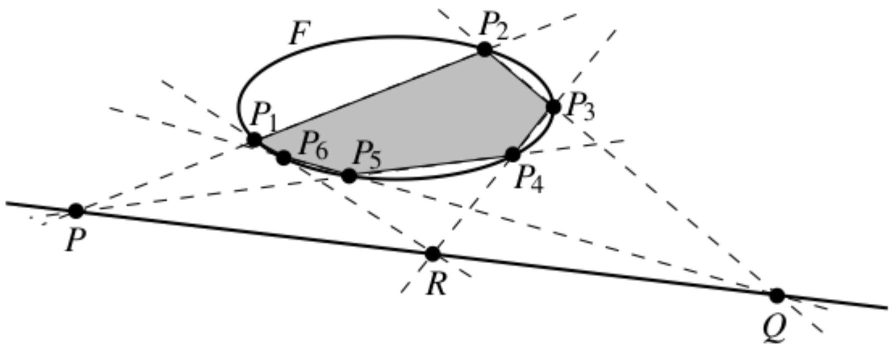
Then Apply lemma 1 and lemma 3 to F^i and G^i

$$\sum_{P\in F\cap G} \mu_P(F,G) = \sum_{P\in F^i\cap G^i} \mu_P(F^i,G^i) \underset{\text{lemma 1 } ①}{\le} \dim K[x,y]/\langle F^i, G^i\rangle \underset{\text{lemma 3 } ②}{\le} \deg F^i \cdot \deg G^i = \deg F \cdot \deg G$$

And let K be algebraically closed then we have "=" at ①

then by lemma 3, we have "=" at ②

Pascal's Theorem Let F be an irreducible projective conic with infinitely many points over an algebraically closed field or an ellipse over $\mathbb{R}$. Pick 6 distinct points $P_1 \dots P_6$ on F (that can be thought of as the vertices of a hexagon inscribed in F). Then the intersection points of the opposite edges of the hexagon (e.g. $P = \overline{P_1P_2} \cap \overline{P_4P_5}$ $Q = \overline{P_2P_3} \cap \overline{P_5P_6}$ and $R = \overline{P_3P_4} \cap \overline{P_6P_1}$ where $\overline{P_iP_j}$ denotes the line through P_i and P_j) lie on a line



pf Consider 2 cubics $G_1 = \overline{P_1P_2} \cup \overline{P_3P_4} \cup \overline{P_5P_6}$ and $G_2 = \overline{P_2P_3} \cup \overline{P_4P_5} \cup \overline{P_6P_1}$

Then by Bezout Theorem they intersect in 9 points $P_1, \dots P_6, P, Q, R$

pick any point $S \in F$ not equal to the previously chosen ones. Then $\exists\, \lambda_1, \lambda_2 \in K$ not both 0. such that $G = \lambda_1 G_1 + \lambda_2 G_2$ vanishes at S (since $G(S) = 0$ is one homogeneous linear equation in 2 variables λ_1, λ_2) Then F meets G in 7 points $P_1 \dots P_6, S$ and so by Bezout theorem, these 2 curves must have a common component. As $\deg F = 2$ $\deg G = 3$ and F is irreducible, the only possibility for this is that G contains factor F, so $G = F \cdot L$ for a line L.

But P, Q, R lie on G since they lie on G_1 and G_2, but not on F so they must be on the line L.

Example $C_0 : F = Y - X^2 \Rightarrow ZY - X^2$

$C_3 : J = X - 2 \Rightarrow X - 2Z$

there is an intersection point at infinity $[0:1:0]$