Outline

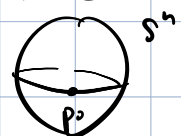
- Classical Cohomology theories
  - Simplicial (Singular, cellular, deRham)
  - Mayer-Vietoris
- Cech Cohomology
  - Connection to classical cohomology theories.
  - Connection to vector bundles $\rightsquigarrow$ If true, Picard Group.

§1 Cohomology

Motivation

Recall: $\pi_1(X)$ is the homotopy class of loops on X, which only depends on its low-dimensional structure.

Q. How to study high-dim. spaces? e.g. distinguish between S^2 and S^3 (or even: show that S^2 is not contractible).

A1 Higher homotopy groups 

$\to (X, x_0)$.

$$\pi_n(X, x_0) := \{\text{based maps } (S^n, p_0) \to (X, x_0)\} / \text{based homotopy}.$$

Involved Thm $\pi_n(S^n) = \mathbb{Z}$, $\pi_k(S^n) = 0$ for $k < n$.

Problem Hard to compute! ($\pi_k(S^n)$, $k > n$ is one of the greatest mysteries of alg top. e.g. $\pi_3(S^2) = \mathbb{Z}$).

A2 (Co) Homology H_n, H^n: $\mathrm{Top} \to \mathrm{Ab}$. (In AG, we care more about cohomology).

Advantage, Much more computable, e.g.

Thm $$H_k(S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, n \\ 0 & \text{otherwise.} \end{cases} \quad (n > 0).$$

$\Rightarrow$ distinguishes between S^n and S^m.

Disadvantage: More complicated to define.

Cochain Complexes

There are many kinds of cohomology theories (simplicial, singular, cellular, de-Rham) but in all of them, we will be dealing with sequences of ab. groups

$$(C^*, \delta^*) \qquad \to C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \to \cdots \qquad \text{with } \delta^n \circ \delta^{n-1} = 0 \quad (*).$$

"cochain" "coboundary operators"

Such a sequence is called a **cochain complex**.

Rmk If the indices of C^n are decreasing, the sequence is called a "chain complex".

Since $\delta^n \circ \delta^{n-1} = 0$, we have

nth coboundary: $B^n(C) \equiv \operatorname{Im}(\delta^{n-1}) \subseteq \operatorname{Ker} \delta^n \equiv Z^n(C)$. nth cocycle

The **nth-cohomology group** of (C^*, δ^*) is

$$H^n(C^*, \delta^*) = Z^n(C) / B^n(C) = \operatorname{Ker} \delta^n / \operatorname{Im} \delta^{n-1}.$$

A stronger condition than $(*)$ is

$$\operatorname{Im} \delta^{n-1} = \operatorname{Ker} \delta^n, \quad \forall n \geq 0. \qquad (**).$$

A sequence satisfying $(**)$ is called an **exact sequence**.

Ex. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ is **short exact**.

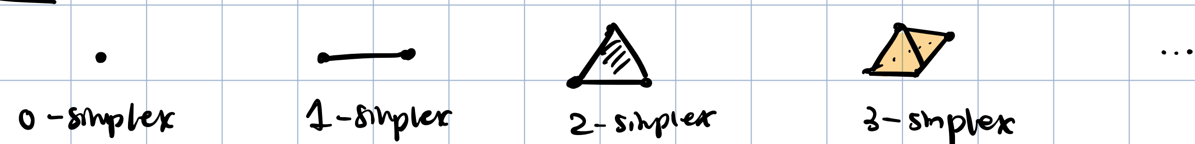
Ex. $0 \to Z^n \to C^n \xrightarrow{\delta^n} B^{n+1} \to 0$ is SES.

If (C^*, δ^*) is exact, then $H^n(C^*, \delta^*) = 0 \ \forall n$.

$\Rightarrow$ Algebraically, cohomology measures the extent to which a cochain complex fails to be exact. Different cohomology theories construct cohomology groups through different cochain complexes.

Simplicial Cohomology $H^n_\Delta(X; \mathbb{Z})$.

Idea Simplicial complexes are spaces built out of "simplices" triangles

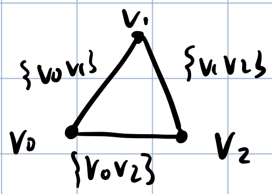


Def The standard n-simplex is

$$\Delta^n := \{(t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, \ t_i \geq 0\}.$$

Nontrivial fact: Manifolds & varieties can be built by gluing simplicies.

Ex We can build S^1 as a simplicial complex



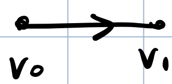
3 0-simplices

3 1-simplices

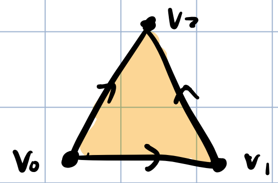
Simplices are also oriented.

We'll denote an oriented simplex with vertices $v_0, \dots v_n$ by $[v_0, \dots, v_n]$

Ex.



$[v_0, v_1]$.



$[v_0, v_1, v_2]$.

For each simplicial complex K and integer n, we'll define an abelian group $C^n_\Delta(K)$ called the simplicial n-cochain of K

Def $C^n_\Delta(K;\mathbb{Z}) := \bigoplus_{\substack{\sigma:\sigma\to\mathbb{Z}.\\ \sigma\text{ n-simplices}\\ \text{in } K}} \mathbb{Z}e.$ free abelian group generated by homomorphisms $\sigma \to \mathbb{Z}$.

(Note: Can replace $\mathbb{Z}$ with your favorite ring, e.g. $\mathbb{Z}/2$.)

Given an n-cochain F, we define an $(n+1)$-cochain $\delta^n F$ by

$$\delta^n F([v_0, \dots v_n]) = \sum_i (-1)^i F([v_0, \dots \hat{v}_i, \dots, v_n]).$$

omit v_i

Check 1. (Extending δ by linearity). δ is a group homo

2. $\delta^n \circ \delta^{n-1} = 0$

We have the simplicial cochain complex of K

$$(C^*_\Delta, \delta^*) = \cdots \to C^{n-1}_\Delta(K) \xrightarrow{\delta} C^n_\Delta(K) \xrightarrow{\delta} C^{n+1}_\Delta(K) \to \cdots$$

The nth-simplicial cohomology group of K is

$$H^n(K;\mathbb{Z}) := H^n(C^*_\Delta, \delta^*)$$

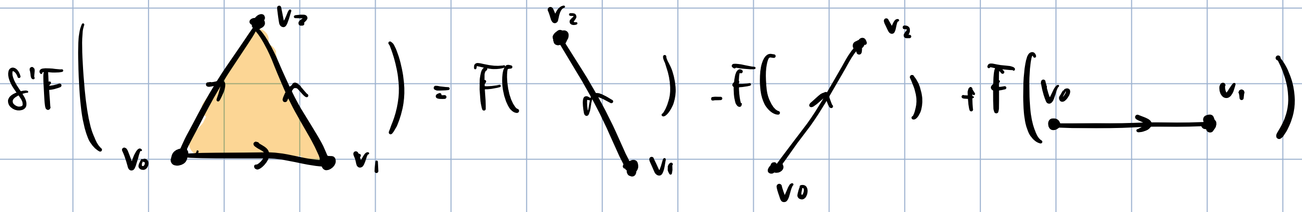
Rmk $H^n(K)$ is actually a graded ring, with multiplication given by "$\cup$"!

Ex $\delta^0: C^0_\Delta(K) \to C^1_\Delta(K)$. $\delta^0 F([v_0, v_1]) = F(v_1) - F(v_0)$

$F \mapsto \delta^0 F$

$\delta^1: C^1_\Delta(K) \to C^2_\Delta(K)$. $\delta^1 F([v_0, v_1, v_2]) = F([v_1, v_2])$

$F \mapsto \delta F$ $\qquad -F([v_0,v_2]) + F([v_0,v_1])$.



Thm (Homotopy invariance)

If $f: X \to Y$ ctns map, then it induces a group homo $f^*: H^n_\Delta(Y) \to H^n(X)$ $\forall n$.
If f is a homotopy equivalence, then f_* is an isomorphism for each n.

Cor $H^k(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & k=0 \\ 0, & \text{otherwise.} \end{cases}$

Rmk There are other cohomology theories, but in some sense they are all equal.

Instuhl Thm X sm mfd. $H^n_{dR}(X;\mathbb{R}) \cong H^n_{sing}(X;\mathbb{R}) \cong H^n_\Delta(X;\mathbb{R})$

We denote them as $H^n(X)$

Mayer-Vietoris – key computational tool in cohomology

Recall: Exact, SES, LES.

Thm X mfd. U, V open cover of X. Then we have a LES

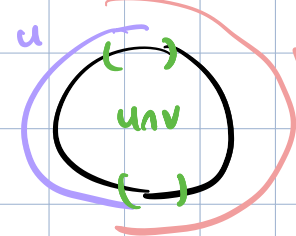
$$\cdots \to H^n(X) \to H^n(U) \oplus H^n(V) \to H^n(U \cap V) \to H^{n+1}(X) \to \cdots$$

The M-V sequence allows us to treat $H^n(X)$ as a "function" of $H^n(U)$, $H^n(V)$ and $H^n(U \cap V)$. Especially useful when some terms of the LES is 0, in which case exactness implies that the neighboring maps are injections, surjections or isomorphisms.

Exercise Use M-V to show that

$$H^k(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ 0, & \text{o.w.} \end{cases}$$

Hint: Consider the open cover



and that more generally.

$$H^k(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k=0, n \\ 0, & \text{o.w.} \end{cases}$$

Rmk Intuitively, cohomology counts the "n-dimensional holes".

Here's a more interesting example. Let $\mathbb{P}^n$ be the complex projective space.

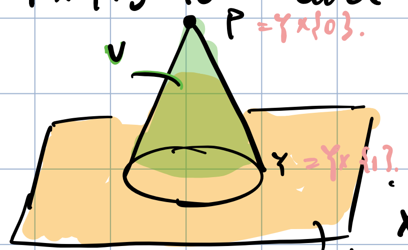
Ex / Thm $H^k(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } 0 \le k \le 2n \text{ even} \\ 0, & \text{o.w.} \end{cases}$

$(\cong \mathbb{Z}[x]/x^{2n+1})$.

For a "good" pair of top. spaces (X, Y), $Y \subseteq X$, define the mapping cone of Y to be

$$C = X \sqcup (Y \times I)/(Y \times \{0\}).$$

where we glue the base of the cylinder $Y \times \{1\}$ to X and collapse the top to a point p



Let $U = C - P$, $V = Y \times [0,1)/Y \times \{0\}$

Then (orange) (green).

$U \cup V = C$, $U \simeq X$, $V \simeq *$, $U \cap V \simeq Y$. M-V gives LES

$$\cdots \to H^i(C) \to H^i(X) \to H^i(Y) \to H^{i+1}(C) \to \cdots$$

$$\overset{\text{"}}{H^i(C)} = H^i(X/Y).$$ collapsing Y to a point

Apply this to $(X, Y) = (\mathbb{P}^n, \mathbb{P}^{n-1})$. The complement $X - Y = \mathbb{C}^n$.

Collapsing Y to a point $\Leftrightarrow$ adding a point of infinity to $\mathbb{C}^n$, so $X/Y \simeq S^{2n}$. We get a LES

$$H^{2n+1}(\cancel{S^{2n}})^{0} \to H^{2n+1}(\mathbb{P}^n) \xrightarrow{\sim} H^{2n+1}(\mathbb{P}^{n-1})$$

$$H^{2n}(\cancel{S^{2n}})^{\mathbb{Z}} \to H^{2n}(\mathbb{P}^n) \to H^{2n}(\cancel{\mathbb{P}^{n-1}})_{0}$$

$$\cdots \qquad \cdots$$

$$0 \qquad\qquad\qquad\qquad 0$$

$$H^1(\cancel{S^{2n}})^{0} \to H^1(\mathbb{P}^n) \xrightarrow{\sim} H^1(\mathbb{P}^{n-1})$$

$$H^0(S^{2n}) \to \underset{\mathbb{Z}}{\underset{\wr\!\wr}{H^0(\mathbb{P}^n)}} \to \underset{\mathbb{Z}}{\underset{\wr\!\wr}{H^0(\mathbb{P}^{n-1})}}$$

$\Rightarrow$ $H^k(\mathbb{P}^n) = H^k(\mathbb{P}^{n-1})$ when $0 < k < 2n$ and $k > 2n$.

$\Rightarrow$ $H^k(\mathbb{P}^n) = H^k(\mathbb{P}^{n-1}) = \cdots = H^k(\mathbb{P}^0) = 0$ for $k > 2n$.

$\Rightarrow$ $H^{2n}(\mathbb{P}^n) = \mathbb{Z}$

$\Rightarrow H^k(\mathbb{P}^n) = H^k(\mathbb{P}^{n-1}) = \cdots = H^k(\mathbb{P}^{2k}) = \mathbb{Z}$ for $k < 2n$.

Also, $H^0(\mathbb{P}^n)$ since projective spaces are connected. □

§2 Čech Cohomology

Sheaf

Def X top. space. A presheaf $\mathcal{F}$ of abelian groups on X consists of the following data

- $\forall$ open $U \subseteq X$, assigns abelian group $\mathcal{F}(U)$.
- $\forall V \subseteq U$, $\exists$ restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$.

such that $\mathcal{F}(\phi) = 0$,

$$\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(U),$$

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV} \qquad \text{if } W \subseteq V \subseteq U.$$

Rmk • We can replace the word "abelian group" by other categories.

e.g. rings, Vect_k, Mod_A, etc.

- The elements of $\mathcal{F}(U)$ are called "sections of $\mathcal{F}$ over U."

Ex $\mathcal{F}(U)$ = ring of differentiable fns on U. $\equiv \mathcal{O}_X(U)$.

"smooth" "continuous" "C^k-dif."

For $V \subseteq U$, we have a restriction map

$$\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$$
$$f \mapsto f|_V.$$

($\mathcal{F}(\phi) = 0$, identity, compatibility are easy to check).

A sheaf is a presheaf with some global gluing conditions.

Def Presheaf $\mathcal{F}$ over X is a sheaf if

- If $\{V_i\}_{i\in I}$ open cover of U and $f_1, f_2 \in \mathcal{F}(U)$ s.t. $f_1|_{V_i} = f_2|_{V_i} = 0$ for all i, then $f_1 = f_2$.
- If $\{V_i\}_{i\in I}$ open cover of U and $\exists f_i \in \mathcal{F}(V_i)$ for all i such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{V_i} = f_i \ \forall i$.

Ex $\mathcal{O}_X$ is a sheaf, called the **structure sheaf**.

Ex (Constant sheaf) X top. space. A ab. group with discrete topology. The **constant sheaf** A_X is defined as follows: for open $U \subseteq X$, $A_X(U) := \{ \varphi : U \to A \mid \varphi \text{ cts} \}$. Note that if U cted, then $A_X(U) \cong A$. In general, if X is locally connected, then $A(U) \cong A^{\oplus J}$, where J is the number of connected components of U.

Now we're ready to define Čech cohomology, which has the advantage of being quite explicit and computable. As we will see, it is defined in a way similar to simplicial cohomology (this will be made formal through nerve complex). Roughly speaking, Čech $\sim$ sheaf cohomology as simplicial $\sim$ singular cohomology (though we won't introduce sheaf cohomology today).

X top. space. $\mathcal{U} = \{U_i\}_{i \in I}$ open covering. Denote $U_{i_0, \dots, i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$

I is a totally ordered set

$\mathcal{F}$ sheaf on X. We define the **Čech n-cochains** as

$$C^n(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_n} \mathcal{F}(U_{i_0, \dots, i_n}).$$

abelian group associated to the intersection of open sets.

An element $\alpha \in C^n(\mathcal{U}, \mathcal{F})$ is determined by giving an element $f_{i_0, \dots, i_n}$ for each $(n+1)$-tuple $(i_0 < \cdots < i_n)$ of elements in I. Coboundary map defined as

$$(\delta f)_{i_0, \dots, i_{n+1}} = \sum_k (-1)^k f_{i_0, \dots, \hat{i_k}, \dots, i_{n+1}} \big|_{U_{i_0, \dots, i_{n+1}}}$$

(Note: $U_{i_0, \dots, i_{n+1}} \subseteq U_{i_0, \dots, \hat{i_k}, \dots, i_{n+1}}$ $\leadsto$ Restriction map on $f_{i_0 \dots \hat{i_k} \dots i_{n+1}} \in \mathcal{F}(U_{i_0 \dots \hat{i_k} \dots i_{n+1}})$)

Check: $\delta^2 = 0$.

The **nth-Čech cohomology group**

$$\check{H}^n(\mathcal{U}, \mathcal{F}) := H^n(C^\bullet(\mathcal{U}, \mathcal{F}), \delta^\bullet).$$

Ex $\check{H}^0(\mathcal{U}, \mathcal{F}) = \{ f_i \in \prod \mathcal{F}(U_i) \mid f_i = f_j \text{ on } U_{ij} \}$.

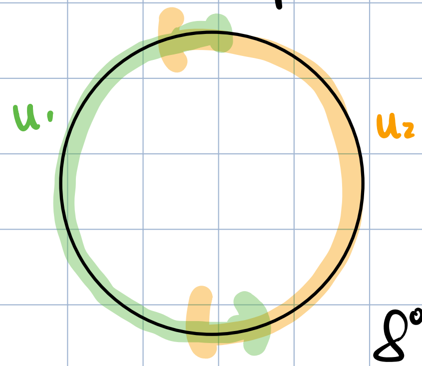
$$\check{H}^1(\mathcal{U}, \mathcal{F}) = \frac{\{ f_{ij} \in \prod \mathcal{F}(U_{ij}) \mid f_{ik} = f_{ij} + f_{jk} \text{ on } U_{ijk} \}}{\{ f_{ij} \mid \exists \varphi_i,\ f_{ij} = \varphi_i - \varphi_j \}}.$$

This resembles simplicial cohomology. We make this formal by introducing the nerve of the covering $N(\mathcal{U})$. The set of simplices in $N(\mathcal{U})$ is given by $\Sigma = \{\{i_0,\dots,i_n\} \mid U_{i_0\dots i_n} \neq \emptyset\}$.

Thm If each $U_{i_0\dots i_n}$ is connected, then $C^n(\mathcal{U}, A_X) = C^n_\Delta(X; A)$

<u>Slogan</u> : Cech cohomology of a constant sheaf gives ordinary cohomology w/ these coefficients.

<u>Ex</u>



$\mathcal{U} = \{U_1, U_2\}$.

Compute $\check{H}^*(\mathcal{U}, \mathbb{Z})$ ← constant sheaf on S^1 def. by $\mathbb{Z}$.

$C^0(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}(U_1) \times \mathbb{Z}(U_2) = \mathbb{Z} \times \mathbb{Z}$

$C^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \times \mathbb{Z}$

$\delta^0 : C^0 \to C^1$, $\langle a, b \rangle \mapsto \langle b-a, b-a \rangle$.

$\Rightarrow \check{H}^0(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$, $\check{H}^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$. Matches with simplicial!

Our next goal is to relate $\check{H}^1(\mathcal{U}, \mathcal{F})$ with the language of vector bundles

Crash Course in Vector Bundles

A complex (resp. real) vector bundle of rank r over a mfd X consists of

- A mfd E called the total space
- A surjective map $\pi : E \to X$

such that

(i) (local trivialization) There is an open cover $\{U_i\}_{i \in I}$ of X and diffeo. $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ s.t. the following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "h_i"] \arrow[d, "\pi_i"'] & U_i \times \mathbb{C}^r \\
U_i & \arrow[ul, leftarrow, "pr_1"']
\end{tikzcd}$$

(where $\pi_i = \pi|_{\pi^{-1}(U_i)}$, pr_1 is projection onto the first factor)

(ii) (transition functions). If $U_i \cap U_j \neq \emptyset$, then the map

$$h_j \circ h_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$$

is a diffeo. of the form $h_j \circ h_i^{-1}(x, v) = (x, g_{ji}(x)(v))$

these are called transition functions

where $g_{ji} : U_i \cap U_j \to GL(r, \mathbb{C})$.

transition fn is a linear isomorphism.

Rmk When rank $r=1$, $E \to X$ is called a line bundle.

Ex $E = X \times \mathbb{C}^r$ is the product vector bundle
$$\begin{array}{c} E = X \times \mathbb{C}^r \\ \downarrow pr_1 \\ X \end{array}$$

Def $\pi: E \to X$ is a trivial vector bundle if there exists a global trivialization $h: E \to X \times \mathbb{C}^r$.

A vector bundle is determined by the "gluing data" between local trivializations.

Fact: Given an open cover $\{U_i\}_{i \in I}$ of X More generally... $GL(\mathcal{O}_X, r)$.
$$\left\{ \begin{array}{l} \text{vector bundle } \pi: E \to (X, \mathcal{O}_X) \\ \text{of rank } r \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{transition maps } g_{ij}: U_i \cap U_j \to GL(\mathbb{C}, r) \\ \text{s.t. } g_{ii} = id,\ g_{ij} = g_{ji}^{-1};\ g_{ik} = g_{ij} g_{jk} \end{array} \right\}$$

Recall that X top. space, $\mathcal{U}$ cover, $\mathcal{F}$ sheaf of ab. grp on X, we have
$$\check{H}^1(\mathcal{U}, \mathcal{F}) = \frac{\{(f_{ij}) \in \prod \mathcal{F}(U_{ij}) \mid f_{ik} = f_{ij} + f_{jk} \text{ on } U_{ijk}\}}{\{f_{ij} \mid \exists \phi_i,\ f_{ij} = \phi_i - \phi_j\}}.$$
We may extend this to a sheaf of arbitrary group $\mathcal{G}$.
$$\check{H}^1(\mathcal{U}, \mathcal{G}) = \{(g_{ij}) \in \prod \mathcal{G}(U_{ij}) \mid g_{ik} = g_{ij} g_{jk} \text{ on } U_{ijk}\} / \sim$$

where $g_{ij} \sim \bar{g}_{ij}$ if $\exists (\sigma_i) \in \prod \mathcal{G}(U_i)$ s.t. $g_{ij} = \sigma_i \bar{g}_{ij} \sigma_i^{-1}$.

Considering $\mathcal{G}: U \mapsto GL(\mathcal{O}_X(U))$

Thm There is a bijection between the following sets

1. The set of isomorphism classes of rank r vector bundles over $(X, \mathcal{O}_X)$. trivializable over $\mathcal{U}$.
2. $\check{H}^1(\mathcal{U}, GL(\mathcal{O}_X, r))$

Pf $\square$

When $n=1$, the set of isomorphism classes of line bundles carries the structure of a group, i.e. $\check{H}^1(X, \mathcal{O}_X^*)$. This group is called the Picard group $\mathrm{Pic}(X)$.

Exponential Exact Sequence

SES $\quad 0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{e^{2\pi i \cdot}} \mathbb{C}^* \to 0$

$\Rightarrow$ SES $\quad 0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{e^{2\pi i}} \mathcal{O}_X^* \to 0.$

Recall that SES gives LES in homology. This gives us a LES

$$0 \to \check{H}^1(X, \mathbb{Z}) \to \check{H}^1(X, \mathcal{O}_X) \to \check{H}^1(X, \mathcal{O}_X^*) \to \check{H}^2(X, \mathbb{Z}) \to \check{H}^2(X, \mathcal{O}_X) \to \cdots$$

Let's use this to compute $\mathrm{Pic}(\mathbb{P}^n) = \check{H}^1(\mathbb{P}^n, \mathcal{O}^*)$.

<u>Fact</u>: $\check{H}^1(\mathbb{P}^n, \mathcal{O}) = \check{H}^1(\mathbb{P}^n, \mathcal{O}) = 0$ (algebraic computation).

This immediately implies that $\check{H}^1(\mathbb{P}^n, \mathcal{O}^*) = \check{H}^2(\mathbb{P}^n, \mathbb{Z}) = H^2_\Delta(\mathbb{P}^n; \mathbb{Z})$

$\Rightarrow \mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}.$ $\qquad = \mathbb{Z}.$