October 4, 2024

Introduction

- A History of Algebraic and Differential Topology, 1900 - 1960 by Jean Dieudonné
- REU paper by Christopher Stith
- calculation of $\pi_1(S^1)$
- covering spaces of (X, x_0)
- $\pi_1(X, x_0)$

0. History

- 18^{th} and 19^{th} centuries
- Poincaré (1854 - 1912)
- prehistory of algebraic topology
- 1873 $\Rightarrow$ Riemann and Betti
- after 1910
- Brouwer's work on approximation
- Weyl's treatment of the theory of Riciann surfaces
- Frechet (1906)
- Hausdorff (1914)

1. Introduction to Algebraic Topology

- 1.1: topological space (X, T)
  - (1) $\emptyset, X \in T$
  - (2) The intersection of finitely many sets in T is again in T
  - (3) The union of any collection of sets in T is again in T
- elements of T and open
- T_X, T_Y
- 1.2: closed points
- 1.3: connected space
- continuity: $\mathbb{R}$, $\mathbb{R}^n$, topology
- 1.4: $f: X \to Y$ is continuous ...
- 1.5: continuity for a path
- 1.6: path-connected space
- local properties, "zoomed in", "small"
- 1.7: neighborhood of x
- 1.8: local path-connectedness
- circle = square
- homeomorphism
- 1.9: $f \circ g = id_Y$ & $g \circ f = id_X$
- 1.10: homeomorphic
- homotopy equivalence
- "continuous deformation"
- homeomorphic $\Rightarrow$ homotopy equivalent
- 1.11: $H: X \times I \to Y$ s.t. $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ $\forall x \in X$
- homotopic: $f \simeq g$
- $h_t: X \to Y$
- $h_t := H|_{X \times \{t\}}$

- 1.12: $\simeq$ is an equivalence relation on the set $C(X, Y)$ of continuous functions from X to Y
- Proof follows in lecture

- 1.14: homotopy equivalent w.r.t $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$
- 1.16: contractible
- 1.17: null-homotopic
- $h_t(0) = f(0) = g(0)$ and $h_t(1) = f(1) = g(1)$ $\forall t \in I$
- 1.18: homotopy relative

2. The Fundamental Group

- topological invariants
- fundamental group
- "group of loops"
- 2.1: loop in X is a path f with $f(0) = f(1)$
- 2.2: constant loop in X based at x_0 ...
- 2.4: path multiplication $\Rightarrow$ $f \cdot g = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$
- 2.5: $f^{-1}(t) = f(1-t)$
- $fg = f \cdot g$
- $ff^{-1} \simeq c$
- is loop multiplication associative?
- $(f \cdot g) \cdot h \overset{?}{=} f \cdot (g \cdot h)$
- $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$
- $[f] \cdot [g] = [f \cdot g]$
- $[f] \cdot [c] = [f]$
- $[f] \cdot [f^{-1}] = [c]$
- 2.6: fundamental group $\pi_1(X, x_0)$
- 2.7: homomorphism $p_*([f]) = [p \circ f]$
- 2.8: Let X be a space and let $x_0, x_1 \in X$ in the same path-connected component. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$
- Proof follows in lecture
- 2.9: If X is path-connected, then $\forall x_0, x_1 \in X$, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. We then refer unambiguously to this isomorphism class as "the fundamental group of X," denoted $\pi_1(X)$
- 2.10: simply-connected space

3. The Fundamental Group of S^1

- $\pi_1(S^1) \cong (\mathbb{Z}, +)$
- $f: I \to S^1$
- winding number
  - $n(f)$
  - $(+) \Rightarrow$ counterclockwise
  - $0 \Rightarrow$ never makes full loop
  - $(-) \Rightarrow$ clockwise
- 2 loops in S^1 are homotopic $\Leftrightarrow$ same winding number
- $\mathbb{R}$ projected onto the circle by the (complex) exponential map
- lifts
- $\mathbb{R} \Rightarrow$ covering space
- 3.1: $p: \mathbb{R} \to S^1$; $p(x) = e^{ix} = (\cos(x), \sin(x))$; p is an open map
- 3.2: restriction $p|_{(x, x+2\pi)}$ of p to an open interval of the form $(x, x+2\pi)$ is a homeomorphism onto $S^1 \setminus \{p(x)\}$
- 3.3: unique path lifting property
- 3.4: winding number $= n(f) = \dfrac{\tilde{f}(1) - \tilde{f}(0)}{2\pi}$ where $\tilde{f}$ is a lift of f
- 3.6: Let $f, g: I \to S^1$ be loops in S^1 with identical basepoint. Then:
  - (1) $n(fg) = n(f) + n(g)$
  - (2) $f \simeq g \Rightarrow n(f) = n(g)$
  - (3) $n(f) = n(g) \Rightarrow f \simeq g$

(4) Let $x \in S^1$. $\forall k \in \mathbb{Z}$, $\exists$ loop $h: I \to S^1$ based at x s.t. $n(h) = k$

- Proof of 1,4 follows in lecture
- 3.7: $\pi_1(S^1) \cong (\mathbb{Z}, +)$

4. Covering Spaces

- 4.1: A covering space of a space X is a space $\tilde{X}$ together with a map $p: \tilde{X} \to X$, called a covering map, s.t. each $x \in X$ is contained in an open neighborhood V whose preimage $p^{-1}(V)$ is the union of disjoint open sets in $\tilde{X}$, each of which is projected homeomorphically by p onto V. That is, $p^{-1}(V) = \bigcup_{\alpha \in A} U_\alpha$, where the U_α are disjoint open sets in $\tilde{X}$ s.t. $p|_{U_\alpha} : U_\alpha \to V$ is a homeomorphism
- $p: \tilde{X} \to X$
- $p^{-1}(x) \Rightarrow$ fiber
- degree n
- n-fold covering
- evenly covered
- sheets
- e^{ix}, $e^{i(x-\delta, x+\delta)}$
- 4.2: $p: \tilde{X} \to X$, $f: Y \to X$, $\tilde{f}: Y \to X$, $p \circ \tilde{f} = f$
- 4.3: homotopy lifting property
- 4.4: based map, $f: (X, x_0) \to (Y, y_0)$
- 4.5: covering space, injective, ...
- 4.6: lifting criterion
- 4.7: unique lifting property
- isomorphism
- 1-to-1 correspondence
- 4.8:

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\Phi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

5. Covering Spaces and the Fundamental Group

- 5.1: semilocally simply-connected space X
- 5.2: Simply connected covering space of X
- 5.3: $H \leq \pi_1(X, x_0)$, $\exists\, p_H : (\tilde{X}, x_0) \to (X, x_0)$ s.t. $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$
- Finished surjectivity, now for the question of injectivity up to isomorphism
- 5.4: isomorphic covering spaces $\Leftrightarrow p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$
- 5.5: 1-to-1 correspondence b/w the set of all based isomorphism classes of path-connected covering spaces $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of all subgroups of $\pi_1(X, x_0)$
- Proof follows in lecture