Representations of the fundamental group:

Some definitions:

A path in a topological space X is a continuous map $\gamma : [0,1] \to X$.

A loop is a path γ where $\gamma(0) = \gamma(1)$.

A path homotopy is a continuous map $F : [0,1] \times [0,1] \to X$ where:

$$F(s,0) = \gamma_1(s)$$
$$F(s,1) = \gamma_2(s)$$
$$F(0,t) = \gamma_1(0) = \gamma_2(0)$$
$$F(1,t) = \gamma_1(1) = \gamma_2(1).$$

The fundamental group $\pi_1(X, x_0)$ of a space X is the set $\{\text{all loops based at } x_0\}/\text{homotopy}$, with the group operation of concatenation.

A representation of a group is a homomorphism $\rho : G \to GL(V)$, where V is a vector space.

A covering of X is a continuous map $\pi : Y \to X$ such that for all $x \in X$, there exists a neighborhood U_x such that $\pi^{-1}(U) = \bigsqcup_{d \in D_x} V_d$, where D_x is a discrete space, and $\pi|_{V_d} : V_d \to U_x$ is a homeomorphism for all $d \in D_x$.

The universal covering of X is the covering $\pi : \tilde{X} \to X$ where $\tilde{X}$ is simply connected, i.e. $\pi_1(\tilde{X})$ is trivial.

Presheaves and Sheaves

For a topological space X, a *presheaf* $\mathcal{F}$ on X consists of:

(1) an association of all open, non-empty sets $U \subset X$ with a group $\mathcal{F}(U)$, and

(2) for all $V \subseteq U$, a morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$.

Additionally,

(1) $\mathcal{F}(\emptyset) = 0$

(2) ρ_{UU} is an identity map

(3) If $W \subset V \subset U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

A presheaf $\mathcal{F}$ is a *sheaf* if the following two conditions also hold:

(1) Given nonempty, open U, and an open cover $\{U_i\}_{i\in I}$ of U, if $s, t \in \mathcal{F}(U)$, and $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

(2) Given sections $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$, where $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ when $U_i \cap U_j \neq \emptyset$, there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

For a point $x \in X$, the *stalk* $\mathcal{F}_x$ is the disjoint union of the sets $\mathcal{F}(U)$ for all open $U \ni x$, mod the relation $s \sim t$ for $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$ if there exists open $W \subset U \cap V$ with $s|_W = t|_W$.

A sheaf $\mathcal{F}$ is *constant* and associated to a set A if $\mathcal{F}(U) \cong A$ for all open, connected $U \subset X$.

A sheaf is *locally constant* if for all $x \in X$, there exists open $U \ni x$ such that $\mathcal{F}|_U$ is constant. A locally constant sheaf is also called a local system.

Thm. For X connected, there is a correspondence
$$\left\{\begin{matrix}\text{local systems on} \\ X \text{ up to isomorphism}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{representations of} \\ \pi_1(X) \text{ up to isomorphism}\end{matrix}\right\}.$$

Pf. local system $\longrightarrow$ representation

Let $\gamma \in \pi_1(X, x_0)$, and let $\mathcal{L}$ be a local system on X, so that every $x \in X$ has an open neighborhood $\mathcal{U}_x$ where for all $u \in \mathcal{U}_x$, $\mathcal{L}_u \cong L$. $\{\mathcal{U}_x\}_{x \in \gamma([0,1])}$ is an open cover of $\gamma([0,1])$.

$[0,1]$ compact and γ continuous $\Rightarrow$ $\gamma([0,1])$ compact, so we can choose a finite subcover $\{\mathcal{U}_i\}$ of $\gamma([0,1])$ such that $x_0 \in \mathcal{U}_0 \cap \mathcal{U}_n$, and $\mathcal{U}_i \cap \mathcal{U}_{i+1} \neq \emptyset$. Furthermore, we can choose $x_i \in \mathcal{U}_i \cap \mathcal{U}_{i+1}$.

We have isomorphisms
$$\mathcal{L}_{x_0} \xrightarrow{\sim} \mathcal{L}(\mathcal{U}_1) \xrightarrow{\sim} \mathcal{L}_{x_1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{L}(\mathcal{U}_n) \xrightarrow{\sim} \mathcal{L}_{x_0}.$$
By composing these isomorphisms, we get an automorphism $T: \mathcal{L}_{x_0} \to \mathcal{L}_{x_0}$. We call T a monodromy along γ.

The monodromy of a local system defines a map
$$\rho: \pi_1(X, x_0) \longrightarrow GL(\mathcal{L}_{x_0}).$$

This map we can show is well-defined. Let $H: [0,1] \times [0,1] \to X$ be a homotopy, where $H(t,0) = \gamma(t)$, $H(t,1) = \gamma'(t)$. Then choose partitions $0 = a_1 < \cdots < a_n = 1$ and $0 = b_1 < \cdots < b_m = 1$ of $[0,1]$ such that $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$ is good for $\mathcal{L}$, i.e. for all i,j, there exists open $\mathcal{U}$ containing $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$ where $\mathcal{L}|_{\mathcal{U}}$ is constant.

We then have the following diagram:

$$\begin{tikzcd}
\mathcal{L}|_{H(0,b_{j+1})} \arrow[r] \arrow[d, equal] & \mathcal{L}|_{H(a_1,b_{j+1})} \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \mathcal{L}|_{H(1,b_{j+1})} \arrow[d, equal] \\
\mathcal{L}|_{H(0,b_j)} \arrow[r] & \mathcal{L}|_{H(a_1,b_j)} \arrow[r] & \cdots \arrow[r] & \mathcal{L}|_{H(1,b_j)}
\end{tikzcd}$$

$H([a_i,a_{i+1}]\times[b_j,b_{j+1}])$ for all i,j, so each square in the diagram commutes. Thus $\rho(H(t,b_j))=\rho(H(t,b_{j+1}))$ for all j, and $\rho(\gamma)=\rho(\gamma')$, i.e. ρ is well-defined.

representation $\longrightarrow$ local system

Let $\rho: \pi_1(X,x_0)\longrightarrow GL(L)$ be a representation of $\pi_1(X)$, and $F:\tilde{X}\to X$ be the universal covering of X.
We can define a vector bundle $E=\tilde{X}\times L/\sim$, where $(\tilde{b}_1,v_1)\sim(\tilde{b}_2,v_2)$ if there exists $\sigma\in\pi_1(X)$ such that $b_2=\sigma b_1$, $v_2=\rho(\sigma^{-1})(v_1)$.
The sheaf of sections of this bundle, $\mathcal{L}$, is such that, given $v\in L$, the local sections of an open subset $U\subset X$ consist of $\bar{v}(z)=(F|_{W_\alpha}^{-1}(z),v)$, $z\in U$, where $F^{-1}(U)=\bigsqcup_{\alpha\in D} W_\alpha$, D discrete.
Clearly, this gives us a local system, since these sections are constant.