Riemann - Hilbert Correspondance:

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① Introduction of sheaves, rings, manifolds, tangent spaces

Manifolds: n-manifold is a topological space with the property that each point has a neighbourhood that is homeorphic to an open set of n-dimensional Euclidean-space -

Sheaves: A tool for systematically tracking data (such as sets. Abelian groups, rings) attached to the open sets of a topological space and defined locally with regard to them

Rings: A ring $R = (R, +, \cdot)$ consists of a set R together with two binary operations $+$ and $\cdot$ on R s.t.

(i) R together with operation $+$ is an abelian group.

(ii) The binary operation $\cdot$ is associative

(iii) Distributive law hold for all $r, s, t \in R$

$$(r+s)t = rs + rt$$

$$t(r+s) = rt + ts$$

Tangent spaces:

A germ of functions of a C^∞ function at p in Manifold M. is defined to be an equivalence class of C^∞ functions in a neighbourhood of p in M. The set of germs is denoted as $C_p^\infty(M)$. $C_p^\infty(M)$ is a ring with the addition and multiplication.

A derivation at a point in a manifold M:

A linear map $D: C_p^\infty(M) \to \mathbb{R}$ s.t.

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Tangent vector at p is the derivation at p.

And these tangent vectors at p in manifold form a vector space or tangent space denoted as $T_p(M)$

② Differential 1-form.

Let M be the smooth manifold and p a point in M. The cotangent space of M at p denoted by $T_p^*(M)$ is defined to be the dual space of the tangent space $T_p(M)$

→ consisting all linear forms on $T_p(M)$ and together with pointwise addition and multiplication by constants

$$T_p^*(M) = (T_p(M))^\vee$$

an element in $T_p^*(M)$ is called a covector at p.

So a covector w_p at p is a linear function

$$w_p : T_p(M) \to \mathbb{R}$$

Then, a covector field, which is differential 1-form is a function w that assigns every point p in M with a covector w_p.

Then, differential 1-form for a function f on a manifold at p denoted as $(df)_p$ is:

$$(df)_p(T_p(M)) = T_p(M)f.$$

③ The Riemann-Hilbert Correspondance:

To begin with, consider the sheaf of rings $\mathcal{O}$ of holomorphic functions on the open domain $D \subset \mathbb{C}$. A sheaf of $\mathcal{O}$-modules is a sheaf of abelian groups $\mathcal{F}$ on X s.t for every open $U \in X$ the group $\mathcal{F}(U)$ is equipped with an $\mathcal{O}(U)$-module structure $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ making the diagram below:

$$\begin{array}{ccc} \mathcal{O}(U)\times\mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(V)\times\mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commute for each inclusion of opensets $V \subset U$.

Also, we need the sheaf Ω^1_D of differential 1-form on D.

One possible example of Ω^1_D is defined as:

Given a holomorphic function f on some $U \subset D$, consider the function $df: U \to \mathbb{C}$ given by $z \to f'(z)$. Ω^1_D is defined as a complex function $w_U : U \to \mathbb{C}$ s.t. each $z \in U$ has an open neighborhood $V \in U$ mapped isomorphically onto an open disk around 0 in $\mathbb{C}$ by a holomorphic function $g \in \mathcal{O}(V)$ s.t $w_{U|V} = f\,dg$ with some $f \in \mathcal{O}(V)$

Finally, $\mathcal{F}$ is defined locally free sheaf if every point of D has an open neighbourhood $V \subset D$ s.t. $\mathcal{F}|_V \cong \mathcal{O}^n|_V$, where $\mathcal{O}^n$ denotes the n-fold direct sum of $\mathcal{O}$. The integer n is called the rank of $\mathcal{F}$, and we say $\mathcal{F}$ is free if there is actually an isomorphism $\mathcal{F} \cong \mathcal{O}^n$ on the whole of D.

Holomorphic connection on D is defined as a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a locally free sheaf on D and $\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D$ (tensor product.), which is a morphism of sheaves of $\mathbb{C}$-vector spaces satisfying 'Leibniz rule'

$$\nabla(fs) = df \otimes s + f\,\nabla(s)$$

for all $U \subset D$, $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. we call ∇ as the connection map. And we call $s \in \mathcal{E}(U)$ horizontal if it satisfies $\nabla(s) = 0$, and these horizontal s will form a subsheaf $\mathcal{E}^{\nabla} \subset \mathcal{E}$.

There is a morphism $(\mathcal{E}, \nabla) \to (\mathcal{E}', \nabla')$ which is also a morphism of $\mathcal{O}$-modules $\phi: \mathcal{E} \to \mathcal{E}'$ making the diagram commute:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\nabla"] \arrow[d, "\phi"'] & \mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D \arrow[d, "\phi \otimes \mathrm{id}"] \\
\mathcal{E}' \arrow[r, "\nabla'"] & \mathcal{E}' \otimes_{\mathcal{O}} \Omega^1_D
\end{tikzcd}$$

Example:

Assume that $\mathcal{E} \cong \mathcal{O}^n$ is a free $\mathcal{O}$-module.

Then we can identify $s \in \mathcal{E}(U)$ with n tuples $(f_1, f_2 \dots f_n)$ of holomorphic functions on U. Then there is an obvious connection map $d: \mathcal{E}$ or $\mathcal{O}^n \to (\Omega^1_D)^{\oplus n}$ direct sum by setting $d(f_1, \dots, f_n) = (df_1, df_2 \dots df_n)$, which satisfies the 'Lebniz rule'.

Lemma 1: The subsheaf $\mathcal{E}^\nabla$ is a local system of dimension equal to the rank of $\mathcal{E}$.

Proposition 1: The functor $(\mathcal{E}, \nabla) \to \mathcal{E}^\nabla$ induces an equivalence between the category of holomorphic connections on D and that of complex local systems on D.

(This lemma is called Riemann Hilbert correspondence)

Proof:

Take a local system $\mathcal{L}$ on D. The rule

$U \to \mathcal{L}(U) \otimes_{\mathbb{C}} \mathcal{O}(U)$ defines a locally free sheaf $\mathcal{E}_{\mathcal{L}}$ on D. We define a connection map $\nabla_{\mathcal{L}}$ on $\mathcal{E}_{\mathcal{L}}$ as follows. Given an open subset U where $\mathcal{L}|_U \cong C^n$, fix a C-basis $s_1, s_2 \dots s_n$ of $\mathcal{L}(U)$. Then each section s_i of $\mathcal{E}_{\mathcal{L}}(U)$ can be uniquely written as a sum $\sum s_i \otimes f_i$ with some $f_i \in \mathcal{O}(U)$. Now Define $\nabla_{\mathcal{L}}|_U$ by setting $\nabla_{\mathcal{L}}(\sum s_i \otimes f_i) = \sum s_i \otimes df_i$.

As two bases of $\mathcal{L}(U)$ differ by a matrix whose entries are in C and hence are eliminated by the differential d, so $\nabla_{\mathcal{L}}|_U$ doesn't depend on the choice of the s_i.

Therefore $\nabla_{\mathcal{L}}|_U$ defined over the various patch together to form a map $\nabla_{\mathcal{L}}$ defined over the whole of D. and because

$(\mathcal{E}, \nabla) \to \mathcal{E}^{\nabla}$ and $\mathcal{L} \to (\mathcal{E}_{\mathcal{L}}, \nabla_{\mathcal{L}})$ induce an equivalence of categories. So by the construction above, it can define over the whole of D $\square$

Remark:

Such Riemann-Hilbert correspondence holds for arbitrary Riemann surfaces, and extends to higher dimensional complex manifolds as well. In the higher dimensional case, one has to impose a further condition on connection $(\mathcal{E}, \nabla)$, we can write as

$$\mathcal{E} \otimes_{\mathcal{O}} \Omega_X^i \to \mathcal{E} \otimes_{\mathcal{O}} \Omega_X^{i+1} \quad \text{for each } i \geq 0$$

and $\nabla(w \otimes s) = dw \otimes s + (-1)^i w \wedge \nabla(s)$

And we define the connection is integrable/flat, if $\nabla \circ \nabla_1$ is trivial

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}} \Omega_X^1 \xrightarrow{\nabla_1} \mathcal{E} \otimes_{\mathcal{O}} \Omega_X^2 \quad \text{is zero (trivial)}$$

Another proposition:

Given a holomorphic connection $(\mathcal{E}, \nabla)$ on D, there is a connection $(\bar{\mathcal{E}}, \bar{\nabla})$ on $\mathbb{P}^1(\mathbb{C})$ with logarithmic poles along S satisfying $(\bar{\mathcal{E}}, \bar{\nabla})|_D \cong (\mathcal{E}, \nabla)$

*Proof:

(Take small open discs $D_i \subset \mathbb{P}^1(\mathbb{C})$ around each x_i, that do not meet, and write n for the rank of the connection $(\mathcal{E}, \nabla)$.

there is a premise that for each i, there exists an $n \times n$ $y' = A_i y$ of linear differential equations having a simple pole at x_i and holomorphic elsewhere. We can write the $(\mathcal{E}_i, \nabla_i)$ for the connection given by $\mathcal{E}_i = \mathcal{O}|_D^n$ and $\nabla_i((f_1, f_2 \cdots f_n)) = (df_1 \cdots df_n) + A_i(f_1 \cdots f_n)$. We can cover each open set $D \cap D_i = D_i \setminus \{x_i\}$ by two simply connected open subsets U_{i+} and U_{i-}. $U_{i+} = D \setminus \{0, i\mathbb{R}\}$, $U_{i-} = D \setminus \{0, -i\mathbb{R}\}$

Over each U_{i+} and U_{i-} the constant sheaves $\mathcal{E}^\nabla$ and $\mathcal{E}_i^{\nabla_i}$ are both trivial of dimension n. Then we may patch the locally free sheaves $\mathcal{E} \cong \mathcal{E}^\nabla \otimes_{\mathbb{C}} \mathcal{O}$ and $\mathcal{E}_i \cong \mathcal{E}_i^{\nabla_i} \otimes_{\mathbb{C}} \mathcal{O}$ together. The restrictions of $\mathcal{E}$ and $\mathcal{E}_i$ to the U_{i+} and U_{i-} are both equipped with the trivial connection map since they correspond to trivial connection map since they correspond to trivial local systems. So the connections also patch.

The construction of patch.