Distinguishing Numbers

Caleb Ji

Washington University in St. Louis

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The Distinguishing Number of a Graph

Definition (Albertson, Collins, 1996)

Given a simple graph G, define the **distinguishing number**, $D(G)$, to be the smallest r for which there exists a labeling of the graph $\phi : V(G) \to \{1, 2, \ldots, r\}$ such that the only element of the automorphism group of G that preserves the labels is the identity.

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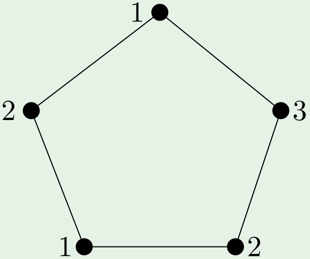
Example

C_5: cycle graph on 5 vertices. $\mathrm{Aut}(C_5) = D_5$, $D(C_5) = 3$.



The Distinguishing Number of a Group Action

Definition (Tymoczko, 2004)

Given a group Γ acting on a finite set X, define the **distinguishing number**, $D_\Gamma(X)$, to be the least positive integer r for which there exists a labeling $\phi : X \to \{1, 2, \dots, r\}$ such that the only element of Γ that preserves the labels is the identity.

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Theorem (Tymoczko, 2004)

If $|\Gamma|$ *is at most* $k!$ *then* $D_\Gamma(X)$ *is at most* k.

Distinguishing numbers for S_n

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Theorem (J.)

This construction gives $D_{S_n}(X) = n-1$, and furthermore, all constructions have to "look like" this in a certain way.

Generalization 1: The consumption poset

Definition (type)

Let $|X| = n$. Given a labeling ϕ of a set X, define the **type** of ϕ to be the partition of n given by the multiplicity of the labels.

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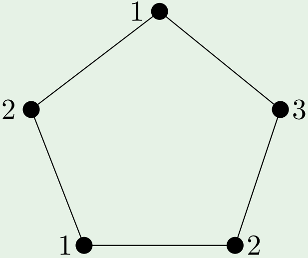
Fact: The distinguishing number of a group Γ acting on a set X is the minimum length of a partition it consumes.

Generalization 1: The consumption poset

Example

D_5, taken as a subgroup of S_5 acting on 5 elements, consumes the partitions $(3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)$.



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Let λ and μ be two partitions of n. We say $\lambda \geq_c \mu$ if every subgroup $H \leq S_n$ that consumes λ also consumes μ.

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Recall the dominance order on partitions: Let λ and μ be partitions of n with parts written in decreasing order. $\lambda \geq \mu$ if and only if $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$ for all k.

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Question (J.)

For which n are the consumption ordering and dominance ordering equivalent?

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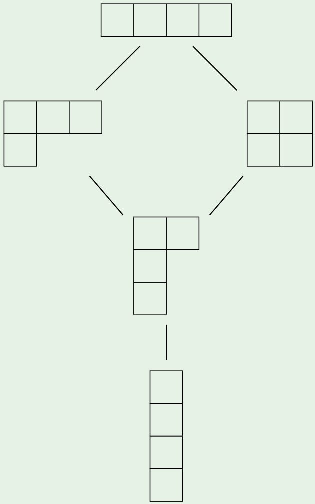
Example ($n = 4$)



Generalization 2: Partitioning Cartesian powers

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Example

If we require our elements to preserve some partition of X, we can only get groups of the form $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$, where the λ_i correspond to a set partition of X.

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Can we get anything interesting by looking at partitions of other things?

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Every subgroup of S_n can be obtained in this manner through a partition of X^{n-1}.

Question (J.)

Given a subgroup $H < S_n$, what is the minimal k for which H can be achieved through a partition of X^k?

Generalization 3: Distinguishing symmetric functions

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Definition (J.)

Let X be a set of size n carrying an action of the group G. Let f_X be the polynomial function such that $f_X(m)$ is the number of distinguishing labelings of X from the set $\{1, 2, \ldots, m\}$. Then f_X is the **distinguishing polynomial** of X.

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Theorem (J.)

Let a_i be the number of distinguishing labelings of X using precisely i colors for $1 \leq i \leq n$. Then $f_X(x) = \sum_{i=1}^{n} a_i \binom{x}{i}$. In particular, f_X is a polynomial of degree n.

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The distinguishing number is the smallest positive integer that is not a root of the distinguishing polynomial.

Generalization 3: Distinguishing symmetric functions

Idea: Generalize further, à la the chromatic symmetric function extending the chromatic polynomial

Definition

A **symmetric function** is an element of the polynomial ring $R[x_1, x_2, \ldots]$ in infinitely many variables that is invariant under the transposition of any two variables.

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Let

$$Y_X = \sum_{(i_1, i_2, \ldots, i_n) \text{ is a distinguishing labeling for } X} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

We call Y_X the **distinguishing symmetric function of X.**

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Y_X is a symmetric function, and when evaluated with $x_1 = x_2 = \cdots = x_m = 1$ and with other variables 0, gives the value of $f_X(m)$.

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- $(\bigcup_{\lambda_i} K_{\lambda_i}) = e_\lambda$ when λ has distinct parts.

Thanks!

- Joe Gallian, for suggesting this project and running the Duluth REU where this research was done
- My advisors Colin Defant, Levent Alpoge, and Aaron Berger