Riemann surfaces are algebraic

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1 Overview

It is a famous fact that complex algebraic curves and compact Riemann surfaces are equivalent notions. Here we will describe three proofs, in increasing generality, that any compact Riemann surface X is algebraic. All of them require some nontrivial analytical input. The first proof uses the existence of a nonconstant meromorphic function on X and to give X the structure of an algebraic curve. The second is to use the Riemann-Roch theorem to embed X holomorphically in projective space, then use Chow's theorem to deduce algebraicity. The third is to use the Kodaira embedding theorem, which generalizes in a suitable way to higher dimensions. Then we show how all these curves can be realized in $\mathbb{P}^3$, or even $\mathbb{P}^2$ are allowed, and speculate on further questions.

First, in the opposite direction, taking the closed points of a curve C (complete and nonsingular) over $\mathbb{C}$ gives a Riemann surface. Indeed, let us assume for now that C is embedded in projective space; we will see later how this can be done. Then by [4], Theorem 8.17, C is a local complete intersection. Then by the implicit function theorem, each of the standard open coverings U_i has some chart given by a coordinate of the form $\frac{x_j}{x_i}$, and it is clear that the transition maps are holomorphic.

Furthermore, we should also mention the Normalization theorem, which states that compact Riemann surfaces correspond to irreducible plane curves with at worst double points. The Riemann surface is obtained by the normalization of the plane curve.

2 Function field proof

Analysis

The first step is to show the existence of a nonconstant meromorphic function on X. There are many ways to do this; one proof is given in [2], chapter 14. In fact, we the meromorphic functions on X separate points, and extend this in the next section to the full Riemann-Roch theorem. Let $\mathcal{O}$ be the sheaf of holomorphic functions on X. The key statement is the following.

Theorem 2.1. *Given $Y_1 \subset Y_2$ open subsets of a Riemann surface X with Y_2 containing the closure of Y_1, the image of the restriction map*

$$H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$$

is finite dimensional.

We will simply state some of the inputs to this theorem. First, one needs a certain generalization of the Schwarz lemma, which gives L^2 functions whose norm on a subset of a domain is sufficiently small in comparison to their norm on the whole domain. Then one uses some functional analysis with this to show that a suitable restriction of H^1 of some cover to some refinement has finite dimensional image. Then recalling that H^1 with respect to a covering with no H^1 on each individual piece gives Cech H^1, the theorem follows.

An easy corollary is that on a compact Riemann surface, $\dim H^1(X, \mathcal{O})$ is finite; of course it is the genus of X. But what we are looking for is the following corollary.

Corollary 2.2. *The meromorphic functions on a compact Riemann surface separate points.*

For this, it suffices to show that for every point $x \in X$, there is a meromorphic function with a pole at a and holomorphic elsewhere. To prove this, cover X with U_1: a coordinate neighborhood of x and U_2: $X\backslash\{x\}$. Since H^1 of this cover always injects into Cech H^1, we know that it is finite dimensional by the theorem. Thus the cocycles $z^{-1}, z^{-2}, \cdots, z^{-k}$ are linearly dependent for some k; i.e. some combination of them agrees with $f_1 - f_2$ for $f_1 \in \mathcal{O}(U_1)$ and $\mathcal{O}(U_2)$. This immediately gives the desired construction.

The rest

Now take some nonconstant meromorphic $f : X \to \mathbb{P}^1$. If f has degree d, then it realizes X as a finite ramified cover over $\mathbb{P}^1$ of degree d. Then one can show [1], 11.1

$$[\mathbb{C}(X) : \mathbb{C}(t)] = d.$$

From here, we recognize that the way to proceed is to show that the category of compact connected Riemann surfaces with non-constant holomorphic maps is equivalent to the category of fields of transcendence degree 1 over $\mathbb{C}$ with field inclusions. This is done in essentially the same way as for algebraic curves – see [1], 11.1 for the Riemann surface case and [4], I.6. for the curve case. The upshot is that we associate to each function field an 'abstract' curve/Riemann surface where the points are the valuations; note that these are naturally in bijection with the points of the curve/Riemann surface we started with. Then we define the curve/Riemann surface structure on these valuations and prove the equivalence of categories. Since going from curve to function field to Riemann surface gives the closed points of the curve, we realize each Riemann surface as a curve.

3 Embedding in projective space

Riemann-Roch

We begin by proving the Riemann-Roch theorem for Riemann surfaces. First, we recall how it is proven for curves. For a smooth projective curve C of genus g and canonical divisor K, we wish to show that for a divisor D, we have

$$l(D) - l(K - D) = \deg D - g + 1.$$

Since Serre duality identifies the vector spaces $H^0(C, \omega_C \otimes L(-D))$ and $H^1(X, L(D))$, it suffices to show that $\chi(L(D)) = \deg D - g + 1$. This is done by induction. The base case follows becuase $\dim H^1(X, L(D)) = g$. Then for any point P, we have an exact sequence coming from its ideal sheaf

$$0 \to L(-P) \to \mathcal{O}_C \to \mathbb{C}_P \to 0,$$

which gives the exact sequence

$$0 \to L(D) \to L(D + P) \to \mathbb{C}_P.$$

Taking Euler characteristics gives the inductive step.

What changes must be made for compact Riemann surfaces? Of course, if we assume that they were algebraic and that their meromorphic functions were algebraic, nothing would need to be changed. We don't do that here and instead follow the same argument, replacing the objects and their names with their analytical counterparts. We need an exact sequence of sheaves

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \to \mathbb{C}_P \to 0.$$

Indeed, we can construct the last map by taking the appropriate Laurent coefficient over $U \ni P$. Then the same results follow through to show

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^1(X, \mathcal{O}_D) = \deg D - g + 1.$$

We would like to replace $\dim H^1(X, \mathcal{O}_D)$ with $\dim H^0(X, \Omega_{-D})$. This is of course Serre duality for compact Riemann surfaces. It is proven by showing that the composition

$$H^0(\mathcal{O}_D, X) \times H^1(\Omega_{-D}, X) \to H^1(X, \Omega) \to \mathbb{C}$$

given by $(f, \omega) \mapsto f\omega$ and the residue map is a perfect pairing.

The rest

Next, we construct an embedding of X into $\mathbb{P}^n$. Algebraically, this is done in [4], IV.3. Briefly, recall that an ample divisor gives rise to an embedding in $\mathbb{P}^n$, while a divisor that gives rise to a base-point free linear system gives rise to some map to $\mathbb{P}^n$. Being base-point free means that there are no points P for which the natural embedding $|D - P| \hookrightarrow |D|$ is surjective; by Riemann-Roch this is equivalent to $\dim |D - P| = \dim |D| - 1$. Then by Riemann-Roch again, we have $\deg K = 2g - 2$, and so if $\deg D \geq 2g$, then $|D|$ is base-point free.

But we are after an embedding in $\mathbb{P}^n$, so we use the condition that we obtain an embedding if and only if $|D|$ separates points and tangent vectors. This is seen to be equivalent to requiring that $\dim |D - P_Q| = \dim |D| - 2$, so by a similar argument to the above, if $\deg D \geq 2g + 1$, then D is very ample.

Now the analytic case is similar. If $\deg D \geq 2g + 1$, then we take a basis of $H^0(X, \mathcal{O}_D)$, which give a map into $\mathbb{P}^n$. Geometrically, an embedding is an injective immersion, and we use the Riemann-Roch theorem to show that this is the case.

In fact, every curve, or Riemann surface, can be embedded into $\mathbb{P}^3$. Indeed, take $X \subset \mathbb{P}^n$ and consider projecting down to $\mathbb{P}^{n-1}$ from some point O. This is the map to $\mathbb{P}^{n-1}$ given by the linear system cut out by hyperplanes passing through O. This is an embedding if it separates points an tangent vectors: this means that O must not be on any line passing through two (possibly the same) points of X. Such points make up a variety of dimension at most 3, so if $n \geq 3$, we can keep projecting down until we get to $\mathbb{P}^3$. If we try to go further to $\mathbb{P}^2$, then we may get double points.

So we have an embedding $X \to \mathbb{P}^n$ (where we can take $n = 3$) by meromorphic sections $f_0, \ldots, f_n$, and the quotient of each of these (say by f_0) is a meromorphic function g_i. Then we use a fact that we glossed over in the previous section; namely, that any two meromorphic functions are algebraically dependent. Indeed, we can directly construct a polynomial relating them using some symmetric polynomials, based on the fact that outside some ramification points, a meromorphic function of degree d gives a degree d covering of $\mathbb{P}^1$. Then X is cut out by these equations, which relate the ratios of the coordinates x_i/x_0 of $\mathbb{P}^n$. One can show that these are irreducible, from which one concludes that the variety cut out by all of them is also irreducible. We have an embedding from X onto this variety, and since X is compact the image must be surjective onto an irreducible component, so it is indeed surjective. Thus these equations give X as a projective variety.

It is worth mentioning that what we have just discussed is essentially a special case of Chow's theorem, which is itself a special case of GAGA. Chow's theorem states that analytic subvarieties of projective space are algebraic, and that in fact all meromorphic functions on a projective variety are rational.

4 Kodaira embedding theorem

This is a generalization to higher dimensions. When is a compact complex manifold X algebraic? One answer is when it is Kähler and Moishezon, where the latter condition means that the transcendence degree of its function field is equal to iits own dimension. Another, which we will describe a bit more, is when it is Kähler with integral Kähler form. Really, what these prove is that X can be embedded in projective space. Then one uses the proper mapping theorem or the holomorphic constant rank theorem (it does seem to be glossed over in the literature!) to show that this realizes X as an analytic subvariety of projective space. Then Chow's theorem states that X is algebraic. For a proof of Chow's theorem, see [3], Section 1.3. We only give the basic ideas, and everything here is taken from [3]. See also [5].

4.1 Basic notions

Recall that a Kähler manifold is a complex manifold with a Hermitian metric h giving rise to a closed non-degenerate 2-form ω. Now we can describe ampleness in terms of positivity, an analytic condition of line bundles.

First, let $E \to M$ be a Hermitian vector bundle with a connection D. Then given a frame $e_1, \ldots, e_n$ for E, we can write $De_i = \sum_j \theta_{ij} e_j$. Then θ_{ij} is the associated **connection matrix**. There is a canonical connection D compatible with the metric and the Hermitian structure, the **Chern connection**. Writing

$$D^2 e_i = \sum_j \Theta_{ij} \otimes e_j,$$

the Θ_{ij} comprise the **curvature matrix** of D for the given frame. The Cartan structure equation states that

$$\Theta = d\theta - \theta \wedge \theta.$$

The curvature matrix is a Hermitian matrix of $(1,1)$-forms.

In the case of the holomorphic tangent bundle $T'M$, a Hermitian metric is just called a Hermitian metric on M itself. In this case, such a metric is a $(1,1)$-form after multiplying by i. A Hermitian complex manifold is called **Kähler** if the associated $(1,1)$-form is closed.

Given a line bundle L, its first Chern class $c_1(L) \in H^2(M;\mathbb{Z})$ (or in $H^2_{dR}(M)$) may be defined as the image of its representative in the connecting homomorphism

$$H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z})$$

arising from the exponential sequence. We can also interpret the Chern class in terms of the curvature form. Indeed, for line bundles the curvature matrix is 1×1 and the transition functions are trivial, so we may speak of Θ as the curvature (2-)form of E. Then one may show that in fact,

$$c_1(L) = \left[\frac{i}{2\pi}\Theta\right] \in H^2_{dR}(M).$$

Now recall that a $(1,1)-$form ω is called **positive** if all its associated Hermitian forms on holomorphic tangent spaces $T'_z(M)$ are positive definite. That is, we have

$$i \cdot \langle \omega(z), v \wedge \overline{v} \rangle > 0.$$

Definition 4.1. *A holomorphic line bundle $L \to M$ is called* ***positive*** *if there is a Hermitian metric on L whose curvature form Θ is positive.*

Recall that whatever Hermitian metric we take on L, the resulting curvature form is going to represent the same cohomology class, namely the Chern class $c_1(L)$. A converse statement holds: if ω is a $(1,1)$ form representing $c_1(L)$, then there is a Hermitian metric on L giving ω as the curvature form. This means that L is positive if and only if $c_1(L)$ can be represented by a positive form in $H^2_{dR}(M)$.

4.2 The Kodaira vanishing and embedding theorems

We now state the Kodaira vanishing theorem.

Theorem 4.2 (Kodaira vanishing)**.** *Let M be a compact Kähler manifold and let $L \to M$ be a positive line bundle. Then*

$$H^p(M, \Omega^q(L)) = 0 \textit{ for } p + q > n.$$

The proof involves harmonic theory. One application is to the Lefschetz hyperplane theorem, which states that if $V \subset M$ is a hypersurface of an $n$dimensional compact complex manifold with $[V]$ positive (e.g. a hyperplane section), then $H^q(V, \mathbb{Q}) \to H^q(M, \mathbb{Q})$ is an isomorphism for $q \leq n-2$ and injective for $q = n-1$.

Anyways, we are still interested in embedding complex manifolds into projective space. This positivity condition is the analog of ampleness in algebraic geometry.

Theorem 4.3 (Kodaira embedding theorem)**.** *Let M be a compact complex manifold and let $L \to M$ be a positive line bundle. Then for sufficiently large k, we have that the complete linear system of L^k defines an embedding into $\mathbb{P}^N$.*

Another way of stating this is the following, which applies to compact Riemann surfaces.

Theorem 4.4 (Kodaira embedding theorem restated)**.** *A compact complex manifold M is a complex algebraic variety if and only if it has a closed, positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational.*

Indeed, a complex projective space possesses such a Kähler form which can be used in one direction. In the other, we have $[k\omega] \in H^2(M; \mathbb{Z})$ for some k. Then looking at the exponential exact sequence, we obtain a line bundle L with $c_1(L) = [k\omega]$. Then L will be positive.

To prove the embedding theorem, one can try to apply the vanishing theorems to a cohomological interpretation of the condition of separating poitns and tangent vectors. This does not quite directly work, because we don't directly have enough divisors if $\dim M > 1$. This can be solved by blowing up.

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