Weil II

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Overview

Contents

I Fundamentals of étale cohomology **2**

1 The étale topos **2**
- 1.1 Definition and cohomology . . . 2
- 1.2 Stalks in the étale site . . . 4
- 1.3 Sheafification, direct and inverse images . . . 5
- 1.4 Spectral sequences . . . 6
- 1.5 Morphisms of sites and comparisons of topology . . . 6

2 Eilenberg-Steenrod and Weil cohomology axioms **6**

3 Examples and calculations **6**
- 3.1 The étale site $\mathrm{Spec}(k)_{\text{ét}}$ . . . 6
- 3.2 Cohomology of curves . . . 7
- 3.3 Cohomology of number fields . . . 9
- 3.4 Cohomology of surfaces . . . 10
- 3.5 Other . . . 10

4 Constructible sheaves **10**
- 4.1 Constructible sets . . . 10

5 Proper base change **10**
- 5.1 Statement and initial reductions . . . 10
- 5.2 Dévissage I: reduction to case of a curve . . . 11
- 5.3 Dévissage II: reduction to constant coefficients . . . 12
- 5.4 Formal geometry . . . 14
- 5.5 Proof in the case of a curve . . . 14
- 5.6 Applications . . . 16

6 Purity, Gysin sequence, and the weak Lefschetz theorem **17**
- 6.1 Purity . . . 17
- 6.2 Gysin sequence . . . 18
- 6.3 Vanishing and finiteness theorems . . . 18
- 6.4 Weak Lefschetz theorem . . . 19

7 Fundamental classes, cycle classes, and Chern classes **19**
- 7.1 Fundamental class . . . 19
- 7.2 Cycle class . . . 20
- 7.3 Chern classes . . . 21

8 Poincaré duality 21
8.1 Statements 21
8.2 The trace morphism 21
8.3 The cup product 21
8.4 Poincaré duality for curves 21
8.5 Poincaré duality for varieties 22
8.6 Relative Poincaré duality 23

9 The Lefschetz trace formula 23
9.1 Recollection of Weil cohomology axioms 23
9.2 Generalizing the Gysin map 24
9.3 The Lefschetz trace formula 24
9.4 The many Frobenii 25
9.5 Application to the Weil conjectures 25

10 The Grothendieck trace formula 26
10.1 Statement 26
10.2 Perfect complexes and filtered derived categories 27
10.3 Dévissage to curves 27
10.4 Reduction to the Lefschetz trace formula 27
10.5 In terms of L-functions 27
10.6 Application: exponential sums 27

II Weights and Weil II 27

11 Weil sheaves and weights 27
11.1 Weil sheaves 27
11.2 Semicontinuity of weights 28
11.3 Sheaf-function correspondence and radius of convergence 29
11.4 Determinant weights 30

References 30

Part I
Fundamentals of étale cohomology

1 The étale topos

1.1 Definition and cohomology

Recall that in ordinary sheaf cohomology, one computes the cohomology of a sheaf $\mathcal{F}$ over a topological space X by taking an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots,$$

taking global sections, then taking cohomology. By treating a scheme X as a topological space with the Zariski topology, we obtain the usual cohomology groups of X. We can consider different Grothendieck topologies on X. For example, the objects of the big fppf site X_{fppf} are schemes over X and its covering families are those that are jointly surjective, flat, and locally of finite presentation. The objects of the small fppf site $X_{\text{ét}}$ are étale schemes over X and its covering families are those that are jointly surjective[1]. We recall the definition of étale morphism.

Definition 1.1. *A morphism of schemes $f : X \to Y$ is étale if it satisfies the following equivalent conditions.*

1. *f is flat and unramified.*[2]
2. *f is smooth and unramified.*
3. *f is smooth of relative dimension 0.*

Example 1.2 (Non-example)**.** *Normalizations are generally not flat. The classic example is the normalization*

$$\mathbb{A}^1 \to \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3 - x^2)$$

defined by $(x, y) \mapsto (t^2 - 1, t^3 - t)$. Indeed, flatness is a local condition, so in fact $f_*\mathcal{O}_X$ should be locally free. But we see that the corresponding extension of modules here is an isomorphism localized away from 0, but has rank 2 at 0.

Example 1.3. *Standard étale maps. These are of the form $\operatorname{Spec} A[x]_f/(g) \to \operatorname{Spec} A$, where g is monic and g' is invertible in $A[x]_f/(g)$. The main result here is that every étale map is locally standard étale.*

The small étale site $X_{\text{ét}}$ is the category whose objects are étale morphisms $U \to X$ and whose morphisms are morphisms of schemes over X. By a 2-out-of-3 property, such morphisms are themselves étale. A covering of an object is a collection of morphisms in this category $\{U_i \to U\}$ which are étale (this is automatic) and jointly surjective. This allows us to define sheaves on $X_{\text{ét}}$. The étale topos of X is defined as the category of sheaves on the small étale site $X_{\text{ét}}$. These are the presheaves $\mathcal{F}$ that satisfy the sheaf property:

$$0 \to F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_i \times_U U_j)$$

for all étale coverings $\{U_i \to U\}$. Since all Zariski coverings are étale coverings, this is stronger than the condition of being a Zariski sheaf. We have the following criterion.

Proposition 1.4. *Let P be a presheaf on X_{fppf} or $X_{\text{ét}}$. Then P is a sheaf if and only if*

- *P is a Zariski sheaf.*
- *$0 \to P(U) \to P(U') \rightrightarrows P(U' \times_U U')$ is exact given any surjective morphism of affine schemes $U' \to U$.*

We define the étale cohomology groups (or those from a different site) as the derived functors of the global sections functor $\Gamma(X, -) : \operatorname{Sh}(X_t) \to \mathbf{Ab}$. We note that étale sheaves may well produce different cohomology groups in the Zariski topology, since injective Zariski sheaves may not even be sheaves, let alone injective! What we see, however, is that the category of sheaves is what determines the cohomology. Probably motivated by this, and certainly many more deep considerations, Grothendieck formulated his definition of a topos.

Definition 1.5. *The étale topos of X is the category of sheaves on $X_{\text{ét}}$.*

More generally a topos is a category equivalent to the category of sheaves over some site. Unfortunately we will not (due to the current ignorance of the author) say much more about topos theory.

[1]By the 2/3 property, such maps are automatically étale

[2]Technically we may want to say G-unramified: locally of finite presentation and $\Omega_{X/Y} = 0$, but this doesn't matter much for us.

1.2 Stalks in the étale site

Definition 1.6. *An étale neighborhood of a point $x \in X$ is a pair (Y, y) and an étale morphism $f : Y \to X$ with $f(y) = x$ and $k(y) = k(x)$[3].*

In particular, étale neighborhoods of a geometric point are commutative diagrams that look like the following.

$$\begin{array}{ccc} & & U \\ & \nearrow & \downarrow \\ \bar{x} & \longrightarrow & X \end{array}$$

Definition 1.7. *The stalk of a sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\acute{e}t})$ at a point $x \in X$ is defined as*

$$\varinjlim_{(Y,y)} \mathcal{F}(Y)$$

where the colimit is taken over all étale neighborhoods of x. The stalk at a geometric point is defined similarly.

The Henselization of a local ring is the smallest ring containing it that satisfies Hensel's lemma, while the strict Henselization is the same thing with the added condition that the residue field is separably closed. The main result is the following.

Proposition 1.8. *The stalk of the structure sheaf in the étale topology is the Henselization of the stalk in the Zariski topology. If we work over a geometric point, we get the strict Henselization.*

A key property of étale neighborhoods is that they are cofiltered. In particular, is we have two étale neighborhoods $(U, \bar{u}), (V, \bar{v})$ of some geometric point $\bar{x} \to X$, then taking the fiber product gives another étale neighborhood which factors through $(U, \bar{u}), (V, \bar{v})$. Note this uses that étale morphisms are preserved under composition and base change.

Proposition 1.9. *A sequence of sheaves in $\mathrm{Sh}(X_{\acute{e}t})$*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is exact if and only if it is exact on all stalks of geometric points

$$0 \to \mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}} \to \mathcal{H}_{\bar{x}} \to 0.$$

Proof. Let us first prove left-exactness. As we will soon see, the inclusion $i : \mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{Sh}(Y_{\acute{e}t})$ is left exact (because its left adjoint is sheafification). So say $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact. Taking direct limits of exact sequences preserves exactness. Now the other direction boils down to $s_{\bar{u}} = 0 \to s = 0 \in \mathcal{F}(U)$. But the former implies that s is zero on an étale cover of U, so it must be 0.

For right-exactness, one shows that $\mathcal{G} \xrightarrow{\alpha} \mathcal{H} \to 0$ is equivalent to α being locally surjective. It is not hard to see that this implies surjectivity on geometric stalks. In the other direction, by what we have shown above we have that in fact $\mathcal{G}_{\bar{u}} \to \mathcal{H}_{\bar{u}}$ is surjective for every geometric $\bar{u} \to U \to X$ with $U \to X$ étale. Then α is clearly locally surjective by definition. □

Generally, being exact on the level of sheaves implies being exact on the level of stalks. The other direction is essentially the statement that a given topos has enough points.

[3]Some sources, e.g. Stacks, omit the last condition. Then we get the strict Henselization for stalks, which we would get if we were working over geometric points. But we will take stalks over geometric points anyways

1.3 Sheafification, direct and inverse images

We claim that the natural inclusion $i : \mathrm{PSh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ has an exact left adjoint, sheafification: $\mathrm{sh} : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{PSh}(X_{\text{ét}})$. That is,

$$\mathrm{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(\mathrm{sh}\,\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{PSh}(X_{\text{ét}})}(\mathcal{F}, i(\mathcal{G}))$$

The construction is similar to the classical case. The sheafification has the same stalks as the original presheaf, so sh is exact. This implies that i also preserves injectives.

Given a morphism $f : X \to Y$, there are several associated functors between $\mathrm{Sh}(X_{\text{ét}})$ and $\mathrm{Sh}(Y_{\text{ét}})$. The two most basic ones, the direct and inverse image, form an adjoint pair (f^*, f_*). First, we define them on presheaves.

Definition 1.10. *Take $\mathcal{F} \in \mathrm{PSh}(X_{\acute{e}t})$. Then $f_*\mathcal{F} \in \mathrm{PSh}(Y_{\acute{e}t})$ is defined by $f_*\mathcal{F}(U) = F(U \times_Y X)$. We use the same exact definition for $f_* : \mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{Sh}(Y_{\acute{e}t})$.*

Indeed, these definitions are consistent in that π_* of a sheaf in the presheaf category is a sheaf, as it clearly satisfies the sheaf conditions.

Definition 1.11. *Take $\mathcal{G} \in \mathrm{PSh}(Y_{\acute{e}t})$. Then $f^*\mathcal{G} \in \mathrm{PSh}(X_{\acute{e}t})$ is defined as the direct limit*

$$f^*\mathcal{G}(V) = \varinjlim \mathcal{G}(V)$$

where we take the direct limit over all commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with étale columns. Since π_* of a sheaf is not necessarily a sheaf, in the sheaf category we define it to be the sheafification $\mathrm{sh}(\pi^*\mathcal{G})$.

Now we have the adjunction

$$\mathrm{Hom}_{X_{\text{ét}}}(\pi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{Y_{\text{ét}}}(\mathcal{G}, \pi_*\mathcal{F})$$

because both index families of maps $\mathcal{G}(V) \to \mathcal{F}(U)$ by the definition of fiber product and the universal property of sheafification. One sees immediately from the definition of π^* that if π is étale, then π^* is just the restriction map.

In general, π^* is exact because it preserves stalks. Indeed, given $\bar{x} \xrightarrow{i} X \xrightarrow{\pi} Y$ where the composition is denoted $\bar{y}$, we have

$$(\pi^*\mathcal{F})_{\bar{x}} = i^*(\pi^*\mathcal{F})(\bar{x}) = \mathcal{F}_{\bar{y}}.$$

Therefore, π_* preserves injectives.

Definition 1.12. *Let $j : U \hookrightarrow X$ be an open immersion. Then for $\mathcal{F} \in \mathrm{PSh}(U_{\acute{e}t})$, we have $j_!\mathcal{F} \in \mathrm{PSh}(X_{\acute{e}t})$ defined by $j_!\mathcal{F}(\pi : V \to X) = \mathcal{F}(V)$ if $\mathrm{im}\,\pi \subset U$ and 0 otherwise. This does not always turn a sheaf into a sheaf. Thus in the category of sheaves, we define it to be $\mathrm{sh}(j_!\mathcal{F})$. This is known by extension by 0, or lower shriek or proper direct image.*

We have an adjunction $(j_!, j^*)$ because

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(j_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{PSh}(X_{\text{ét}})}(j_!\mathcal{F}, \mathcal{G}) &\cong \mathrm{Hom}_{\mathrm{PSh}(U_{\text{ét}})}(\mathcal{F}, \mathcal{G}|_U) \cong \mathrm{Hom}_{\mathrm{PSh}(U_{\text{ét}})}(\mathcal{F}, j^*\mathcal{G}) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}(U_{\text{ét}})}(\mathcal{F}, j^*\mathcal{G}), \end{aligned}$$

as desired. As we will soon see, by looking at stalks we have that $j_!$ is exact, so j^* preserves injectives.

Proposition 1.13. *For $\pi : X \to Y$ an open immersion, stalks are given by $(\pi_{\mathcal{F}})_{\bar{y}} = \mathcal{F}_{\bar{y}}$ for $y \in X$; otherwise it is not clear. For π a closed immersion, $(\pi_{\mathcal{F}})_{\bar{y}} = \mathcal{F}_{\bar{y}}$ for $y \in X$ and 0 otherwise. Ditto for $\pi_!$ if π is an open immersion.*

In particular, π_* is exact for closed immersions. The closed immersion one is the only one that is not immediate; it follows from the description of étale morphisms as locally standard étale.

Take $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$. Given $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ a decomposition of X into open and closed subsets, we have the short exact sequence of sheaves on $X_{\text{ét}}$

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$

This follows immediately from the adjunctions and checking stalks. This exact sequence will be very useful for computations.

1.4 Spectral sequences

Let us list the principal spectral sequences we will use for calculations.

Theorem 1.14 (Grothendieck spectral sequence)**.** *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be left-exact functors between abelian categories such that F sends injectives to acyclics. Then for $A \in \mathrm{Ob}(\mathcal{A})$, there is a spectral sequence*

$$R^p G \circ R^q F(A) \Rightarrow R^{p+q}(G \circ F)(A).$$

Corollary 1.15 (Leray spectral sequence)**.** *Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. Applying the Grothendieck spectral sequence to*

$$\mathrm{Sh}(X_{\text{ét}}) \xrightarrow{\pi_*} \mathrm{Sh}(Y_{\text{ét}}) \xrightarrow{\Gamma} \mathbf{Ab},$$

we have a spectral sequence

$$H^p(Y, R^q \pi_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

1.5 Morphisms of sites and comparisons of topology

TODO

2 Eilenberg-Steenrod and Weil cohomology axioms

TODO

3 Examples and calculations

In this section we do a lot of computations of étale cohomology groups. For fun, we sometimes also do them for other sites (mainly Zariski).

3.1 The étale site $\mathrm{Spec}(k)_{\text{ét}}$

The goal of this section is to describe the sheaves on $\mathrm{Spec}(k)_{\text{ét}}$ and show that their cohomology is given by the Galois cohomology of k.

We claim that the category of sheaves on $\mathrm{Spec}(k)_{\text{ét}}$ is equivalent to the category of discrete G_k modules. Recall that the latter refers to G_k-modules that are discrete given the discrete topology, which is equivalent to stabilizers being open.

Take $\mathcal{F} \in \mathrm{Sh}(\mathrm{Spec}(k)_{\text{ét}})$. The étale maps to $\mathrm{Spec}\, k$ are given by finite disjoint unions of $\mathrm{Spec}\, L \to \mathrm{Spec}\, k$ where L/k is finite separable. For each such L, let $M_L = \mathcal{F}(\mathrm{Spec}\, L)$ and define $M = \varinjlim_L \mathcal{F}(\mathrm{Spec}\, L)$. Alternatively, we can view M as the stalk $\mathcal{F}_{\mathrm{Spec}\, K^{\text{sep}}}$. First we claim that this is discrete. Indeed, we have that $M = \cup M^H$, where H runs over the open subgroups of G_k. This implies that the stabilizer of a point is a union of open subgroups of G_k, and is thus open.

To go further, it will be useful to identify the category $(\mathrm{Spec}\, k)_{\text{ét}}$ itself with continuous G_k sets through the functor sending $X \to \mathrm{Spec}\, k$ to $\mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k^{\text{sep}}, X)$. Then we see that if M is a discrete G_k-module, the corresponding sheaf h_M on the category of G_k-modules translates to the sheaf $F_M \in \mathrm{Sh}(X_{\text{ét}})$ defined by

$$F_M(U) = \mathrm{Hom}_{G_k}(\mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k^{\text{sep}}, U), M).$$

This gives the desired quasi-inverse. Now to compute cohomology, one notes that

$$H^0_{\text{ét}}(\mathrm{Spec}\, k, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Spec}(k)_{\text{ét}})}(h_{\mathrm{Spec}\, k}, \mathcal{F}) = \mathrm{Hom}_{G_k}(\{*\}, M_{\mathcal{F}}) = M^{G_k}.$$

Then the derived functors are those of Galois cohomology, as desired.
Now let us compute cohomology. We would like to identify $H^i_{\text{ét}}(\mathrm{Spec}\, k, \mathcal{F})$ with $H^i(G_k, M_{\mathcal{F}})$.

3.2 Cohomology of curves

3.2.1 Smooth projective curves over $k = \overline{k}$

Recall that if A is an integrally closed domain, then $A_{\mathfrak{p}}$ is a DVR for $\mathrm{ht}(\mathfrak{p}) = 1$ and $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$. We begin with the Weil-divisor sequence. Let $g : \eta \to X$ be the generic point of an integral normal scheme X. Let $K = K(X)$.

Proposition 3.1. *For connected integral Noetherian X, there is a left-exact sequence*

$$0 \to \mathbb{G}_m \to g_* \mathbb{G}_{m,K} \to \bigoplus_{\mathrm{codim}\, z=1} i_* \mathbb{Z} \to 0$$

which is right exact if X is regular.

Proof. For left-exactness, one sees that for any connected étale $U \to X$ and open affine $\mathrm{Spec}\, A \subset U$, this just becomes

$$0 \to A^* \to K^* \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=1} \mathbb{Z}.$$

This is exact by the commutative algebra result we cited, because A is an integrally closed domain. For right-exactness, we can check on stalks. If X is regular, then U is regular. Regular local rings are UFDs, and in UFDs every prime ideal of height 1 are prinicpal. The result follows. □

We are currently interested in the case that X/k is a smooth projective curve over an algebraically closed field. We have

$$H^0(X_{\text{ét}}, \mathbb{G}_m) = k^*, H^0(X_{\text{ét}}, g_* \mathbb{G}_{m,K}) = K^*, H^0(X_{\text{ét}}, \bigoplus_{\mathrm{codim}\, z=1} i_* \mathbb{Z}) = \mathrm{Div}(X)$$

Indeed, the sheaf $\bigoplus_{\mathrm{ht}(\mathfrak{p})=1} \mathbb{Z}$ is given by Div_X. Thus one obtains the long exact sequence

$$\begin{aligned}
0 &\to k^* \to K^* \to Div(X) \\
&\to H^1(X_{\text{ét}}, \mathbb{G}_m) \to H^1(X_{\text{ét}}, g_*\mathbb{G}_{m,K}) \to H^1(X_{\text{ét}}, \mathrm{Div}_X) \\
&\to H^2(X_{\text{ét}}, \mathbb{G}_m) \to H^2(X_{\text{ét}}, g_*\mathbb{G}_{m,K}) \to H^2(X_{\text{ét}}, \mathrm{Div}_X) \\
&\to H^3(X_{\text{ét}}, \mathbb{G}_m) \to \cdots
\end{aligned}$$

Now we will show higher vanishing of both $H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,K})$ and $H^r(X_{\text{ét}}, \mathrm{Div}_X)$. This will imply that $H^0(X_{\text{ét}}, \mathbb{G}_m) = \mathcal{O}_X(X)^* = k^*$, $H^1(X_{\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}(X)$, and higher groups are 0.

The case of Div_X is easier. Indeed, recall that i_* is exact for closed immersions i. Since we are working on a curve, the codimension 1 points are closed. Thus each copy

$$H^r(X_{\text{ét}}, i_*\mathbb{Z}) = H^r(G_k, \mathbb{Z}) = 0$$

for $r > 0$, because k is algebraically closed. Thus $H^r(X_{\text{ét}}, \mathrm{Div}_X) = 0$.

In the other case, the Leray spectral sequence gives

$$H^p(X_{\text{ét}}, R^q g_*\mathbb{G}_{m,K}) \Rightarrow H^{p+q}((\mathrm{Spec}\, K)_{\text{ét}}, (K^{\text{sep}})^*).$$

We claim that all higher (i.e. $q > 0$) $R^q g_*\mathbb{G}_{m,K}$ are 0. This is nontrivial. We have

$$(R^q g_*\mathbb{G}_{m,K})_{\bar{y}} = H^q(\mathrm{Spec}\, K_{\bar{y}}, \mathbb{G}_m)).$$

Here, $K_{\bar{y}}$ is the field of fractions of $\mathcal{O}_{X,\bar{y}}$. For example, when the underlying point of y is the generic point, this cohomology group is indeed 0 for $q > 0$. In general, one has to use Lang's theorem, which gives us that $K_{\bar{y}}$ is quasi-algebraically closed. Then theorems of Galois cohomology give us that

$$H^q(\mathrm{Spec}\, K_{\bar{y}}, \mathbb{G}_m)) = H^q(G_{K_{\bar{y}}}, (K^{\text{sep}})^*) = 0$$

for $q > 0$. For example, $q = 1$ is just Hilbert theorem 90, and $q = 2$ is a Brauer group calculation. We note that these theorems don't apply to e.g. $K = \mathbb{F}_q$, which we will consider in future calculations.

All this proves that the Leray spectral sequence degenerates and

$$H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,K}) = H^r((\mathrm{Spec}\, K)_{\text{ét}}, (K^{\text{sep}})^*).$$

Again by our Galois cohomology results, this is 0 for $r > 0$ as desired.

By using the Kummer sequence and basic results about the Picard scheme of a curve, we get that if $(n, \mathrm{char}(k)) = 1$, then

$$H^0(X, \mu_n) \cong \mathbb{Z}/n, H^1(X, \mu_n) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}, H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}, H^r(X, \mu_n) = 0$$

for $r > 2$.

3.2.2 Open subschemes

3.2.3 Singularities

3.2.4 Curves over non-algebraically closed fields

3.3 Cohomology of number fields

Cohomology of a DVR

Recall the localization exact sequence for $U \overset{j}{\hookrightarrow} X \overset{i}{\leftarrow} Z$ a decomposition of X into open and closed subsets.

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$

Then the category of sheaves on $X_{\text{ét}}$ is equivalent to the category of triples $(\mathcal{G}, \mathcal{H}, \phi)$ with $\mathcal{G} \in \mathrm{Sh}(Y_{\text{ét}}), \mathcal{H} \in \mathrm{Sh}(Z_{\text{ét}})$, and $\phi : \mathcal{G} \to i^* j_* \mathcal{H}$, by the functor

$$\mathcal{F} \mapsto (\mathcal{F}_Z, \mathcal{F}_U, \phi).$$

The starting point is again the Weil-divisor exact sequence, which we reproduce here for convenience.

Proposition 3.2. *For connected normal X, there is a left-exact sequence ...*

In particular, this holds for $X = \operatorname{Spec} R$ where R is a DVR.

Cohomology of $\operatorname{Spec} \mathcal{O}_K$

Let $X = \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a totally imaginary number field. The Weil-divisor exact sequence takes the form

$$0 \to \mathbb{G}_m \to g_* \mathbb{G}_{m,K} \to \bigoplus_{0 \neq \mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K} i_* \mathbb{Z} \to 0.$$

Of course we have $H^0(X_{\text{ét}}, \mathcal{G}_m) = \mathcal{O}_K^*$. By Hilbert theorem 90, we have $H^1((\operatorname{Spec} \mathcal{O}_K)_{\text{ét}}, g_* \mathcal{G}_{m,K}) = H^1(G_K, \overline{K}^*) = 0$. So we have the exact sequence

$$0 \to \mathcal{O}_K^* \to K^* \to \bigoplus_{\mathfrak{p}} \mathbb{Z} \to H^1(X_{\text{ét}}, \mathcal{G}_m) \to 0$$

from which we conclude that $H^1(X_{\text{ét}}, \mathcal{G}_m) = \mathrm{Pic}(\mathcal{O}_K) = \mathrm{Cl}(\mathcal{O}_K)$.

So we have

$$\begin{aligned} 0 &\to \mathcal{O}_K^* \to K^* \to Div(X) \\ &\to \mathrm{Cl}(\mathcal{O}_K) \to H^1(X_{\text{ét}}, g_* \mathbb{G}_{m,K}) \to H^1(X_{\text{ét}}, \mathrm{Div}_X) \\ &\to H^2(X_{\text{ét}}, \mathbb{G}_m) \to H^2(X_{\text{ét}}, g_* \mathbb{G}_{m,K}) \to H^2(X_{\text{ét}}, \mathrm{Div}_X) \\ &\to H^3(X_{\text{ét}}, \mathbb{G}_m) \to H^3(X_{\text{ét}}, g_* \mathbb{G}_{m,K}) \to H^3(X_{\text{ét}}, \mathrm{Div}_X) \\ &\to H^4(X_{\text{ét}}, \mathbb{G}_m) \to \cdots \end{aligned}$$

Let us compute $H^r(X_{\text{ét}}, \mathrm{Div}_X)$. Well, this is just

$$H^r(X_{\text{ét}}, \coprod_{\mathfrak{p}} i_* \mathbb{Z}) = \bigoplus_{\mathfrak{p}} H^r((\operatorname{Spec} \mathbb{F}_q)_{\text{ét}}, \mathbb{Z}).$$

This can be computed to be $\mathbb{Q}/\mathbb{Z}$ for $r = 2$ and 0 for all other $r > 0$.

Next, we again have a Leray spectral sequence which can be shown to degenerate to give

$$H^r(X_{\text{ét}}, g_*\mathbb{G}_{m,K}) \cong H^n((\operatorname{Spec} K)_{\text{ét}}, \overline{K}^*) = H^n(G_K, \overline{K}^*).$$

We currently have the exact sequence

$$0 \to H^2(X_{\text{ét}}, \mathbb{G}_m) \to H^2(G_K, \overline{K}^*) \to \bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \to H^3(X_{\text{ét}}, \mathbb{G}_m) \to H^3(G_K, \overline{K}^*) \to 0$$

and

$$H^r(X_{\text{ét}}, \mathbb{G}_m) \to H^r(G_K, \overline{K}^*)$$

for $r > 3$. Now we use the exact sequence of global class field theory

$$0 \to H^2(G_K, \overline{K}^*) \to \bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

and the result that $H^r(G_K, \overline{K}^*) = 0$ for $r > 2$. The maps to $\bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z}$ are the same, so at last we obtain

$$H^3(X_{\text{ét}}, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}, H^r(X_{\text{ét}}, \mathbb{G}_m) = 0$$

for all other $r > 1$.

3.4 Cohomology of surfaces

3.5 Other

3.5.1 Affine and projective space

4 Constructible sheaves

4.1 Constructible sets

We recall that a constructible set is one that is a finite disjoint union of locally closed subsets. By some combinatorial manipulation, one sees that constructible sets are closed under finite unions and complements. The importance of constructible sets may is illustrated by Chevalley's theorem.

Theorem 4.1 (Chevalley's theorem). *Let $f : X \to Y$ be a quasi-compact morphism locally of finite presentation where Y is quasicompact. Then the image of a constructible set in X is a constructible set in Y.*

Let us outline a proof. Note that it suffices to prove that the image of X is constructible. We first note that by restricting to Noetherian schemes, we just need to prove that the image of a morphism of finite type is constructible. Such a proof is outlined through in different ways in Hartshorne both in

5 Proper base change

5.1 Statement and initial reductions

Theorem 5.1. *Let $X \to S$ be a proper morphism of schemes and let $\mathcal{F}$ be a torsion abelian sheaf on X. Then in the following Cartesian diagram:*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

the natural morphism

$$g^* R^q f_* \mathcal{F} \to R^q f'_*(g'^* \mathcal{F})$$

is an isomorphism.

An important case of this arises when S' is some geometric point $g : \bar{s} \to S$. In this case if we take the stalk at $\bar{s}$, the statement becomes

$$(R^q f_* \mathcal{F})_{\bar{s}} \cong H^q(X_s, \mathcal{F}|_{X_s}).$$

In fact, we want to perform a series of reductions that reduces the proof to an even more simple case. We first note that since there is a map between the two étale sheaves we are interested in, we just have to check that there is an isomorphism on the stalks of geometric points.

Looking at the left hand side, we can reduce further. Indeed, for a geometric point $\bar{s}$ given by $\operatorname{Spec} k(s)$ of the point s it lies over, we have

$$(R^q f_* \mathcal{F})_{\bar{s}} = \varinjlim_{V} H^q(X \times_S V, \mathcal{F}) = H^q(X \times_S \operatorname{Spec}(\mathcal{O}_{S,\bar{s}}), \mathcal{F}).$$

Letting $(A, \mathfrak{m}, k)$ be the strictly Henselian local ring $\mathcal{O}_{S,\bar{s}}$, we have the following Cartesian diagrams for every geometric point $\operatorname{Spec} k' \to S$ lying over s.

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} A & \longrightarrow & S \end{array}$$

Relabeling $X = X'$, We conclude that it suffices to show that

$$H^q(X, \mathcal{F}) \cong H^q(X'', \mathcal{F})$$

where we simply write $\mathcal{F}$ for $\mathcal{F}|_{X_s}$. By a limiting argument, we reduce to the case that $k' = k$. Thus we would like to show that cohomology is preserved on passing from $\operatorname{Spec} A$ (a strictly Henselian local ring) to the special fiber:

$$H^q(X, \mathcal{F}) \cong H^q(X_0, \operatorname{Spec} k).$$

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

5.2 Dévissage I: reduction to case of a curve

We want to reduce to the case where the fibers X_s are curves. Furthermore, we would like to reduce to the case where $\mathcal{F} = \mathbb{Z}/n$ is constant. We do this by dévissage. Our first order of business will be to reduce to the case of $X = \mathbb{P}^1_S \to S$. This is done through the use of the following three lemmas.

Lemma 5.2. *Cohomology commutes with base change for finite morphisms.*

This is because finite morphisms are exact for the étale topology, and the $q = 0$ case can be checked by hand.

Lemma 5.3. *Let $f : X \to Y$ be surjective and proper, $g : Y \to Z$ be proper. If cohomology commutes with base change for f and $g \circ f$, then it does so for g.*

Lemma 5.4. *Let $f : X \to Y, g : Y \to Z$ be proper. If cohomology commutes with base change for f and g, then it does so for $g \circ f$.*

Proving these two involves the Leray spectral sequence.

Assuming these, we can reduce to the case of $\mathbb{P}^1_S \to S$ as follows. For convenience, say a morphisms has property C if cohomology commutes with base change for it. We first reduce to showing that $\mathbb{P}^n_S \to S$ has C. By Chow's lemma, there is a birational, proper and surjective S-morphism $X' \to X$ such that X' is projective over S.

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbb{P}^n_Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} & \swarrow & \\ & & S & & \end{array}$$

By our lemmas we just need to show that g and π have C. Since π factors as $X' \to X \times \mathbb{P}^n_Y = \mathbb{P}^n_X \to X$, we see that both factor as a closed immersion after something of the form $\mathbb{P}^n_S \to S$. The first is finite, so from our lemmas we are reduced to showing that $\mathbb{P}^n_S \to S$ has C.

From here, we can construct a finite surjective morphism $(\mathbb{P}^1_S)^n \to \mathbb{P}^n_S$. Thus again from our lemmas it suffices to show that $(\mathbb{P}^1_S)^n \to S$ has C, and this reduces to showing that $\mathbb{P}^1_S \to S$ has C. From the previous section, we see that we just need to prove that

$$H^q(\mathbb{P}^1_A, \mathcal{F}) \cong H^q(\mathbb{P}^1_k, \mathcal{F}).$$

This is still not that easy! In fact, as we will soon see, we will have to prove this for curves in general.

5.3 Dévissage II: reduction to constant coefficients

Recall that $\mathcal{F}$ is a torsion sheaf. Torsion sheaves are a pretty general class, and we will want to reduce to the case of constructible, then finite constant sheaves to prove our desired equality. Clearly proving the result for finite constant sheaves is equivalent to proving it for sheaves of the form $\mathbb{Z}/n$. In the second reduction, we will have to prove results for finite morphisms to $\mathbb{P}^1$, so we end up dealing with curves in general. We may as well just start with trying to prove that $H^q(X, \mathcal{F}) \cong H^q(X_0, \mathcal{F})$ for curves X. As a matter of fact, we will reduce the statement to the map being an isomorphism for $q = 0$ and a surjection for $q > 0$.

The first reduction is from torsion sheaves to constructible sheaves. For this, we should define constructible sheaves.

Definition 5.5.

- *A sheaf $\mathcal{F} \in \mathrm{Sh}(X_{ét})$ is locally constant constructible (lcc) if it is finite and locally constant.*
- *Let X be Noetherian (or qcqs). Then $\mathcal{F}$ is constructible if there is a stratification of X into locally closed subsets $X = \coprod_i X_i$ such that $\mathcal{F}|_{X_i}$ is lcc for all i.*

Remark. Finite refers to $F(U)$ being finite for all quasi-compact U. We will be working with Noetherian schemes here, so this implies finite stalks. In the locally constant case, the opposite implication also holds. Also, $\mathcal{F}|_{X_i}$ refers to the pullback of $\mathcal{F}$ to X_i.

We now note that torsion sheaves $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ are filtered direct limits of constructible sheaves. Indeed, for n-torsion elements $x \in \mathcal{F}(U)$, consider the subsheaf $j_!(U) \subset \mathcal{F}$. This is clearly constructible, and $\mathcal{F}$ is the filtered direct limit of these sheaves as desired. Now cohomology commutes with filtered colimits ([2, Tag 073D]), so we have reduced to the case of constructible sheaves.

The situation is now the following. In the following sections, we will show that for X a curve, the map $H^q(X, \mathbb{Z}/n) \to H^q(X_0, \mathbb{Z}/n)$ is an isomorphism for $q = 0$ and surjective for $q > 0$. So let us assume this. Then we will use the following three lemmas, which combined with an inductive argument involving a long exact sequence will show the result for constructible sheaves.

Lemma 5.6. *Let $p_i : X_i \to X$ be a finite set of finite morphisms to X. If C_i is a finite constant sheaf on X_i, then $H^q(X, \prod_i (p_i)_* C_i) \to H^q(X_0, \prod_i (p_i)_* C_i)$ is bijective for $q = 0$ and surjective for $q > 0$.*

Proof. We are assuming the result for the C_i. But the pushforwards of finite morphisms are exact, so this statement is immediate. □

Recall that an effaceable functor between abelian categories $F : C \to D$ is one such that for all $A \in \mathrm{Ob}(C)$, there is a monomorphism $u : A \to M$ such that $F(u) = 0$.

Lemma 5.7. *The functors $H^q(X, -)$ for $q > 0$ are effaceable in the category of constructible sheaves on X.*

Proof. See [1], Lemma IV.3.5. □

Lemma 5.8. *Every constructible sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ injects onto a finite product $\prod (p_i)_* C_i$, where the p_i are finite morphisms $X_i \to X$ and C_i is a finite constant sheaf on X_i.*

Details omitted. Being locally constant constructible is, by descent theory, equivalent to being represented by a finite étale group scheme. Since étale morphisms are local isomorphisms in the étale topology, one obtains finite étale $U_{ij} \to X_i$ for each $X_i \to X$ in the stratification over which $\mathcal{F}$ is finite constant. This gives the desired injection. □

We are now going to show that $H^q(X, \mathcal{F}) \cong H^q(X_0, \mathcal{F})$ given that $\mathcal{F}$ is constructible. We have the surjectivity statement (and isomorphism for $q = 0$) for constant sheaves. We will induct on q, so that we can use the result for $q - 1$ while for q we only have surjectivity for constant sheaves (and thus for suitable products of pushforwards of constant sheaves by one of our lemmas). By our lemmas we embed $\mathcal{F}$ in a product of pushforwards of constant sheaves; call this $\mathcal{G}$. The category of constructible sheaves is abelian, so $\mathcal{G}/\mathcal{F}$ is constructible. We have exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{G}) & \longrightarrow & H^0(X, \mathcal{G}/\mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & H^0(X_0, \mathcal{F}) & \longrightarrow & H^0(X_0, \mathcal{G}) & \longrightarrow & H^q(X_0, \mathcal{G}/\mathcal{F})
\end{array}$$

By a diagram chase the left arrow is injective, so the right is as well, so they are also isomorphisms by the five lemma.

For larger q we have the following exact sequences.

$$\begin{array}{ccccccc}
H^{q-1}(X,\mathcal{G}/\mathcal{F}) & \longrightarrow & H^q(X,\mathcal{F}) & \longrightarrow & H^q(X,\mathcal{G}) & \longrightarrow & H^q(X,\mathcal{G}/\mathcal{F}) \\
\downarrow\cong & & \downarrow & & \downarrow & & \downarrow \\
H^{q-1}(X_0,\mathcal{G}/\mathcal{F}) & \longrightarrow & H^q(X_0,\mathcal{F}) & \longrightarrow & H^q(X_0,\mathcal{G}) & \longrightarrow & H^q(X_0,\mathcal{G}/\mathcal{F})
\end{array}$$

Diagram chasing gives surjectivity. For injectivity, we need to use effaceability of $H^q(X,-)$. We embed $\mathcal{F}$ into an effacing constructible sheaf and diagram chase the corresponding diagrams, and we win.

5.4 Formal geometry

We list the main theorems from formal geometry which we will use here.

Theorem 5.9 (Formal functions). *Let $f : X \to Y$ be a projective morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, then*

$$\widehat{R^i f_*(\mathcal{F})}_y \cong \varprojlim_n H^i(X_n, \mathcal{F}_n).$$

Corollary 5.10. *Taking $i = 0$, we have*

$$\Gamma(X_{\widehat{A}}, \mathcal{O}_{X_{\widehat{A}}}) = \varprojlim_n \Gamma(X_n, \mathcal{O}_{X_n})$$

Theorem 5.11 (Formal GAGA). *Let X be a Noetherian scheme, separated and of finite type over $Y = \operatorname{Spec} A$, where A is an adic*[4] *Noetherian ring. Let $\widehat{X}$ be its I-adic completion. Then the functor $F \mapsto \widehat{F}$ is an equivalence of categories between the category of coherent sheaves on X whose support is proper over Y to the category of coherent sheaves on $\widehat{X}$ whose support is proper over $\widehat{Y}$.*

A Noetherian local ring B has the *approximation property* if, given some system of polynomials $P_i \in B[Y_1, \ldots, Y_m]$, a solution $(\widehat{b_1}, \widehat{b_2}, \ldots, \widehat{b_m}) \in \widehat{B}^m$, and some $N \geq 1$, then one can find a solution $(b_1, b_2, \ldots, b_m) \in B^m$ such that $b_j \equiv \widehat{b_j} \pmod{\mathfrak{m}^N}$.

Theorem 5.12 (Artin approximation). *The Henselization of a finitely generated algebra over a field or excellent Dedekind domain has the approximation property.*

5.5 Proof in the case of a curve

We have calculated that étale cohomology of curves with coefficients $\mathbb{Z}/n$ vanishes in degrees higher than 2. Thus from our reductions, it suffices to show that $H^q(X, \mathbb{Z}/n) \to H^q(X_0, \mathbb{Z}/n)$ under the natural map in the diagram below for is an isomorphism for $q = 0$ and surjective for $q = 1, 2$.

$$\begin{array}{ccc}
X_0 & \longrightarrow & X \\
\downarrow & & \downarrow \\
\operatorname{Spec} A/\mathfrak{m} & \longrightarrow & \operatorname{Spec} A
\end{array}$$

[4] Hausdorff and complete wrt I^n

$q = 0$

Because the connected components of a scheme X correspond to the primitive idempotents of $\Gamma(X, \mathcal{O}_X)$, we must show that the map $X_0 \to X$ induces a bijection

$$\operatorname{Idem} \Gamma(X, \mathcal{O}_X) \to \operatorname{Idem} \Gamma(X_0, \mathcal{O}_{X_0}).$$

Let $X_k = X \times_A \operatorname{Spec} A/m^{k+1}$ for all $k \geq 0$. Then all X_k have the same topological space, so we have

$$\operatorname{Idem} \varprojlim_k \Gamma(X_k, \mathcal{O}_{X_k}) \cong \operatorname{Idem} \Gamma(X_0, \mathcal{O}_{X_0}).$$

Now by the formal functions theorem, we have that

$$\varprojlim_k \Gamma(X_k, \mathcal{O}_{X_k}) = \Gamma(X_{\widehat{A}}, \mathcal{O}_{X_{\widehat{A}}}).$$

Now this latter ring is simply $\Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A}$ by definition. Thus it suffices to show that the idemportents of $\Gamma(X, \mathcal{O}_X)$ are in natural bijection with those of $\Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A}$. Recall that A is a strictly Henselian local ring. Then $\Gamma(X, \mathcal{O}_X)$ is an A-algebra and a finite A-module, which implies that it is a direct product of local rings. Then $\Gamma(X, \mathcal{O}_X)$ has the same idempotents as any $\Gamma(X, \mathcal{O}_X) \otimes_A A/\mathfrak{m}^{k+1}$, and thus has the same idempotents as $\Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A}$ as desired.

$q = 1$

We wish to show that $H^1(X, \mathbb{Z}/n) \to H^1(X_0, \mathbb{Z}/n)$ is surjective. As we have that $H^1_{\text{ét}}(X, \mathbb{Z}/n)$ classifies the finite étale $\mathbb{Z}/n$-torsors over X, it suffices to show that the $\mathbb{Z}/n$-torsors $Y_0 \to X_0$ can be lifted to torsors $Y \to X$.

Now we use the topological invariance of the étale site [2, Tag 04DY]. This gives that every $\mathbb{Z}/n$-torsor lifts uniquely to a $\mathbb{Z}/n$-torsor $Y_k \to X_k$, and thus to one over the formal scheme $\widehat{X}$. Now we apply formal GAGA, which allows us to lift coherent sheaves, and thus finite étale torsors (as they are finite morphisms) from $\widehat{Y} \to \widehat{X}$ to $Y' \to X \otimes_A \widehat{A}$. Our goal is to obtain an appropriate $Y \to X$ itself. This is done through Artin approximation.

We won't literally lift $Y' \to X \otimes_A \widehat{A}$ to $Y \to X$ (this might not be possible); we simply need some finite étale $\mathbb{Z}/n$-torsor $Y \to X$ that induces the same special fiber over X_0. The idea is that finite étale maps can be described by finitely many equations, so we can use Artin approximation to find an appropriate approximate solution.

To formalize this, we consider the functor

$$F : R \mapsto \{\text{finite étale covers of } X \otimes_A R\}/\sim.$$

Now because X is Noetherian, this functor is *locally of finite presentation*, which just means that

$$\varinjlim F(A_i) \cong F(\varinjlim A_i)$$

for A-algebras $\{A_i\}$. Pick the A_i to be the finitely generated A-subalgebras of $\widehat{A}$. Then any element of $F(\widehat{A})$ comes from an element of some $F(A_i)$. If $A_i = A[X_1, \ldots, X_t]/(f_1, \ldots, f_r)$, then we have a solution to the f_i in $\widehat{A}$. Then by Artin approximation, there is a solution in A that gives the same result when quotienting out by the maximal ideals. Thus we are done.

$q = 2$

We want to show that the map

$$H^2(X, \mathbb{Z}/n) \to H^2(X_0, \mathbb{Z}/n)$$

is surjective. We may assume n is a power of a prime. Then if $(n, \mathrm{char}\, k) > 0$, then Artin-Schreier theory shows that $H^2(X_0, \mathbb{Z}/n) = 0$. Otherwise, we use the Kummer sequence and reduce the problem to showing that $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ is surjective.

For this, one may give an elementary argument as in [1]. One may also give a similar argument to the one in the previous case as follows. First, we can lift line bundles to X_n, because we can do so locally and the global obstruction lies in $H^2(X_0, \mathcal{O}_{X_0}) = 0$. Thus we obtain the corresponding line bundle $\tilde{L}$ on $\tilde{X}$, and by formal GAGA we have the corresponding line bundle L' on $X \otimes_A \widehat{A}$. Now to apply Artin approximation, we consider the functor

$$F : R \mapsto \{\mathrm{Pic}(X \otimes_A R)\}.$$

Using the same argument as in the previous case, it suffices to show that this functor is locally of finite presentation. As before, this is true because X is Noetherian.

5.6 Applications

There are a lot of applications of proper base change, and we will see it used extensively later. But here we list a few important ones.

5.6.1 Compactly supported cohomology

Nagata's compactification theorem states that if X is a separated scheme of finite type over a field k[5], then there is an open immersion $j : X \hookrightarrow \overline{X}$ into a proper k-scheme $\overline{X}$.

Definition 5.13. *Let $\mathcal{F}$ be a torsion sheaf on X. Then we define cohomology with proper support as*

$$H^q_c(X, \mathcal{F}) := H^q(\overline{X}, j_! \mathcal{F}).$$

The fact that this is well-defined depends on the proper base change theorem. Indeed, take two different compactifications j_1, j_2. Then we have a factorization

$$\begin{array}{ccccccc} X & \longrightarrow & \overline{X}_3 & \longrightarrow & \overline{X}_1 \times \overline{X}_2 & \longrightarrow & \overline{X}_1 \\ & & & & \downarrow & & \downarrow \\ & & & & \overline{X}_2 & \longrightarrow & \mathrm{Spec}\, k \end{array}$$

where $\overline{X}_3$ is the closure of the image of X. Since $\overline{X}_3/k$ is proper, we reduce to showing that $H^q(\overline{X}_1, j_{1!}\mathcal{F}) = H^q(\overline{X}_3, j_{3!}\mathcal{F})$.

$$\begin{array}{ccc} X & \xrightarrow{j_{3!}} & \overline{X}_3 \\ & {\scriptstyle j_{1!}} \searrow & \downarrow {\scriptstyle p \text{ proper}} \\ & & \overline{X}_1 \end{array}$$

5 or any qcqs base, following Deligne

Since $j_{1!} = p_* j_{3!}$, by the Leray spectral sequence we have

$$H^q(\overline{X}_1, R^q p_*(j_{3!}\mathcal{F})) \Rightarrow H^{p+q}(\overline{X}_3, j_{3!}\mathcal{F}).$$

Thus it suffices to show that $R^q p_*(j_{3!}\mathcal{F}) = 0$ for $q > 0$. Now we use the proper base change theorem on the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{j_{3!}} & \overline{X}_3 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[j_{1!}]{} & \overline{X}_1 \end{array}$$

We have

$$j_1^*(R^q p_*(j_{3!}\mathcal{F})) = R^q \operatorname{id}(j_3^*(j_{3!}\mathcal{F})) = 0.$$

Looking at stalks, the result follows immediately.

Remark. More generally, for separated morphisms of finite type of Noetherian schemes $f : X \to S$, we may define

$$R^q f_! \mathcal{F} := R^q \overline{f}_*(j_! \mathcal{F}).$$

Then we have the base change formula

$$g^*(R^q f_! \mathcal{F}) \cong R^q f'_!(g'^* \mathcal{F}).$$

5.6.2 Constructible sheaves

Theorem 5.14. *Let $f : X \to S$ be a separated morphism of finite type and let $\mathcal{F}$ be a constructible sheaf on X. Then the sheaves $R^q f_! \mathcal{F}$ are constructible.*

5.6.3 Computations with dévissage

Theorem 5.15. *Let $f : X \to S$ be a separated morphism of finite type whose fibers are of dimension $\leq n$ and let $\mathcal{F}$ be a torsion sheaf on X. Then $R^q f_! \mathcal{F} = 0$ for $q > 2n$.*

Theorem 5.16. *Let $f : X \to S$ be a separated morphism of schemes of finite type over $\mathbb{C}$ and let $\mathcal{F}$ be a torsion sheaf on X. Then*

$$(R^q f_! \mathcal{F})^{\mathrm{an}} \cong R^q f_!^{\mathrm{an}} \mathcal{F}^{\mathrm{an}}.$$

In particular,

$$H_c^q(X, \mathbb{Z}/n) \cong H_c^q(X^{\mathrm{an}}, \mathbb{Z}/n).$$

Theorem 5.17. *Let X be an affine scheme of finite type over a separably closed field and $\mathcal{F}$ a torsion sheaf on X. Then $H^q(X, \mathcal{F}) = 0$ for $q > \dim X$.*

6 Purity, Gysin sequence, and the weak Lefschetz theorem

6.1 Purity

Let $Z \to X$ be a closed immersion of smooth schemes over a separably closed field k whose characteristic is prime to some positive integer n. If Z has codimension c, we call (Z, X) be a smooth pair of codimension c.

Theorem 6.1. *Let (Z, X) be a smooth pair of codimension c. and let $\mathcal{F}$ be a locally constant sheaf of Λ-modules. Then there is a canonical isomorphism*

$$H^{r-2c}(Z, \mathcal{F}(-c)) \xrightarrow{\cong} H^r_Z(X, \mathcal{F})$$

for $r \geq 0$.

The existence of such a canonical isomorphism has a lot of content. Indeed, abstract cohomological purity only gives an isomorphism, and the map itself represents a choice of fundamental class. When Z is not smooth, we still have the following result.

Proposition 6.2 (Semi-purity)**.** *For any closed $Z \subset X$ of codimension c, we have $H^r_Z(X, \Lambda) = 0$ for $r < 2c$.*

6.2 Gysin sequence

Recall the long exact sequence of a pair.

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1_Z(X, \mathcal{F}) \to \cdots.$$

Applying purity immediately gives the following results.

Proposition 6.3. *For $0 \leq r \leq 2c - 1$, we have $H^r_Z(X, \mathcal{F}) = 0$ and $H^r(X, \mathcal{F}) \cong H^r(U, \mathcal{F})$.*

Proposition 6.4 (Gysin sequence)**.** *We have a long exact sequence*

$$0 \to H^{2c}(X, \mathcal{F}) \to H^{2c}(U, \mathcal{F}) \to \cdots$$
$$\to H^{r-2c}_Z(X, \mathcal{F}) \to H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F}) \to \cdots.$$

6.3 Vanishing and finiteness theorems

Here is an application of the Gysin sequence.

Proposition 6.5. *Let X/k be a smooth variety and let $\mathcal{F}$ be a finite locally constant sheaf whose torsion is relatively prime to $\mathrm{char}(k)$. Then $H^i(X, \mathcal{F})$ is finite.*

Sketch of proof. ??? □

Proposition 6.6. *Let X/k be an affine variety of dimension d over a separably closed field and let $\mathcal{F}$ be a torsion sheaf. Then $\mathrm{cd}(X) \leq d$.*

Sketch of proof. We only give the proof for curves. The general case is more complicated but is an induction from this base case using the proper base change theorem. ???

First, since every torsion sheaf is a direct limit of constructible sheaves, if suffices to prove the statement for constructible $\mathcal{F}$. Then $\mathcal{F}$ is locally constant on some nonempty open $U \subset X$. We have an exact sequence

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to Q \to 0.$$

Then Q is just a finite direct sum of points $i_{x*}(Q_x)$, which has no higher cohomology. After passing to the normalization, we can assume X is smooth. So we've reduced to the case of smooth X and $\mathcal{F} = j_!(\mathcal{G}|_U)$ for $\mathcal{G}$ finite locally constant. One can now conclude with Poincaré duality and general vanishing for $i \geq 3$. ???

□

6.4 Weak Lefschetz theorem

Theorem 6.7. *Let X be a smooth projective variety over a separably closed field of dimension d and let H be a hyperplane section. If $\mathcal{F}$ is a torsion sheaf, then the morphisms*

$$H^i(X, \mathcal{F}) \to H^i(Z, \mathcal{F})$$

are isomorphisms for $i \leq d-2$ and is injective for $i = d-1$.

Theorem 6.8. *Let $U = X - X \cap H$; this is affine. The long exact sequence associated to the localization exact sequence is of the form*

$$H^{i-1}(Z, \mathcal{F}) \to H^i_c(U, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(Z, \mathcal{F}) \to H^{i+1}_c(U, \mathcal{F}) \to \cdots$$

We will use Poincaré duality, which gives that

$$H^i_c(U, \mathcal{F}) \cong H^{2d-i}(U, \hat{F}(d)).$$

By the vanishing theorem above for affines, this is 0 for $i \leq d-1$. The result follows.

7 Fundamental classes, cycle classes, and Chern classes

The cycle class map allows us to analyze cohomology classes coming from subvarieties. It can be fairly easily defined through the Gysin sequence, along with the purity statement in the previous section. However, it will be useful to develop a theory of fundamental classes to prove key properties, such as the fact that intersection product corresponds to cup product. These will be further complemented by Chern classes in étale cohomology.

7.1 Fundamental class

Let $Z \to X$ be a closed immersion of smooth schemes over a separably closed field k whose characteristic is prime to some positive integer n. If Z has codimension c, we call (Z, X) be a smooth pair of codimension c. Let $\Lambda = \mathbb{Z}/n$. We will define an associated **fundamental class**

$$s_{Z/X} \in H^{2c}(X, \Lambda(c)).$$

We will first define an element in $H^{2c}_Z(X, \Lambda(c))$ and take $s_{Z/X}$ to be its image in $H^{2c}(X, \Lambda(c))$. Recall that purity implies that for $0 \leq r \leq 2c-1$, we have $R^r\Gamma_Z(X, \mathcal{F}) = H^r_Z(X, \mathcal{F}) = 0$. Therefore in the Leray spectral sequence of

$$\mathrm{Sh}(X) \xrightarrow{i^!} \mathrm{Sh}(Z) \xrightarrow{\Gamma} \mathbf{Ab}$$

we get

$$H^i(Z, \mathcal{H}^j_Z(X, \mathcal{F})) \Rightarrow H^{i+j}_Z(X, \mathcal{F}) \quad \text{so} \quad H^i(Z, \mathcal{H}^{2c}_Z(X, \mathcal{F})) \cong H^{2c+i}_Z(X, \mathcal{F}).$$

In particular, when $i = 0$, we have $H^{2c}_Z(X, \Lambda(c)) \cong \Gamma(Z, \mathcal{H}^{2c}_Z(X, \Lambda(2c)))$ which is locally isomorphic to Λ. Our fundamental class will have order n, and thus show that this sheaf is indeed isomorphic to the constant sheaf Λ.

We proceed by first defining the fundamental class in the case that $c = 1$. In this case we use the Kummer sequence as follows. Assume Z is irreducible.

$$H^1_Z(X, \mathcal{G}_m) \xrightarrow{n} H^1_Z(X, \mathcal{G}_m) \xrightarrow{\delta} H^2_Z(X, \Lambda(1))$$

The exact sequence of a pair shows that $H^1_Z(X, \mathcal{G}_m) \cong \mathbb{Z}$, so we see that the image of 1 under δ has order n as desired.

In general, there is a unique way of assigning a fundamental class

$$s_{Z/X} \in H^{2c}_Z(X, \Lambda(c))$$

of order n extending what we just did in a functorial manner. More precisely, if $(\phi : (Z', X') \to (Z, X)$ is a morphism of smooth pairs of codimension c, then $\phi^*(s_{Z/X}) = s_{Z'/X'}$. Furthermore, if (Z, Y, X) is a smooth triple, we have $s_{Z/Y} \otimes s_{Y/X} = s_{Z/X}$. These conditions imply, using induction, that the fundamental class is unique. Existence follows from showing that they can be patched from local data.

We can now identify $s_{Z/X}$ with $1 \in H^0(Z, \Lambda)$ and thus their images in $H^{2c}(X, \Lambda(c))$. Through some homological algebra, we have the following commutative diagram.

$$\begin{array}{ccccc} H^r(Z, \Lambda) & \times & H^s(Z, \Lambda) & \xrightarrow{\smile} & H^{r+s}(Z, \Lambda) \\ {\scriptstyle i^*}\uparrow & & {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ H^r(X, \Lambda) & \times & H^{s+2c}(X, \Lambda) & \xrightarrow{\smile} & H^{r+s+2c}(X, \Lambda) \end{array}$$

The key is that $i_* i^*(x) = x \smile i_*(1_Z)$ for $x \in H^r(X, \Lambda)$.

7.2 Cycle class

Recall the Gysin sequence for smooth pairs (Z, X) of codimension c:

$$\begin{aligned} &0 \to H^{2c-1}(X, \mathcal{F}) \to H^{2c-1}(U, \mathcal{F}) \\ &\to H^0_Z(X, \mathcal{F}) \to H^{2c}(X, \mathcal{F}) \to H^{2c}(U, \mathcal{F}) \to \cdots \\ &\to H^{r-2c}_Z(X, \mathcal{F}) \to H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F}) \to \cdots . \end{aligned}$$

When Z is smooth, we can define the cycle class $\mathrm{cl}_X(Z) \in H^{2c}(X, \Lambda(c))$ as the image of $1 \in H^0_Z(X, \Lambda) \cong H^{2c}_Z(X, \Lambda(c)) \to H^{2c}(X, \Lambda(c))$.

When Z is not smooth, we can define it using semi-purity. Let Y be the singular locus of Z. The exact sequence of the triple $(X, X \backslash Y, X, \backslash Z)$ is of the form

$$\cdots \to H^r_Y(X, Z) \to H^r_Z(X, \Lambda) \to H^r_{Z \backslash Y}(X \backslash Y, \Lambda) \to \cdots .$$

By semi-purity for Y, which has codimension at least $c+1$, we obtain $H^{2c}_Z(X, \Lambda) \cong H^{2c}_{Z\backslash Y}(X \backslash Y, \Lambda)$. Thus we can define $\mathrm{cl}_X(Z)$ to be the image of 1 under the composite

$$\Lambda \cong H^0(Z \backslash Y, \Lambda) \cong H^{2c}_{Z \backslash Y}(X \backslash Y, \Lambda(c)) \cong H^{2c}_Z(X, \Lambda(c)) \to H^{2c}(X, \Lambda(c)).$$

The key property we would like to have is that the intersection product on cycles is compatible, at least in nice scenarios, with the cup product on cohomology. We begin with a few lemmas.

Lemma 7.1. *Let $\pi : Y \to X$ be a morphism of varieties and let Z be a cycle on X. Assuming $Y \times_X Z$ is smooth, we have $\mathrm{cl}_Y(\pi^* Z) = \pi^* \mathrm{cl}_X(Z)$.*

Indeed, this is true by functoriality of the fundamental class. (e.g. the LHS is just the image of $s_{Z \times Y/Y}$.

Lemma 7.2. *For $W \in C^*(X)$ and $Z \in C^*(Y)$, we have*

$$\mathrm{cl}_{X\times Y}(W \times Z) = p^* \mathrm{cl}_X(W) \cup q^* \mathrm{cl}_Y(Z).$$

This uses the key fact that $i_* i^*(x) = x \smile i_*(1_Z)$ for $x \in H^r(X, \Lambda)$.

From these, we get the following proposition, noting that in the transverse case, we have $W \cdot Z = W \times_X Z$.

Proposition 7.3. *Let W and Z be cycles on X whose primes intersect each other transversally. Then*

$$\mathrm{cl}_X(W \cdot Z) = \mathrm{cl}_X(W) \smile \mathrm{cl}_X(Z).$$

7.3 Chern classes

8 Poincaré duality

8.1 Statements

There are multiple different levels of generality of Poincaré duality in étale cohomology: constant sheaves, locally constant sheaves, and constructible sheaves, and curves, varieties, relative version.

8.2 The trace morphism

8.3 The cup product

8.4 Poincaré duality for curves

Theorem 8.1. *Let U be a smooth curve over $k = \overline{k}$. For all constructible sheaves $\mathcal{F}$ of $\mathbb{Z}/n$-modules on U and all $r \geq 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U, \mathcal{F}) \times \mathrm{Ext}^{2-r}_{U,n}(\mathcal{F}, \mu_n) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n.$$

Corollary 8.2. *With the same conditions, if $\mathcal{F}$ is locally free we have a perfect pairing*

$$H^r_c(U, \mathcal{F}) \times H^{2-r}(U, \mathcal{F}^\vee(1)) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n$$

Proof. Consider the Grothendieck spectral sequence associated to

$$\mathrm{Sh}(U) \xrightarrow{\mathrm{Hom}(\mathcal{F},-)} \mathrm{Sh}(U) \xrightarrow{\Gamma} \mathbf{Ab}.$$

Applying this to μ_n gives the spectral sequence

$$H^p(U, \underline{\mathrm{Ext}}^q(\mathcal{F}, \mu_n)) \Rightarrow H^{p+q}(U, \mu_n).$$

Since $\mathcal{F}$ is locally free, $\mathrm{Ext}^q(\mathcal{F}, \mu_n) = 0$ for $q > 0$ and we get the desired result. □

Actually, we will prove the corollary properly and use it to deduce the theorem. We will not give the details of why the cup product is the pairings we are using.

Proof. We are assuming that $\mathcal{F}$ is locally free. First, let us rephrase the statement to say that

$$\phi^r : H^r(U, \mathcal{F}) \to H^{2-r}_c(U, \mathcal{F}^\vee(1))^\vee$$

is an isomorphism for all r. For $r \neq 0, 1, 2$, we can use the general result on cohomological dimension.

1. If $\pi : U' \to U$ is finite, then if the statement is true for $\mathcal{F}$ on U' then it is true for $\pi_*\mathcal{F}$ on U. This is true because π_* is exact.

2. $r = 0$. Since $\mathcal{F}$ is locally constant, there is some finite étale $\pi : U' \to U$ over which $\mathcal{F}$ is constant, and thus can be embedded into some $\mathcal{G} = (\mathbb{Z}/n\mathbb{Z})^s$. So the injection $\pi^*\mathcal{F} \hookrightarrow \mathcal{G}$ gives an injection $\mathcal{F} \hookrightarrow \pi_*\mathcal{G}$. Then we have an exact sequence of sheaves $0 \to \mathcal{F} \to \pi_*\mathcal{G} \to \mathcal{H}$, and $\mathcal{H}$ is also locally constant. We then have the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X,\mathcal{F}) & \longrightarrow & H^0(X,\pi_*\mathcal{G}) & \longrightarrow & H^0(X,\mathcal{H}) \\
\downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & H^0_c(X,\mathcal{F}^\vee(1))^\vee & \longrightarrow & H^0_c(X,\pi_*\mathcal{G}^\vee(1))^\vee & \longrightarrow & H^0_c(X,\mathcal{H}^\vee(1))^\vee
\end{array}$$

   Diagram chasing gives the desired result.

3. Reduction to the case $\mathcal{F} = \mathbb{Z}/n$ and injectivity of ϕ^1. We induct using the same method as above.

$$\begin{array}{ccccccc}
H^0(X,\mathcal{H}) & \longrightarrow & H^1(X,\mathcal{F}) & \longrightarrow & H^1(X,\pi_*\mathcal{G}) & \longrightarrow & H^1(X,\mathcal{H}) \\
\downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \\
H^0_c(X,\mathcal{H}^\vee(1))^\vee & \longrightarrow & H^1_c(X,\mathcal{F}^\vee(1))^\vee & \longrightarrow & H^1_c(X,\pi_*\mathcal{G}^\vee(1))^\vee & \longrightarrow & H^1_c(X,\mathcal{H}^\vee(1))^\vee
\end{array}$$

   The right arrow is assumed to be injective, so by the four lemma ϕ^1 is also surjective. The case $r = 2$ is another four lemma argument on the following diagram.

$$\begin{array}{ccccccccc}
H^1(X,\pi_*\mathcal{G}) & \longrightarrow & H^1(X,\mathcal{H}) & \longrightarrow & H^2(X,\mathcal{F}) & \longrightarrow & H^2(X,\pi_*\mathcal{G}) & \longrightarrow & H^2(X,\mathcal{H}) \\
\cong \downarrow & & \cong \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
H^1(X,\pi_*\mathcal{G}) & \longrightarrow & H^1_c(X,\mathcal{H}^\vee(1))^\vee & \longrightarrow & H^2_c(X,\mathcal{F}^\vee(1))^\vee & \longrightarrow & H^2_c(X,\pi_*\mathcal{G}^\vee(1))^\vee & \longrightarrow & H^1_c(X,\mathcal{H}^\vee(1
\end{array}$$

4. Injectivity of ϕ^1. Interpret $H^1(U,\mathbb{Z}/n) = \mathrm{Hom}_c(\pi_1(U),\mathbb{Z}/n)$. An element s is determined by its kernel, which corresponds to a Galois covering $\pi : U' \to U$. Then s maps to 0 in $H^1(U',\mathbb{Z}/n\mathbb{Z}) \cong H^1(U,\pi_*\mathbb{Z}/n\mathbb{Z})$. Then use the long exact sequence associated to

$$0 \to \mathbb{Z}/n \to \pi_*\mathbb{Z}/n \to \mathcal{H} \to 0.$$

   This shows that if s is in the kernel of ϕ^1, then $s = 0$.

5. Final step: isomorphisms for $\mathcal{F} = \mathbb{Z}/n$. For projective X, we know they have the same size for $r = 1$, which seals the deal by injectivity. It is easy to check the result for $r = 2$. Now we need to be able to remove points. We do this by the exact sequence of the pair $(U, U\backslash x)$ and the localization exact sequence.

□

8.5 Poincaré duality for varieties

Theorem 8.3. *Let X/k be a smooth variety of dimension d over a separably closed field and let $(n, \mathrm{char}\, k) = 1$. If $\mathcal{F}$ is a constructible sheaf of $\Lambda = \mathbb{Z}/n$-modules, then the cup product gives a perfect pairing*

$$H^i_c(X,\mathcal{F}) \times H^{2d-i}(X,\mathcal{F}^\vee(d)) \to H^{2d}_c(X,\Lambda(d)) \xrightarrow{\cong} \Lambda.$$

8.5.1 Derived categories and the 6 functors

The six operations naturally arise in the setting of derived categories, but let us begin by reviewing the constructions on the ordinary sheaf level.

Let $f : X \to Y$ be a morphisms of schemes. Then we will define

- direct image $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$
- inverse images: $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$
- proper direct image: $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$

8.6 Relative Poincaré duality

9 The Lefschetz trace formula

9.1 Recollection of Weil cohomology axioms

Let us list the main properties of étale cohomology that make it a Weil cohomology theory. Let X be a variety of dimension d over a separably closed field k.

1. (Cycle class) Let $Z \subset X$ be a smooth closed subscheme of codimension r; then there is an associated fundamental class $s_{Z/X} \in H^{2r}_Z(X, \Lambda(r))$. In general, there is a cycle class map
$$\mathrm{cl}_X(C^*(X) \to H^*(X))$$
that agrees with the fundamental class.

2. (Trace map) There is an isomorphism
$$H^{2d}_c(X, \Lambda(d)) \xrightarrow{\cong} \Lambda$$
sending $\mathrm{cl}_X(P)$ to 1 for all closed points P.

3. (Poincaré duality) There is a perfect pairing for locally constant constructible sheaves $\mathcal{F}$ of Λ-modules on X:
$$H^i_c(X, \mathcal{F}) \times H^{2d-i}(X, \tilde{F}(d)) \to H^{2d}_c(X, \Lambda(d)) \xrightarrow{\cong} \Lambda$$

4. (Künneth formula) Let X, Y be proper and assume that $\mathcal{F}$ and $H^r(X, \mathcal{F})$ are flat. Then the cup product gives isomorphisms
$$H^*(X, \mathcal{F}) \otimes H^s(Y, \mathcal{G}) \xrightarrow{\cong} H^*(X \times Y, p^*\mathcal{F} \otimes q^*\mathcal{G}).$$

5. (Weak Lefschetz theorem) Let X be projective and let $i : Z \to X$ be a hyperplane section. Then the map
$$H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F})$$
is an isomorphism for $i \geq d + 2$ and a surjection for $i = d + 1$. ...

6. (Hard Lefschetz theorem) The Lefschetz operator $L : H^i(X, \mathcal{F}) \to H^{i+2}(X, \mathcal{F})$ is given by taking the cup product of the cycle class of some hyperplane. Then
$$L^i : H^{d-i}(X, \mathcal{F}) \to H^{d+i}(X, \mathcal{F})$$
is an isomorphism.

9.2 Generalizing the Gysin map

Recall that the Gysin map

$$H^r(Z,\mathcal{F}) \to H^{r+2c}(X,\mathcal{F}(c))$$

applied to $1 \in H^0_Z(X,\Lambda)$ gives the cycle class $\mathrm{cl}(Z)$. Using Poincaré duality we can generalize this to arbitrary proper morphisms $\pi : Y \to X$ of smooth varieties where $\dim X = \dim Y + c$. The associated map on cohomology $\pi^* : H_c^{2d-r}(X,\Lambda(d)) \to H_c^{2d-r}(Y,\Lambda(d))$ is dual to the corresponding map

$$\pi_* : H^{r-2c}(Y,\Lambda(-c)) \to H^r(X,\Lambda).$$

When π is a closed embedding, then π_* is the Gysin map. This map π_* acts well functorially, and one obtains a projection formula:

$$\pi_*(y \smile \pi^*(x)) = \pi_*(y) \smile x.$$

9.3 The Lefschetz trace formula

Using these wonderful properties, we will prove the Lefschetz trace formula for étale cohomology.

Theorem 9.1 (Lefschetz trace formula). *Let X be a smooth projective variety over an algebraically closed field of dimension d and let $\phi : X \to X$ be an endomorphism. We have*

$$\Delta \cdot \Gamma_\phi = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(\phi, H^i_c(X,\mathbb{Q}_l)).$$

The Künneth formula allows us to identify $H^*(X\times X)$ with $H^*(X)\otimes H^*(X)$. Here, $p^*(a) \smile q^*(b)$ is identified with $a\otimes b$. If we let $\{e^r_i\}$ be a basis for $H^r(X) = H^r_c(X)$, then we may set $\{f^r_i\}$ to be the dual basis as a basis of $H^{2d-r}(X)$. One can ask: what is the class of Γ_ϕ in these terms?

Proposition 9.2. *We have* $\mathrm{cl}_{X\times X}(\Gamma_\phi) = \sum_{r,i} \phi^*(e^r_i) \otimes f_i^{2d-r}$.

Proof. Let $\mathrm{cl}_{X\times X}(\Gamma_\phi) = \sum_{r,i} a_i \otimes f_i^{2d-r}$; we must show that $\phi^*(e_i) = a_i$. Well,

$$\begin{aligned}
\phi^*(e_i) &= p_*(1,\phi)_*(1 \smile (1,\phi)^* q^*(e_i)) \\
&= p_*((1,\phi)_*(1) \smile q^*(e_i)) \quad \text{(projection formula)} \\
&= p_*(\mathrm{cl}_{X\times X}(\Gamma_\phi) \smile q^*(e_i)) \\
&= a_i,
\end{aligned}$$

as desired. □

Proof of the Lefschetz trace formula. Since Δ and Γ intersect transversely, we can compute their intersection product by computing the cup product of their cycle classes. From the lemma above, we have

$$\mathrm{cl}_{X\times X}(\Gamma_\phi) = \sum_{r,i} e^r_i \otimes f_i^{2d-r}, \quad \mathrm{cl}_{X\times X}(\Gamma_\phi) = \sum_{r,i} \phi^*(e^r_i) \otimes f_i^{2d-r} = \sum_{r,i} (-1)^{r(2d-r)} f_i^{2d-r} \otimes \phi^*(e^r_i).$$

Thus we have

$$\begin{aligned}
\mathrm{cl}_{X\times X}(\Delta\cdot\Gamma) &= \mathrm{cl}_{X\times X}(\Delta) \smile \mathrm{cl}_{X\times X}(\Gamma) \\
&= \Big(\sum_{r,i} e^r_i \otimes f_i^{2d-r}\Big) \smile \Big(\sum_{r,i} (-1)^{r(2d-r)} f_i^{2d-r} \otimes \phi^*(e^r_i)\Big) \\
&= \sum_{r,i} (-1)^r \big(e^r_i \smile f_i^{2d-r}\big) \otimes \big(\phi^*(e^r_i) \smile f_i^{2d-r}\big)
\end{aligned}$$

Taking the trace of both sides yields the desired result.

□

9.4 The many Frobenii

9.5 Application to the Weil conjectures

Recall the Weil conjectures.

10 The Grothendieck trace formula

10.1 Statement

The Grothendieck trace formula is an important generalization of the Lefschetz trace formula to constructible $\mathbb{Q}_l$ sheaves. For instance, it is a key component of Deligne's completion of the proof of the Weil conjectures.

Theorem 10.1 (Grothendieck trace formula). *Let $X_0/k = \mathbb{F}_q$ be a variety of dimension d and $\mathcal{F}_0$ be a constructible $\mathbb{Q}_l$-sheaf on X_0. Set $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ and let $\mathcal{F}$ be the pullback of $\mathcal{F}_0$ to X. Let π_x be the geometric Frobenius. Then we have*

$$\sum_{x \in X(k)} \mathrm{Tr}(\pi | \mathcal{F}_{\overline{x}}) = \sum_{i=0}^{2d} \mathrm{Tr}(\pi | H^i_c(X, \mathcal{F})).$$

When $\mathcal{F} = \mathbb{Q}_l$, we recover the Lefschetz trace formula (which is stated for an arbitrary endomorphism ϕ of X), which we recall here.

Theorem 10.2 (Lefschetz trace formula). *Under the same conditions, we have*

$$|\Delta \cdot \Gamma_\phi| = \sum_{i=0}^{2d} \mathrm{Tr}(\phi | H^i_c(X, \mathbb{Q}_l).$$

Let us make some comments on the statement of the Grothendieck trace formula. We recall that the action of the geometric Frobenius on the stalks is given by

$$\mathrm{Spec}\, \overline{\mathbb{F}_q} \xrightarrow{(x \mapsto x^q)^{-1}} \mathrm{Spec}\, \overline{\mathbb{F}_q} \xrightarrow{x} X.$$

This changes the étale neighborhoods in the definition of the stalk, and thus acts on $\mathcal{F}_{\overline{x}}$. The geometric Frobenius acts on $X = X_0 \times_{\mathbb{F}_q} \mathrm{Spec}\, \overline{\mathbb{F}_q}$ by $\mathrm{id} \times \mathrm{Fr}^{-1}_{\overline{\mathbb{F}_q}}$, which gives the same action as the relative Frobenius: $\mathrm{Fr}_{X_0} \times \mathrm{id}$.

As a sanity check, let's see what this formula says for $\mathbb{P}^1(\mathbb{F}_q)$. On one hand, the left hand side is just $q+1$. We have $H^0_c(\mathbb{P}^1, \mathbb{Q}_l) = \mathbb{Q}_l$ which gives a trace of 1 and $H^2_c(\mathbb{P}^1, \mathbb{Q}_l) = \mathbb{Q}_l(-1)$ which gives a trace of q.

Next, let us discuss what it means to take the trace on the RHS. While $H^i_c(X, \mathcal{F})$ is a $\mathbb{Q}_l$ vector space, to prove this we are really proving the statement for each $\mathcal{F}_n$ which make up $\mathcal{F}$. Indeed, recall that a $\mathbb{Q}_l$-sheaf is obtained from tensoring by $\mathbb{Q}_l$ the system of $\mathbb{Z}/l^n$-sheaves $\mathcal{F}_n$ which satisfy $\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z} \xrightarrow{\cong} \mathcal{F}_n$.

If M is a projective module over some ring Λ, then we can take a trace of an endomorphism $\phi : M \to M$. Indeed, we can write $\Lambda^N = M \oplus M'$ and extend by 0 to extend ϕ to an endomorphism of Λ^n. Then the trace of ϕ on Λ^n is well-defined, and we take this to be our trace. More generally, if Λ is non-abelian, e.g. $\mathbb{Z}/l^n Z[G]$, then we can still embed M inside Λ^n with a section, and can take the trace to be the trace of the appropriate composition. The main issue we now have is that even if the sheaf $\mathcal{F}$ in our scenario is flat and thus the stalks are free, its cohomology groups may not be projective. For this we will need the formalism of perfect complexes and filtered derived categories.

10.2 Perfect complexes and filtered derived categories

10.3 Dévissage to curves

10.4 Reduction to the Lefschetz trace formula

10.5 In terms of L-functions

10.6 Application: exponential sums

Part II
Weights and Weil II

11 Weil sheaves and weights

11.1 Weil sheaves

Recall that if $E/\mathbb{Q}_l$ is a finite extension, then we define étale E-sheaves in the same way as $\mathbb{Q}_l$ sheaves. The category of $\overline{\mathbb{Q}}_l$-sheaves is obtained from taking the direct limit of the category of E-sheaves.

Let $X_0/\mathbb{F}_q$ be a scheme and let $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Let $\mathcal{G}_0$ be a $\overline{\mathbb{Q}}_l$ sheaf on X_0; it extends to a $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{G}$ on X. Recall there is the relative Frobenius Fr_X defined by the absolute Frobenius on X_0 and the identity on scalars, and the geometric Frobenius F_X (also known as the Frobenius automorphism) acting by the inverse of the arithmetic Frobenius on scalars. We have isomorphisms

$$\mathrm{Fr}_X^*(\mathcal{G}) \cong \mathcal{G} and F_X^*(\mathcal{G}) \cong \mathcal{G}$$

that allow us to define equivalent maps on cohomology by the Frobenius. The latter one is used to generalize to Weil sheaves.

Definition 11.1. *A Weil sheaf $\mathcal{G}_0$ on X_0 consists of a $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{G}$ on X with an isomorphism $F_X^*(\mathcal{G}) \cong \mathcal{G}$.*

The Weil group $W(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \mathbb{Z}$ is the subgroup of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ generated by the geometric Frobenius. The preimage of it in the homotopy exact sequence gives rise to an exact sequence

$$1 \to \pi_1(X, \overline{x}) \to W(X_0, \overline{x}) \to W(\overline{\mathbb{F}_q}/\mathbb{F}_q) \to 1.$$

Recall that by taking the stalk at $\overline{x}$, lisse $\overline{\mathbb{Q}}_l$-sheaves correspond to finite-dimensional continuous representations of $\pi_1(X, \overline{x})$ on $\overline{\mathbb{Q}}_l$-vector spaces (without lisse, they don't need to be finite-dimensional). If we use lisse Weil sheaves instead, we get finite-dimensional continuous representations of $W(X, \overline{x})$ on $\overline{\mathbb{Q}}_l$-vector spaces.

For example, lisse rank 1 Weil sheaves on $\operatorname{Spec} \mathcal{F}_q$ correspond to characters $\mathbb{Z} \to \overline{\mathbb{Q}}_l^*$. Thus for each $b \in \overline{\mathbb{Q}}_l^*$, we obtain a Weil sheaf $\mathcal{L}_b$ on $\operatorname{Spec} \mathcal{F}_q$. We will also use $\mathcal{L}_b$ to denote its pullback to X_0.

Given an irreducible lisse Weil sheaf $\mathcal{G}_0$ on a normal and geometrically connected $X_0/\mathcal{F}_q$, there is an étale sheaf $\mathcal{F}_0$ and an element $b \in \overline{\mathbb{Q}}_l^*$ such that $\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b$. In general, one can always give a filtration of $\mathcal{G}_0$ with each factor of this form. Using this, we can extend the Grothendieck trace formula to Weil sheaves. We give the L-function version.

Theorem 11.2. *Let $\mathcal{G}_0$ be a lisse Weil sheaf on X_0. Then we have*

$$L(X_0, \mathcal{G}_0, t) := \prod_{x \in |X_0|} \det\left(1 - t^{d(x)} F_x, \mathcal{G}_x\right)^{-1} = \prod_{i=0}^{2 \dim X} \det(1 - tF, H^i_c(X, \mathcal{G}))^{(-1)^{i+1}}.$$

Recall $\overline{\mathbb{Q}}_l$-sheaves. The two isos you get from X_0 sheaves. Def. Weil sheaves. Equiv. of cat. reps, Weil group action on stalks. Grothendieck trace formula.

11.2 Semicontinuity of weights

Let $X_0/\mathcal{F}_q$ be a scheme and let $\mathcal{G}_0$ be a Weil sheaf on X_0. Fix an isomorphism (by the axiom of choice)

$$\tau : \overline{\mathbb{Q}}_l \to \mathbb{C}.$$

Take a geometric point $\overline{x} : \operatorname{Spec} \overline{\mathbb{F}_q} \to X_0$ to lie over each closed point $x \in |X_0|$. Recall that the Weil group $W(\overline{\mathbb{F}_q}/k(x)) \ni F_x$ acts on the stalk $(\mathcal{G}_0)_x$.

Definition 11.3.
- *The sheaf $\mathcal{G}_0$ is **τ-pure of weight** β if for all $x \in |X_0|$, each eigenvalue $\alpha \in \overline{\mathbb{Q}}_l$ of F_x acting on $(\mathcal{G}_0)_x$ satisfies*

$$\tau(\alpha) = N(X)^{\beta/2}.$$

- *The sheaf $\mathcal{G}_0$ is **τ-mixed** if there is a finite filtration of subsheaves so that each factor $\mathcal{G}_0^j/\mathcal{G}_0^{j-1}$ is τ-pure.*

Furthermore, we say a sheaf is pure or mixed if these properties hold for any choice of τ.

Weights satisfy some nice functorial properties. For example, if $f : X_0 \to Y_0$ is a morphism over $\mathbb{F}_q$, then if $\mathcal{G}_0$ on Y_0 is τ-pure of weight β, so is $f_0^*(\mathcal{G}_0)$. The other direction holds if f is surjective or finite.

We can define the weight of a sheaf to be the maximum of its weights over all closed points and eigenvalues. Precisely,

$$w(\mathcal{G}_0) = \sup_{x,\alpha} \frac{\log(|\tau(\alpha)|^2)}{\log(N(x))}.$$

If $\mathcal{G}_0$ is the zero sheaf, then we set $w(\mathcal{G}_0) = -\infty$.

It makes sense that the convergence of the L-function of a sheaf is dependent on its weights. Indeed, if $w(\mathcal{G}_0) \leq \beta$, then the L-function

$$\tau L(X_0, \mathcal{G}_0, t) = \prod_{x \in |X_0|} \tau \det(1 - t^{d(x)} F_x, \mathcal{G}_{0x})^{-1}$$

converges for $|t| < q^{-\beta/2 - \dim(X_0)}$ and has no zeroes or poles in this region. The proof is simple: the Grothendieck trace formula shows it is meromorphic, and use the logarithmic derivative and the given bounds to analyze the zeroes and poles.

Next, we have a result essentially saying that under some conditions, if the weight of a sheaf is bounded when pulled back to some dense open subset, then it satisfies the same bound on the whole space.

Proposition 11.4. *Let X_0 be a smooth irreducible curve over $\mathcal{F} - q$, with $j_0 : U_0 \hookrightarrow X_0$ a nonempty open subset with complement S_0. If $\mathcal{G}_0$ is a Weil sheaf on X_0 with $j_0^* \mathcal{G}_0$ lisse and $H^0_S(X, \mathcal{G}) = 0$, then $w(j_0^*)\mathcal{G}_0)) \leq \beta$ implies $w(\mathcal{G}_0) \leq \beta$.*

This allows us to prove the following semicontinuity result.

Theorem 11.5. *Let $\mathcal{G}_0$ be a lisse Weil sheaf on X_0 and let $j_0 : U_0 \hookrightarrow X_0$ be a dense open subscheme.*

1. $w(\mathcal{G}_0) = w(j_0^*(\mathcal{G}_0))$ *and if the latter is pure, so is the former.*
2. *If X_0 is irreducible and normal and $\mathcal{G}_0$ is normal, then if $j_0^*(\mathcal{G}_0)$ is τ-mixed, then $\mathcal{G}_0$ is τ-pure.*
3. *If X_0 is connected and $j_0^*(\mathcal{G}_0)$ is τ-mixed, $\mathcal{G}_0$ is τ-pure of weight β at $x \in |X_0|$, then $\mathcal{G}_0$ is τ-pure of weight β.*

In general, if $\mathcal{G}_0$ is a Weil sheaf on $X/\mathcal{F}_q$, then there is a dense open subscheme $j_0 : U_0 \hookrightarrow X_0$ such that $j_0^*(\mathcal{G}_0)$ is lisse. We define

$$w_{\mathrm{gen}}(\mathcal{G}_0) = w(j_0^*(\mathcal{G}_0)).$$

We say that $\mathcal{G}_0$ is τ-real if its characteristic polynomial $\tau \det(1 - F_x t, \mathcal{G}_{0x}) \in \mathbb{R}[t]$ for all $x \in |X_0|$. A lisse τ-pure sheaf of weight β is a direct summand of a τ-real and τ-pure sheaf of weight β, e.g.

$$F_0 = (\mathcal{G}_0^\vee \otimes \mathcal{L}_{\tau^{-1}(q^\beta)}) \oplus \mathcal{G}_0.$$

11.3 Sheaf-function correspondence and radius of convergence

Let us recall the sheaf-function correspondence. Again, given a Weil sheaf $\mathcal{G}_0$ on X_0, we have the action of the geometric Frobenius F_x on $\mathcal{G}_{0x}$. We obtain a function $f_n^{\mathcal{G}_0} : X_0(\mathbb{F}_{q^n}) \to \mathbb{C}$ defined by

$$f_n^{\mathcal{G}_0}(\overline{x}) = \tau \operatorname{Tr}(F_x^{n/d(x)}, \mathcal{G}_{0x}).$$

We can now do some analysis. Put a scalar product on functions $X_0(\mathbb{F}_{q^n}) \to \mathbb{C}$ by setting

$$(f, g)_n = \sum_{y \in X_0(\mathbb{F}_{q^n}} f(y)\overline{g(y)}.$$

Let us rewrite the logarithmic derivative of the L-function in these terms. We have

$$\frac{\tau L'(X_0, \mathcal{G}_0, t)}{\tau L(X_0, \mathcal{G}_0, t} = \sum_{n=1}^{\infty} (f^{\mathcal{G}_0}, 1)_n t^{n-1}.$$

We now introduce a variant of this.

Definition 11.6. *Given a Weil sheaf $\mathcal{G}_0$, define*

$$\phi^{\mathcal{G}_0}(t) := \sum_{t=1}^{\infty} ||f^{\mathcal{G}_0}||_n^2 t^{n-1}.$$

This is given by

$$\phi^{\mathcal{G}_0}(t) = \sum_{n=1}^{\infty} \left(\sum_{x \in X_0(\mathbb{F}_{q^n})} |\tau \operatorname{Tr}(F_x^{n/d(x)})|^2 \right) t^{n-1}.$$

Like for the L-function, we want to bound these coefficients of this function and understand its radius of convergence.

Proposition 11.7. *There is a constant C independent from n such that*

$$||f^{\mathcal{G}_0}||_n^2 \le C q^{n(w(\mathcal{G}_0) + \dim(X_0))}.$$

Thus, $\phi^{\mathcal{G}_0}(t)$ converges for $|t| < q^{-w(\mathcal{G}_0) + \dim(X_0)}$.

The main result is that this often is exactly the radius of convergence. More precisely, if X_0 is a smooth curve with $H_E^0(X, \mathcal{G}) = 0$ for all closed subsets E of X, then it is.

L2 norm of a sheaf, radius of convergence.

11.4 Determinant weights

It turns out that rank 1 lisse Weil-sheaves are all τ-pure. Let $\mathcal{G}_0$ be such a sheaf on X_0 and let

$$\chi : W(X_0, \overline{x}) \to \overline{\mathbb{Q}}_l^*$$

be the corresponding character.

Theorem 11.8. *The image of $\pi_1(X, \overline{x})$ is χ in $\overline{\mathbb{Q}}_l^*$ is finite.*

The idea of the proof is the following. By dévissage, we reduce to the case where X_0 is a smooth projective, geometrically connected curve. Since the target is abelian, we know everything factors through its abelianization. Now recall we have an exact sequence

$$1 \to \pi_1(X, \overline{x}) \to W(X_0, \overline{x}) \to W(\overline{\mathbb{F}_q}/\mathbb{F}_q) \to 1.$$

The image of $\pi_1(X, \overline{x})$ is therefore contained in the image of the kernel I_K of the map

$$W(X_0, \overline{x})_{\mathrm{ab}} \to W(\overline{\mathbb{F}_q}/\mathbb{F}_q).$$

One can identify I_K with $\mathrm{Pic}^0(X_0)(\mathcal{F}_q)$, which is clearly finite.

If you try to do this with say the nodal cubic, you probably can't get the Pic^0 thing to work, and the image in any case may be infinite.

Looking back at the short exact sequence, we see that we can write $\chi = \chi_1 \cdot \chi_2$, where χ_1 is torsion (as we have just seen) and χ_2 factors through $W(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \mathbb{Z}$. Thus we may factor

$$\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b,$$

where $\mathcal{F}_0$ is torsion and is thus pure of weight 0. Since $\mathcal{L}_b$ is τ-pure of weight $\frac{\log(|\tau(b)|)^2}{\log(q)}$, we get that $\mathcal{G}_0$ is τ-pure of the same weight.

Now let us define determinant weights. Given a smooth Weil sheaf $\mathcal{G}_0$ (of higher rank), we can filter it so that the quotients $\mathcal{F}_0^i := \mathcal{G}_0^i/\mathcal{G}_0^{i-1}$ are irreducible with rank r_i. Then we know that $\wedge^{r_i}\mathcal{F}_0^i$ is τ-pure. We define the determinant weights of $\mathcal{G}_0$ to be

$$\frac{w(\wedge^{r_i}\mathcal{F}_0^i)}{r_i}.$$

References

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