Faltings-Lawrence-Venkatesh

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Contents

Chapter 1

Introduction

Let C be a smooth projective curve defined over a number field K of genus g. The nature of the set of rational points $C(K)$ depends heavily on g. As in many other scenarios, we have a trichotomy corresponding to the cases $g = 0, g = 1$, and $g \geqslant 2$. Let us give an overview of what occurs in each case.

1.1 The Hasse principle ($g = 0$)

In the case $g = 0$, the anticanonical bundle has degree 2. Since $2 \geqslant 2g + 1 = 1$, the anticanonical bundle is very ample and gives a closed embedding $C \to \mathbb{P}^2_K$ of degree 2. Thus C is isomorphic to a conic. Now there are two possibilities: either $C(K) = \varnothing$ or there exists some $P \in C(K)$. In the first case there is nothing more to say regarding $C(K)$; in the second we may project from P onto some copy of $\mathbb{P}^1_K$ not going through P. This map gives an isomorphism of C onto $\mathbb{P}^1_K$. Alternatively, the point P defines a line bundle $\mathcal{L}(P)$ of degree 1. By Riemann-Roch, $h^0(C, \mathcal{L}(P)) = 2$ and thus the two sections define a closed embedding of C into $\mathbb{P}^1_K$, which must be an isomorphism.

We conclude that either C has no rational points or has infinitely many. The Hasse principle gives a criterion for determining which of these cases C satisfies. It states that a quadratic form over a number field K has a solution in K if and only if it has a solution over all completions K_v with respect to all places (including the infinite ones). Since C is isomorphic to a conic, we have the following result.

Theorem 1.1.1. *Let $g = 0$. Then if C has a solution over all completions K_v, then C is isomorphic to $\mathbb{P}^1_K$ and has infinitely many rational points. Otherwise, C is isomorphic to a conic in $\mathbb{P}^2_K$ and has no rational points.*

1.2 The Mordell-Weil theorem ($g = 1$)

If $g = 1$, then C is an elliptic curve which we denote as E. The points of E form an abelian group; one way to see this is by viewing its points as a complex torus, another way is through theory of divisors. Furthermore, the sum of two rational points is rational, so $E(K)$ is an abelian group. Using Galois cohomology and some classical algebraic number theory, one proves the weak Mordell-Weil theorem, which states that $E(K)/nE(K)$ is finite for each n. Then by the theory of heights, we arrive at the following result.

Theorem 1.2.1 (Mordell-Weil theorem)**.** *Let E/K be an elliptic curve. Then $E(K)$ is a finitely generated abelian group.*

In fact, this result holds for all abelian varieties (and this is the full statement of the theorem). The Hasse principle does not hold for cubic forms. Though every elliptic curve has a rational point by definition, there may be curves of genus 1 with rational points at every completion K_v but no global rational point. In fact, one may reformulate the Hasse principle in terms of Galois cohomology and show that the obstruction to its truth is described by the Tate-Shafarevich group $Ш(E/K)$. Indeed, we define

$$Ш(E/K) := \ker(H^1(G_K, E) \to H^1(G_{K_v}, E).$$

Here, $H^1(G_K, E)$ classifies torsors over E, which may be interpreted as curves of genus 1 which are isomorphic to E over $\overline{K}$. Having a rational point is equivalent to being 0 in this cohomology class. Interpreting $H^1(G_{K_v}, E)$ similarly, we see that if C represents a nontrivial element in $Ш(E/K)$, then it has rational points in each K_v) but no rational point in K.

We may write $E(K) \cong \mathbb{Z}^r \oplus G$, where G is some finite abelian group. Both the torsion and the rank of $E(K)$ are of enormous interest. They are described by the famous theorem of Mazur and the Birch and Swinnerton-Dyer conjecture.

Theorem 1.2.2 (Mazur, Merel). *Let E/K be an elliptic curve. Then the torsion part of $E(K)$ is $\mathbb{Z}/n\mathbb{Z}$ with $1 \leqslant n \leqslant 10$ or $n = 12$, or it is $\mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $1 \leqslant n \leqslant 4$.*

Conjecture 1.2.3 (Birch and Swinnerton-Dyer). *Let E/K be an elliptic curve. Then the rank of $E(K)$ is given by the order of the pole of the Hasse-Weil L-function $L(E, s)$ at $s = 1$.*

1.3 Faltings's theorem ($g \geqslant 2$)

The primary goal of this seminar is to understand the Lawrence-Venkatesh proof of the following theorem, previously nown as Mordell's conjecture.

Theorem 1.3.1 (Faltings's theorem). *Let C/K be a smooth projective curve of genus $g \geqslant 2$. Then $C(K)$ is finite.*

We will now sketch Faltings's original proof, which may be found in [1]. We list the main steps. First, let A/K be an abelian variety over a number field, let $G_K = \mathrm{Gal}(\overline{K}/K)$, and let $V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ be the Tate modules of A for some prime l.

1. (Finiteness I) There are finitely many abelian varieties B which are isogenous to A.
2. (Tate conjecture I) a) The representation of G_K on $V_l(A)$ is semisimple.

   b) The natural map $\mathrm{Hom}_K(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \mathrm{Hom}_{G_K}(T_l(A), T_l(B))$ is an isomorphism.
3. (Shafarevich conjecture for AV) Let S be a finite set of places of K and fix a positive integer g. Then there are only finitely many isomorphism classes of abelian varieties A/K of dimension g with good reduction outside S.
4. (Shafarevich conjecture) With the notation above, there are only finitely many isomorphism classes of smooth projective curves C of genus g with good reduction outside S.
5. (Mordell's conjecture) If $g \geqslant 2$, then $C(K)$ is finite.

We will now say something about each step, obviously not trying to give full details.

1. Actually, Faltings originally proved something slightly different from (Finiteness I). At any rate, this is one of the most technically difficult parts of the proof, and involves an intensive analysis of heights. Briefly, we canonically embed the Siegel moduli space of abelian varieties $\mathcal{A}_g$ into $\mathbb{P}^n$; then the height on $\mathbb{P}^n$ gives a height function H defined on $\mathcal{A}_g$. Then Faltings defined a different height function, now known as the Faltings height. First, we take a Neron model $A' \to \operatorname{Spec} \mathcal{O}_K$. Then $\omega_{A'/\mathcal{O}_K}$ is a *metrized line bundle* on $\operatorname{Spec} \mathcal{O}_K$, and we may define

$$h(A) = \frac{1}{[K:\mathbb{Q}]} \deg(\omega_{A'/\mathcal{O}_K}).$$

Now the point of this is to compare these two definitions of height and show that they are not too different. There are serious technical difficulties that arise when analyzing them near the boundary of the moduli space. Then Faltings analyzes the behavior of the Faltings height under isogeny, showing it varies in a controlled way. Then since H and h are not too different and we have finiteness theorems for h, he is able to deduce the finiteness of isogeny classes.

1. ⇒ 2. Next, Faltings proved Tate conjecture I using a similar argument to Tate's own proof of this conjecture for abelian varieties over finite fields. Indeed, Tate proved the same statement for finite fields using the fact that there are only finitely many isomorphism classes of abelian varieties of dimension g over $\mathbb{F}_q$. This fact may be replaced by Finiteness I. The injectivity of part b) is not too difficult. Moreover, we may assume $A = B$ by using this statement on $A \times B$. Now both statements are proved together in the following way.

First, one shows that all finite subgroups of $A(\overline{K})$ stable under G_K arise as the kernels of isogenies $A \to B$. Then every G_K-stable $\mathbb{Z}_l$-submodule of $T_l B$ arises as the image of some isogeny $A \to B$. Now consider any G_K-invariant subspace $W \subset V_l(A)$. We claim there is some $u \in \operatorname{End}(A) \otimes \mathbb{Q}_l$ such that $uV_l A = W$. Indeed, we apply the previous correspondence between representations and isogenies to the $\mathbb{Z}_l$-submodules

$$(T_l(A) \cap W) + l^n T_l(A)$$

for each n. Now using Finiteness I along with a compactness argument, we obtain the desired u. This allows us to construct complementary subspaces to prove semisimplicity. Finally, semisimplicity applied to a suitable graph construction yields

$$\operatorname{End}(A) \otimes \mathbb{Q}_l \cong \operatorname{End}_{G_K}(V_l A),$$

which gives b).

2. ⇒ 3. By Finiteness I, we only need to show finiteness up to isogeny. Recall that the Néron-Ogg-Shafarevich criterion says that if v is a finite place of K not dividing some prime l, then A has good reduction at v if and only if the representation of G_K on $V_l A$ is unramified at v. This implies that isogenous abelian varieties over K have good reduction at the same finite places.

Now if A has good reduction over v, let $A(v)/K(v)$ be the corresponding abelian variety and let $P_v(A,t) := P(A(v),t)$ be the characteristic polynomial. We claim that if $P_v(A,t) = P_v(B,t)$ for all v in a certain finite set T, then the corresponding G_K-representations $V_l A, V_l B$ are isomorphic. This can be proven using some classical algebraic number theory involving Hermite-Minkowski finiteness and the Chebotarev density theorem.

Now by the Tate conjecture, we have that if V_lA and V_lB are isomorphic, then A and B are isogenous. The final step is thus to show that there are finitely many possible polynomials $P_v(A,t)$ where $v \in T$. But $P_v(A,t)$ is a monic polynomial of degree $2g$ whose roots are the eigenvalues of the Frobenius. These are bounded by the Weil conjectures, so the result follows.

3. ⇒ 4. If we have an abelian variety over a field, there are only finitely many isomorphism classes of principal polarizations on it. Furthermore, if C has good reduction at a prime v, then so does its Jacobian. Then using 3) we may apply the Torelli theorem, which for $g \geqslant 2$ tells us that two curves are isomorphic if their principally polarized Jacobians are.

4. ⇒ 5. (This was proved by Parshin [**?**]) The key is the construction of the so-called **Kodaira-Parshin family**, an abelian scheme over C with good ramification properties. To be precise, if S is a finite set of primes containing the ones dividing 2, we can find a finite extension L/K and curves C_P for each $P \in C(K)$ satisfying the following properties. The genus of C_P is bounded, C_P has good reduction outside the places dividing S, and there are finite maps $\phi_P : C_P \to C$ over L ramified at exactly P. Thus every rational point P gives a pair (C_P, ϕ_P), and by Shafarevich's theorem the C_P fall into finitely many isomorphism classes.

   Next, we use de Franchis's theorem, which states that if C' and C/k are fields and $g_C \geqslant 2$, then there are only finitely many nonconstant maps $C' \to C$. In particular, there are only finitely many possibilities for ϕ_P corresponding to a fixed isomorphism class C_P. The Mordell conjecture follows.

1.4 The Chabauty-Kim approach

Recall that the Mordell-Weil theorem implies that $J(K)$ is finitely generated, where J is the Jacobian of C. Chabauty proved the Mordell conjecture in the case that the rank of $J(K)$ is less than g. The ideas is the following. Take $P_0 \in C(K)$; this gives an embedding

$$\phi : C \to J \qquad \phi(P) = [P - P_0],$$

so $C(K) = C \cap J(K)$. Now embed K into L, some finite extension of $\mathbb{Q}_p$. The logarithm map gives a local isomorphism between $U \subset J(L)$ and $\mathrm{Lie}(J) \cong \mathcal{O}_L^g$. Let $\overline{\Gamma}$ be the closure of $J(K)$ in $J(L)$. Since $J(L)$ is compact, if the intersection is infinite then we get a convergent sequence of points in the intersection to one of them, which we may assume to be 0. Note that $J(K) \cap U$ is free of rank less than g. Changing coordinates, we get a function x_1 on the curve with infinitely many zeroes accumulating at 0; thus $x_1 = 0$ in a neighborhood of 0 on C. But dx_1 has at most $2g-2$ zeroes, contradiction.

Minhyong Kim generalized this method by using deeper quotients of the fundamental group. Indeed, T_l may be viewed as the first étale cohomology group of C, which is (more or less) the abelianization of the étale fundamental group. Roughly, one finds a middle ground between the étale cohomology and torsor given by the étale fundamental group as described by the section conjecture, which the rational points of C are mapped to. One then analyzes a p-adic period map, which as we will see is also done in the Lawrence-Venkatesh approach. Kim has made significant progress through this approach, though a complete of Faltings's theorem in this way has not yet appeared.

1.5 Outline of the Lawrence-Venkatesh approach

The method we will be studying arose from Brian Lawrence's PhD thesis under Akshay Venkatesh. It gives a full proof of the Mordell conjecture and can also be applied to give results for higher dimensional varieties. As the authors say, it uses the setup of Faltings's proof but is close in spirit to the methods of Chabauty and Kim.

Let Y/K be a curve of genus $g \geqslant 2$. Actually, we will eventually want to run this argument for other Y. The starting point is a smooth projective family $X \to Y$ based on the Kodaira-Parshin family used in the last step of Faltings's proof. This family satisfies certain desired properties we will now describe.

Let $\mathcal{O}$ be the S-integers of K and say $X \to Y$ extends to $\pi : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{O}$. For every p unramified in K and not dividing any prime in S, every $y \in \mathcal{Y}(\mathcal{O})$ gives a Galois representation

$$\rho_y : G_K \circlearrowright H^*_{et}(\overline{X_y}, \mathbb{Q}_p).$$

Now recall that in the step $2 \to 3$ of Faltings's proof, it was (essentially) proven that there are only finitely many possibilities for semisimple representations ρ_y^{ss} that are unramified outside a finite set of primes that are moreover these kinds of Galois representations on étale cohomology. As a reminder, classical algebraic number theory results such as Hermite-Minkowski finiteness and the Chebotarev density theorem are used to show that the representation is determined by its characteristic polynomial for a finite subset of Frobenius elements. By the Weil conjectures, there are only finitely many such polynomials that come from these representations.

Now Faltings worked with abelian varieties where he showed that every ρ_y is semisimple and determines X_y up to isogeny – this is Tate's conjecture. In the approach we are now considering, we take the semisimplification ρ^{ss} and restrict it to G_{K_v} for a suitable place v. Then it is proven that the fibers of this mapping from $Y(K)$ to these p-adic representations are not Zariski dense.

To prove this last statement, we use p-adic Hodge theory. Using this theory, each point $y \in Y(K_v)$ gives a filtered ϕ-module over K_v:

$$y \mapsto (H_{\text{dR}}(X_y/K_v), \text{Fil}^\bullet, \phi).$$

The Gauss-Manin connection allows us identify $H_{\text{dR}}(X_z/K_v) \cong H_{\text{dR}}(X_y/K_v)$ as we vary z in a residue disk in $Y(K_v)$ around y. What changes is the Hodge filtration. This variation is described by the p-adic period map, which sends points in the residue disk to K_v-points of a certain flag variety of subspaces of $H_{\text{dR}}(X_y/K_v)$. The p-adic period map is injective, but there may be different filtrations which give rise to the isomorphic filtered ϕ-modules. Thus, we must also show that the image of the period map has finite intersection with an orbit on the period domain of the centralizer $Z(\phi)$. For example, when Y is a curve we will show that the $Z(\phi)$-orbit of the filtrations is a proper subvariety of the ambient flag variety, and that the image of the p-adic period map is Zariski dense. Then the fiber is given by the intersection points which are the zeroes of a nonvanishing K_v-analytic function in a residue disc, which is finite.

To check Zariski density, one passes to the corresponding complex period map which satisfies the same differential equation coming from the Gauss-Manin connection. Checking the result here is done through analyzing monodromy representations and mapping class groups. Finally, for higher dimensions Bakker and Tsimerman used o-minimality to prove the Ax-Schanuel theorem for period mappings. This gives us better control about the intersection of the $Z(\phi)$-orbit and Y_v.

Part I

Background

Chapter 2

The Gauss-Manin connection and the period map

2.1 Definitions

Bibliography

[1] Donald E. Knuth. Literate programming. *The Computer Journal*, 27(2):97–111, 1984.