The J-homomorphism

Caleb Ji

Summer 2021

1 Definitions

Definition 1.1 (J-homomorphism). *Define the J-homomorphism*

$$\pi_k(O(n)) \to \pi_{n+k}(S^n)$$

in the following way. Consider the natural action of $O(n)$ on S^{n-1}. After suspension this gives a map $S^n \to S^n$ fixing a basepoint. This induces a map

$$\pi_k(O(n)) \to \pi_k(\Omega^n S^n) = \pi_{n+k}(S^n).$$

Similarly, or by embedding $U(n) \hookrightarrow O(2n)$, we obtain a complex J-homomorphism

$$\pi_k(U(n)) \to \pi_{2n+k}(S^{2n}).$$

By Bott periodicity and by the existence of stable homotopy, for n large enough the J-homomorphism is independent of n. We will be interested in these stable versions

$$J : \pi_k(O) \to \pi_k(\mathbf{S}), \qquad J_{\mathbb{C}} : \pi_k(U) \to \pi_k(\mathbf{S}).$$

Just like in the case of Bott periodicity, Atiyah generalized this phenomenon to all spaces, defining $J(X)$, a certain quotient of $K(X)$. Define two vector bundles to be fiber homotopy equivalent if their sphere bundles are homotopy equivalent.

Definition 1.2 ($J(X)$). *The group $J(X)$ consists of the vector bundles modulo stable fiber homotopy equivalence.*

Thus, we can write $J(X) = K(X)/T(X)$ and $\tilde{J}(X) = \tilde{K}(X)/T(X)$, where $T(X) \subset \tilde{K}(X)$ consists of the classes $E - E'$ with E fiber homotopy equivalent to E'. To connect this back to the J-homomorphism, one shows that

$$\tilde{J}(S^i) = \operatorname{im} J_{\mathbb{C}} \subset \pi_{k-1}(\mathbf{S}).$$

Our primary goal is to understand the image of the J-homomorphism, and thus gain information about $\pi_k(\mathbf{S})$. Recall Bott periodicity:

$k \pmod 8$	0	1	2	3	4	5	6	7
$\pi_k(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

and the fact that $\pi_k(\mathbf{S})$ is finite. It can be shown that for $k \equiv 0, 1 \pmod 8$, then J is injective. We are left with the case $k \equiv 3 \pmod 4$, which is a very interesting case. To state the result, we recall the definition of the Bernoulli numbers.

Definition 1.3 (Bernoulli numbers). *The Bernoulli numbers B_i are the coefficients of the exponential generating function*

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{i\geq 1} B_i \frac{x^{2i}}{(2i)!}.$$

They begin $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \cdots$.

The Bernoulli numbers also show up in algebraic K-theory, special values of L-functions, Eisenstein series ... For $k = 4s-1$, the statement is that the image of J is a cyclic group of order the denominator of $B_s/4s$ and is a direct summand of $\pi_k(\mathbf{S})$.

The proof makes use of the Adams operations. These are the analogue of cohomology operations in K-theory. Recall that a cohomology operation is a natural transformation $H^m(-;G) \to H^n(-;H)$, which amounts to an element of $[h_{K(G,m)}, h_{K(H,n)}] = H^n(K(G,m);H)$. The explicit computation of these is somewhat involved; for example, over $\mathbb{Z}/2$ the stable ones are given by the Steenrod algebra. By the same argument, the natural transformations of K-theory are given by $K(BU)$. They can all be classified by the Adams operations.

Definition 1.4 (Adams operations). *Let s_k be the kth Newton polynomial and let $\Lambda^i E$ be the ith exterior power of the vector bundle E. Then $\psi^k : K(-) \to K(-)$ is defined by*

$$\psi^k(E) = s_k(\lambda^1(E), \cdots, \lambda^k(E)).$$

This definition is actually very natural. Basically, we begin with $\psi^k(L) = L^k$ for a line bundle L, and we want to extend this linearly to all vector bundles. Since $s_k(\sigma_1, \ldots, \sigma_k) = p_k$ (elementary to power sum), we have $\psi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k \oplus \cdots \oplus L_n^k$. Then it is not hard to show by the splitting principle, the ψ^k give natural transformations that also satisfy $\psi^k \circ \psi^l = \psi^{kl}$ and $\psi^p(\alpha) \equiv \alpha^p \pmod p$ for prime p.

Finally, we define the e-invariant, which is a homomorphism $e : \pi_{2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$. Recall that the Chern character is defined by $\mathrm{ch}(L) = e^{c_1(L)}$ and extending linearly by the splitting principle. It induces an isomorphism for finite CW complexes X:

$$\mathrm{ch} : K(X) \otimes \mathbb{Q} \cong H^(X;\mathbb{Q}).$$

Definition 1.5 (e-invariant). *For an element $f \in \pi_{2m-1}(S^{2n})$, construct the corresponding cofiber sequence $S^{2n} \to C_f \to \Sigma S^{2n-1}$ and take the corresponding exact sequence*

$$0 \to \tilde{K}(S^{2m}) \to \tilde{K}(C_f) \to \tilde{K}(S^{2n}) \to 0.$$

Applying the Chern character, we simply get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \\ 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}\oplus\mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \end{array}$$

Pick generators $\alpha, \beta \in \mathbb{Z} \oplus \mathbb{Z}$ that come and go to 1, and do the same with $a, b \in \mathbb{Q} \oplus \mathbb{Q}$ with $\mathrm{ch}(\alpha) = a$. Then $\mathrm{ch}\,\beta = b + ea$, and this $e \in \mathbb{Q}/\mathbb{Z}$ is a well-defined invariant of f.

2 The image of J in $\pi_k(\mathbf{S})$

As alluded to above, we can analyze the image of J to gain information about $\pi_k(\mathbf{S})$.

Theorem 2.1 (Adams, Quillen). *For $k \equiv 0, 1 \pmod 8$, the image of $J : \mathbb{Z}/2 \to \pi_k(\mathbf{S})$ is injective. For $k = 4s-1$, the image of $J : \mathbb{Z} \to \pi_k(\mathbf{S})$ is a cyclic group of order the denominator of $B_s/4s$ and is a direct summand of $\pi_k(\mathbf{S})$.*

We give some indications of the proof. We focus on the image of J in the case $k = 4s-1$.

A lower bound

For a lower bound of the size of the image, we use the e-invariant. Indeed, given a map $f : S^{4s-1} \to U(n)$ representing a generator of $\pi_{4s-1}(U)$, then we claim that

$$e(J_{\mathbb{C}} f) = \pm \frac{B_s}{2s}.$$

This shows that the order of the image of J in $\pi_{4s-1}(\mathbf{S})$ is divisible by the denominator of $\frac{B_s}{2s}$. This is half the number in the theorem – it takes a bit more work to get the extra factor of 2.

To prove this claim, we use the Thom space $T(E) = D(E)/S(E)$ of a vector bundle (quotient of disk bundle by sphere bundle). One shows that the cone $C_{J_{\mathbb{C}} f}$ is the Thom space of the bundle $E_f \to S^{4s}$ determined by $f : S^{4k-1} \to U(n)$ viewed as a clutching function. Then one uses the Thom isomorphism to choose an appropriate generator β, used in the context of the e-invariant. We obtain an equation in terms of Chern classes, which when expanded in a Taylor series gives that the e-invariant is indeed the desired Bernoulli denominator.

An upper bound

The (exact) upper bound can proven through the Adams conjecture, which was proven by Quillen. They concern the Adams operations ψ^k defined earlier.

Theorem 2.2 (Adams conjecture). *For any finite CW complex X and any $\alpha \in K(X)$ and $k \in \mathbb{Z}$, there exists N such that*

$$k^N(\psi^k(\alpha) - \alpha) \in T(X).$$

In other words, if we localize by k in $J(X)$, then $\psi^k(\alpha) = \alpha$. We will assume this. One can then show that $\psi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ is multiplication by k^n. Thus, for sufficiently large N we have that

$$k^N(k^n - 1)\tilde{K}(S^{2n}) \subset T(S^{2n}).$$

This implies that the order of the image of $J_{\mathbb{C}}$ must divide d_n, where d_n is the largest integer that divides $k^N(k^n - 1)$ for all fixed k and any N. By elementary number theory, d_n is indeed the denominator of $\frac{B_s}{4s}$. This gives the desired upper bound.

3 Comments

The material here is taken from [2], [4], and [3]. There one can also find references to the original and more comprehensive accounts. Clausen has developed a p-adic version (after understanding the real version in terms of algebraic K-theory and spectra), and used it to provide a totally new proof of quadratic reciprocity and even Artin reciprocity.

References

[1] Dustin Clausen. *p-adic J-homomorphisms and a product formula*, 2012.
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[3] Akhil Mathew. *Notes on the J-homomorphism*. Expository notes, retrieved online.
http://math.uchicago.edu/~amathew/j.pdf

[4] Allen Hatcher. *Vector Bundles and K-theory*, V (2005).