Cohomological Descent

Caleb Ji

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Recall...

Simplicial objects are functors

$$\Delta^{\text{op}} \to C$$

where C is some category. A simplicial object X comes with face and degeneracy morphisms

$$d_i : X_n \to X_{n-1}, \qquad s_i : X_n \to X_{n+1}$$

for $0 \le i \le n$. These come from $d^i : [n-1] \to [n]$ skipping i and $s^i : [n+1] \to [n]$ repeating i.

We denote by $\mathbf{S}(C)$ the category of simplicial objects. We will also work with $\mathbf{S}_n(C)$ and $\mathbf{S}_+(C)$, the categories of n-truncated simplicial objects and augmented simplicial objects respectively. These are functors from $\Delta^{\le n}$ and $\Delta^{\ge -1}$ respectively. In what follows, we will assume that C has finite products and finite fiber products.

Essentially everything that follows comes from the notes *Cohomological Descent* by Brian Conrad [1].

1 Coskeleta

Given a simplicial space $X_\bullet \in \mathbf{S}(C)$, we can consider its n-skeleton $\operatorname{sk} X_\bullet \in \mathbf{S}_n(C)$. Consider

Definition 1.1. *Take $Y_\bullet \in \mathbf{S}_n(C)$. The n-coskeleton functor is the right adjoint to sk_n. That is,*

$$\operatorname{Hom}_{\mathbf{S}_n(C)}(\operatorname{sk}_n(X_\bullet), Y_\bullet) \cong \operatorname{Hom}_{\mathbf{S}(C)}(X_\bullet, \operatorname{cosk}_n(Y_\bullet)).$$

Of course, we haven't shown that such an adjoint exists. Let us first see what it must be for $n = 0$. Fix $Y_0 \in \mathbf{S}_0(C) = C$. We need to construct $\operatorname{cosk}_0(Y_0)$ such that for all $X_\bullet$, we have $\operatorname{Hom}(X_0, Y_0) = \operatorname{Hom}(X_\bullet, \operatorname{cosk}_0(Y_0))$. In fact, we will soon see that cosk_n does not change the n-skeleton, so cosk_n satisfies the universal property that we can extend any map from an n-skeleton of some simplicial object to the whole object. Here, given some $\operatorname{sk}_0(X'_\bullet) \to Y_0$, we see that we obtain maps $X_1 \to Y_0 \times Y_0$, $X_2 \to Y_0 \times Y_0 \times Y_0$, etc. This shows that we should take $\operatorname{cosk}_0(Y_0)_n = Y^n$ with the natural simplicial structure. Note that this is essentially a Cech covering.

We now construct cosk_m.

Proposition 1.2. *Take $Y_\bullet \in \mathbf{S}_n(C)$. Set*

$$Y_n^m = \varprojlim_{\operatorname{sk}_m(\Delta[n])} Y_\phi = \varprojlim_{[k]\to[n], k \le m} Y_k.$$

Given a morphism $\alpha : [n'] \to [n]$, we leave it to the reader to define the appropriate map $Y_n^m \to Y_{n'}^m$. This gives the construction of $\operatorname{cosk}_m(Y_\bullet)$.

Idea. Given a map $\mathrm{sk}_n(X_\bullet) \to Y_\bullet$, we have all these maps $X_n \rightrightarrows X_k \to Y_k$ for $k \leq m$. These combine to give the inverse limit object described. One can check that all desired functoriality properties hold. □

From this construction, we conclude the following statement.

Proposition 1.3. *For $X_\bullet \in \mathbf{S}_n(C)$, the map $X_\bullet \to \mathrm{sk}_n(\mathrm{cosk}_n(X_\bullet))$ is an isomorphism.*

Indeed, looking at Y_n^m for $n \geq m$, we see that the single copy of Y_n corresponding to $[n] \to [n]$ determines everything.

Remark. It is not hard to see that the *left* adjoint to sk_n is simply the 'inclusion' of n-truncated simplicial spaces into simplicial spaces. Moreover, i_n and cosk_n are the left and right Kan extensions of sk_n.

The adjunction $\mathrm{id} \to \mathrm{cosk}_m \mathrm{sk}_m$ is not generally an isomorphism, but it is indeed an isomorphism on all n-coskeleta for $n \leq m$. That is, for $-1 \leq n \leq m$, we have

$$\rho_{m,n} : \mathrm{cosk}_n \to \mathrm{cosk}_m \mathrm{sk}_m \mathrm{cosk}_n$$

is an isomorphism of functors.

Moreover, if we take X to be a Kan complex, we have that $\mathrm{cosk}_n(X)$ is also a Kan complex. The sequence

$$\cdots \to \mathrm{cosk}_{n+1} X \to \mathrm{cosk}_n X \to \mathrm{cosk}_{n-1} X \to \cdots \to *$$

is then a Postnikov tower for X.

2 Hypercovers

Definition 2.1. *Let* **P** *be a class of morphisms in C containing isomorphisms and stable under base change and composition. A simplicial object $X_\bullet$ in C is a* ***P-hypercovering*** *if for all $n \geq 0$ (or -1), the natural adjunction*

$$X_\bullet \to \mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet))$$

induces a map $X_{n+1} \to \mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet))_{n+1}$ which is in **P**.

In particular, the augmented simplicial object $\mathrm{cosk}_0(S'/S) \to S$ is a **P**-hypercovering if and only if $S' \to S$ is in **P**. If **P** is some surjectivity condition, then taking S' to be the disjoint union of some open cover will yield the Cech construction. A key point of interest is that these can be used to calculate cohomology. Indeed, given a hypercovering of some object K, we may define

$$\check{H}^i(K, \mathcal{F}) = H^i(s(\mathcal{F}(K)),$$

where $s(-)$ denotes taking the Moore complex. If $\mathcal{F}$ is abelian, we have $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$. We can also take direct limits over hypercoverings, and unlike in the ordinary Cech theory, we do indeed get derived functor cohomology! We will not prove this, but we will see a derived version of these ideas later.

There are indeed a lot more examples of hypercovers than just the Cech construction. For example, if we have an m-truncated **P**-hypercovering $Y_\bullet$, then $\mathrm{cosk}_m(Y_\bullet)$ is a **P**-hypercovering. We are particularly interested in the case of hypercoverings of singular schemes. To get there, first we discuss splittings.

Definition 2.2. *Assume C admits finite coproducts. A simplicial object $X_\bullet$ in C is* ***split*** *if there exist subobjects NX_j with*

$$\bigoplus_{\phi:[n]\twoheadrightarrow[m]} NX_m \to X_n$$

an isomorphism.

Recall that the left hand side is essentially the Γ-functor used in the proof of the Dold-Kan theorem. We will now assume C admits finite inverse limits, finite coproducts, and unique complements. This last condition ensures that split simplicial objects have unique (up to unique isomorphism) splittings. While this doesn't hold for abelian categories in general, Dold-Kan does give a canonical splitting for them.

One key property of splittings is the following fact. Given a split n-truncated object $\mathrm{sk}_n(X)$ and NX_{n+1} along with a map $NX_{n+1} \to (\mathrm{cosk}_n \, \mathrm{sk}_n \, X)_{n+1}$ (which is meant to factor through the inclusion into X_{n+1}, we can recover $\mathrm{sk}_{n+1}(X)$. Indeed, we know that X_{n+1} needs to be this sum $\bigoplus_{[n+1]\twoheadrightarrow[m]} NX_m$, and the face and degeneracy maps are defined naturally.

For our purposes, we will restrict to one of the following cases.

- $C =$ spaces over a base, $\mathbf{P} =$ proper surjections
- $C =$ spaces étale over a space, $\mathbf{P} =$ surjective étale maps
- $C =$ a topos, $\mathbf{P} =$ epic morphisms

The flexibility of hypercovers and being split is illustrated in the next statement.

Theorem 2.3. *If $\mathrm{sk}_n X$ is split, there exists a map $f : X' \to X$ with $\mathrm{sk}_n(F)$ an isomorphism and X' split. Furthermore, if X is an augmented* **P***-hypercover, we can take X' to be an augmented* **P***-hypercover.*

The point of this is the following statement.

Corollary 2.4. *Consider the proper/étale surjective case. For $m \geq 0$, an (augmented) m-truncated* **P***-hypercovering Z, the face and degeneracy maps for Z are proper/étale.*

In particular, if $X_\bullet \to S$ is a proper/étale hypercovering, then all structure maps $X_n \to S$ are proper/étale. We can now prove a theorem of interest.

Theorem 2.5. *Let S be a separated scheme of finite type over a field k. Then there exists a dense open immersion $S \hookrightarrow \overline{S}$ into a proper k-scheme and an augmented proper hypercovering $\overline{X}_\bullet$ of $\overline{S}$ such that each $\overline{X}_n$ is a projective k-scheme which is regular and the part of $\overline{X}_n$ lying over $\overline{S} - S$ is a strict normal crossings divisor in $\overline{X}_n$ for all $n \geq 0$.*

Before we give the proof, let us recall the statement of resolution of singularities. Recall that Hironaka proved that if X is a complex algebraic variety, then there is a proper birational map $p : Y \to X$ from some regular Y. Johan proved the following theorem.

Proposition 2.6 (Johan). *Let X be a variety over a field k and let $Z \subset X$ be a proper closed subset. There exist an alteration*

$$\varphi_1 : X_1 \to X$$

and an open immersion $j_1 : X_1 \to \overline{X}_1$ such that

- *$\overline{X}_1$ is a projective variety and is a regular scheme.*

- *The closed subset* $j_1(\varphi_1^{-1}(Z)) \cup \overline{X}_1 \backslash j_1(X_1)$ *is a strict normal crossings divisor in* $\overline{X}_1$.

  If k *is perfect then the alteration* φ_1 *may be chosen to be generically étale.*

Recall that an alteration of X is a dominant proper morphism $X' \to X$ of varieties over k, with $\dim X = \dim X'$. Note that this differs from a resolution as finite extensions of $k(X)$ are allowed. But we can choose X' to be a complement of a divisor with strict normal crossings in some regular projective variety $\overline{X}'$. Johan noted himself at the beginning of his alterations paper (using smooth instead of regular, so he assumes k is perfect) that this proves Theorem 2.5.

Remark. Apparently another definition of alterations is a surjective, generically finite, and proper morphism. Then every variety has an alteration from a regular variety.

Proof of Theorem 2.5. By Nagata's compactification theorem, we have a dense open immersion $S \hookrightarrow \overline{S}$ into a proper k-scheme. We now apply de Jong's theorem to obtain a regular $\overline{X}_0$ with a proper surjection to $\overline{S}$ where the preimage of S has complement a strict normal crossings divisor. This gives us the 0-skeleton $\overline{X}_0$.
Given $X_{\leq m}$, we have that $\operatorname{cosk}_m X_m$ is a proper hypercovering of S. Moreover, each term is S-proper. Now apply de Jong's theorem again to $(\operatorname{cosk}_m X_{m+1})_{m+1}$ toget $\overline{X}'$ proper and generically finite over the $m+1$-term with the part over $\overline{S} - S$ a strict normal crossings divisor. Then using this for NX_{n+1} and taking the construction above of taking $X_{n+1} = \bigoplus_{[n+1] \twoheadrightarrow [m]} NX_m$, we get an $(m+1)$-truncated solution. We are done by induction. □

3 Cohomological descent: definitions

To define cohomological descent, we will first need to define sheaves on simplicial objects. One can view them as either a simplicial object in the category of sheaves, where C is some site:

$$\mathcal{F} : \Delta^{\mathrm{op}} \to \mathrm{Sh}(C)$$

or a simplicial-set valued presheaf which satisfies the sheaf condition in each degree. In particular, we are interested in sheaves on the following site.

Definition 3.1. *Let* C *be a site with a topology generated by* E*-morphisms. Let* $X_\bullet$ *be a simply object of* C*. Then define* $\widetilde{X}_\bullet$ *to be the category of sheaves of sets on the following site.*

- *Objects:* E*-morphisms* $U \to X_n$
- *Morphisms: commutative squares*

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n'} \end{array}$$

  where f *is any morphism in* C.

- *A covering of* $U_i \to X_n$ *is given by a covering of* U_i *in* C.

A map $U_\bullet : X_\bullet \to Y_\bullet$ of simplicial spaces gives pushforward and pullback functors

$$u_{\bullet *} : \widetilde{X}_\bullet \to \widetilde{Y}_\bullet, \quad u_\bullet^* : \widetilde{Y}_\bullet \to \widetilde{X}_\bullet.$$

We are interested in the case $a : X_\bullet \to S$, where we can apply the constant augmentation S to be $S_\bullet$. We obtain functors

$$a_* : \widetilde{X}_\bullet \to \widetilde{S}, \quad a^* : \widetilde{S} \to \widetilde{X}_\bullet.$$

Here, $(a^*\mathcal{F})^n = a_n^*\mathcal{F}$ and $a_*\mathcal{F}^\bullet$ is the equalizer of $\sigma_0^1, \sigma_1^1 : a_{0*}\mathcal{F}^0 \to a_{1*}\mathcal{F}^1$. We obtain derived functors

$$a^* : \mathbf{D}_+(S) \to \mathbf{D}_+(X_\bullet), \quad \mathbf{R}a_* : \mathbf{D}_+(X_\bullet) \to \mathbf{D}_+(S).$$

We now give the key definition.

Definition 3.2. *The adjoint pair $(a_*, a^*) : \widetilde{X_\bullet} \to \widetilde{S}$ (abbreviated $a : X_\bullet \to S$ is a* ***morphism of cohomological descent*** *if the natural transformation*

$$\mathrm{id} \to \mathbf{R}a_* \circ a^*$$

on $\mathbf{D}_+(S)$ is an isomorphism.

Remark. For simplicity, when we talk about abelian sheaves we can restrict to abelian groups, or $\mathbb{Z}/n$-modules over the étale site.

Recall that $(a^*, \mathbf{R}a_*)$ are adjoint on bounded below derived categories. Thus we may restate the definition of morphism of cohomological descent as follows.

Lemma 3.3. *A map $a : X_\bullet \to S$ is a morphism of cohomological descent if and only if $a^* : \mathbf{D}_+(S) \to \mathbf{D}_+(X_\bullet)$ is fully faithful.*

Definition 3.4. *Let $X_\bullet$ be a simplicial space and let $a : X_\bullet \to S$ be an augmentation. We say that a is* ***universally of cohomological descent*** *if every base change map*

$$a_{/S'} : X_\bullet \times_S S' \to S'$$

is of cohomological descent. Furthermore, a map of spaces $a_0 : X_0 \to S$ is a ***map of cohomological descent*** *if*

$$\mathrm{cosk}_0(a_0) : \mathrm{cosk}_0(X_0/S) \to S$$

is a morphism of cohomological descent, and similarly for the universal case.

The fact that the cohomological descent is preserved under composition is extremely non-trivial. In fact, we will show that morphisms universally of cohomological descent can be used to define a site.

Theorem 3.5. *The class of morphisms universally of cohomological descent satisfies the following properties.*

1. *Let π be universally of cohomological descent in the following Cartesian diagram. Then f is universally of cohomological descent if and only if f' is.*

$$\begin{array}{ccc} X' & \xrightarrow{\pi'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\pi]{} & S \end{array}$$

2. *If f and $g \circ f$ are maps universally of cohomological descent, then g is universally of cohomological descent.*

3. *The composition of two maps universally of cohomological descent is universally of cohomological descent.*

4. *The fiber product over some base S of two maps universally of cohomological descent is universally of cohomological descent.*

The first one is the difficult one. Using it, one can show 2, provided one knows that a map with a local section is universally of cohomological descent. 3 and 4 follow formally.

4 Cohomological descent: properties and applications

Let us begin by explaining the relationship between cohomological descent and ordinary descent. Recall that in descent, one takes a cover $p : X' \to X$, e.g. some fppf/fpqc covering, and takes the projections $p_0, p_1 : X'' = X' \times_X X' \to X'$. Given a sheaf $\mathcal{F}$ on X, we can construct its associated descent data, which consists of a sheaf $\mathcal{F}' = p^*\mathcal{F}$ on X', along with an isomorphism $\alpha : p_1^*\mathcal{F}' \cong p_2^*\mathcal{F}'$ that satisfies a cocycle relation $p_{02}^*(\alpha) = p_{12}^*(\alpha) \circ p_{01}^*(\alpha)$ on the triple fiber power.

For instance, if we take the covering to be $\coprod_i U_i \xrightarrow{p_i} X$, then descent datum consists of sheaves $\mathcal{F}_i$ on U_i with an isomorphism $\phi_{ij} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ satisfying the cocycle condition. If such descent data does indeed come from a sheaf on X, then we say that the descent datum is **effective**. Thus, another way to say that all descent data descend is to say that the functor from sheaves to descent data is fully faithful. In such a case, the fibered category of descent data over the underlying site is a **stack**. For example, QCoh is an fpqc stack. Representable functors are fpqc sheaves, which means that morphisms descend.

Now let us revisit cohomological descent in this context. Recall that the condition for $a : X_\bullet \to S$ to have cohomological descent is for $\mathrm{id} \to \mathbf{R}a_* \circ a^*$ to be an isomorphism on $\mathbf{D}_+(S)$, or for $a^* : \mathbf{D}_+(S) \to \mathbf{D}_+(X_\bullet)$ to be fully faithful. Unwinding this definition, on the level of sheaves we need

$$\mathcal{F} \cong a_*a^*\mathcal{F} = \ker(a_{0*}a_0^*\mathcal{F} \to a_{1*}a_1^*\mathcal{F}), \quad R^i a_*(a^*\mathcal{F}) = 0$$

for $i > 0$. The second condition distinguishes this from ordinary descent. Alternatively, from the second formulation we can see that we want a derived version of full faithfulness working with the entire simplicial object, not just the 2-truncation.

Let us look at the simplest example.

Example 4.1. *The augmentation $S_\bullet \to S$ from the constant simplicial space on S is of cohomological descent.*

Here, a_* pulls back an abelian sheaf on S to one on the whole simplicial object. As this is a simplicial object where the category is the category of abelian sheaves over S, this corresponds to chain complexes of abelian sheaves. By Dold-Kan, we can for the associated chain complex of abelian sheaves over S. Taking the normalized complex, we see that we just get

$$a^*(\mathcal{F}) = \mathcal{F}[0].$$

On the other hand, a_* is the equalizer of $\sigma_0^1, \sigma_1^1 : a_{0*}\mathcal{F}^0 \to a_{1*}\mathcal{F}^1$, and under Dold-Kan this corresponds to taking H^0. Now we have that the adjunction $\mathrm{id} \to a_*a^*$ is clearly an isomorphism, and the derived functors are H^j, which on $\mathcal{F}[0]$ are 0 for $j > 0$.

We now describe the cohomology of abelian sheaves on simplicial spaces $X_\bullet$.

Definition 4.2. *Let $\mathcal{F}$ be an abelian sheaf on $X_\bullet$. The global sections of $\mathcal{F}$ are defined by*

$$\Gamma(X_\bullet, \mathcal{F}) := \ker(\Gamma(X_0, \mathcal{F}^0) \to \Gamma(X_1, \mathcal{F}^1)).$$

Then $\Gamma(X_\bullet, -)$ is left exact, and $\mathbf{R}\Gamma(X_\bullet, -)$ is the resulting total derived functor with hypercohomology groups $\mathbb{H}^i(X_\bullet, -)$.

If we have an augmentation $a : X_\bullet \to S$, then $\Gamma(X_\bullet, \mathcal{F}^\bullet) = \Gamma(S, a_*\mathcal{F}^\bullet)$. We have the following spectral sequences.

Theorem 4.3. *Let $X_\bullet$ be a simplicial space (or a truncated one). For any complex K' in $\mathbf{D}_+(X_\bullet)$, there is a natural spectral sequence*

$$E_1^{p,q} = \mathbb{H}^q(X_p, K'|_{X_p}) \Rightarrow \mathbb{H}^{p+q}(X_\bullet, K')$$

with $d_1^{\bullet,q}$ induced by the "associated differential complex" structure along $X_\bullet$.

When we have an augmentation $a : X_\bullet \to S$ of cohomological descent and we let $K' = a^*K$, then $K'|_{X_p} = a_p^*K$ and the spectral sequence becomes

$$E_1^{p,q} = \mathbb{H}^q(X_p, a_p^*K) \Rightarrow \mathbb{H}^{p+q}(S, K).$$

This is functorial.

Given $a : X_\bullet \to S$ inducing a_p, we can work with a_* and a_{p*} instead of $\Gamma(X_\bullet, -)$ and $\Gamma(X_p, -)$. This leads to the following relative spectral sequence that does not require cohomological descent.

Theorem 4.4. *With the notation above, for any K in $\mathbf{D}_+(X_\bullet)$ there is a canonical spectral sequence*

$$E_1^{p,q} = R^q a_{p*}(K|_{X_p}) \Rightarrow R^{p+q} a_*(K)$$

functorial in $a : X_\bullet \to S$.

Next, we give a useful criterion for a map to be universally of cohomological descent.

Theorem 4.5. *Let $f : X \to S$ be a map of spaces which has a section locally on S. Then f is a map universally of cohomological descent.*

This can be proven with the second spectral sequence above. Applying it to topological spaces with $S = \emptyset$ with the first spectral sequence above, we can recover the classical Cech spectral sequence. The main theorem, which we will state soon, allows us to give a vast generalization of this.

5 The main theorem

We begin with the 0-skeleton version of the desired statement.

Theorem 5.1. *Let $f : X \to S$ be a proper surjective map of topological spaces. Then f is a map universally of cohomological descent. The same holds for schemes with the étale topology, working with derived categories of sheaves of $\mathbb{Z}/n$-modules.*

For this, we need to show that if $\mathcal{F}$ is an abelian sheaf on S and if $a : X_\bullet \to S$ is given by $\operatorname{cosk}_0(X_0/S)$, then $\mathcal{F} \to a_*a^*\mathcal{F}$ is an isomorphism and $R^i a_*(a^*\mathcal{F}) = 0$ for $i > 0$. This can be done through the second spectral sequence above and the proper base change theorem. This reduces to checking on geometric fibers, from which the theorem about maps with local sections suffices to finish.

Now we state the main theorem.

Theorem 5.2. *Let $X_\bullet \to S$ be a proper hypercovering of topological spaces or schemes with the étale topology (in which case we replace abelian sheaves with sheaves of $\mathbb{Z}/n$-modules). Then it is universally of cohomological descent.*

By definition, a proper hypercovering $X_\bullet \to S$ induces a proper surjection

$$X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S))_{n+1}.$$

By the previous theorem, these are maps universally of cohomological descent. Thus to prove this theorem, it suffices to prove the next theorem, which does not involve hypercoverings.

Theorem 5.3. *Let $a : X_\bullet \to S$ be an augmented simplicial space with each map of spaces*

$$X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S))_{n+1}$$

universally of cohomological descent. Then a is universally of cohomological descent.

The proof is difficult. Note that using the first spectral sequence above, the following generalization of the Cech spectral sequence results.

Corollary 5.4. *Let $a : X_\bullet \to S$ be an étale hypercovering of a space. Then for any K in $\mathbf{D}_+(S)$, there is a spectral sequence*

$$E_1^{p,q} = \mathbb{H}^q(X_p, a_p^* K) \Rightarrow \mathbb{H}^{p+q}(S, K)$$

where $d_1^{\bullet,q}$ is induced by the simplicial structure on $X_\bullet$.

References

[1] Brian Conrad, *Cohomological Descent*
https://math.stanford.edu/~conrad/papers/hypercover.pdf

[2] Paul G. Goerss, John F. Jardine, *Simplicial Homotopy Theory*.