Bott periodicity

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1 Statements of Bott periodicity

There are several versions of Bott periodicity. Originally, Bott computed the homotopy groups

$$\pi_k U = \pi_k \operatorname{colim}_n U(n), \qquad \pi_k O = \pi_k \operatorname{colim}_n O(n).$$

Indeed, note that these are well-defined, as the fibrations

$$U(n-1) \hookrightarrow U(n) \to S^{2n-1}, \qquad O(n-1) \hookrightarrow O(n) \to S^{n-1}$$

ensure that the maps induced by inclusions $\pi_k(U(n-1)) \to \pi_k(U(n))$ and $\pi_k(O(n-1)) \to \pi_k(O(n))$ are isomorphisms for large enough n. Bott computed them all, showing they are periodic.

Theorem 1.1 (Bott periodicity, version 1)**.** *The homotopy groups of U and O are given by the following table.*

$k \pmod 8$	0	1	2	3	4	5	6	7
$\pi_k(U)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_k(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

By looking at the classifying spaces BU and BO, we can restate Bott periodicity in terms of K-theory. Indeed, note that for a topological group G, we have that $\Omega(BG) \cong G$ is a weak homotopy equivalence (consider EG and path fibrations), so $\pi_k(BG) = \pi_{k+1}(G)$. Next, note that taking complex and real vector bundles of rank n over a compact space X is represented by maps to $BU(n)$ and $BO(n)$, respectively. For example, one could take infinite Grassmannians as a model for them and analyze what maps into them mean. Now if we take the colimit of maps to $BU(n)$, we see that two vector bundles are identified if they are stably isomorphic. Thus we see that

$$KU(X) \cong [X_+, BU \times \mathbb{Z}], \qquad \tilde{KU}(X) \cong [X, BU \times \mathbb{Z}]$$

and similarly for the real case. Thus, the statement that $\pi_k U$ is 2-periodic is equivalent to saying that

$$[S^k, BU] \cong [S^{k+2}, BU] \Leftrightarrow \tilde{KU}(S^k) \cong \tilde{KU}(\Sigma^2 S^k).$$

Recall that topological K-theory is defined by $K^n(X) = K(\Sigma^n X)$. Therefore the following theorem represents a generalization of the first version of Bott periodicity (though it doesn't say what the groups are).

Theorem 1.2 (Bott periodicity, version 2A)**.** *For compact X, we have isomorphisms $\tilde{KU}^n(X) \cong \tilde{KU}^{n+2}(X)$ and $\tilde{KO}^n(X) \cong \tilde{KO}^{n+8}(X)$. In the complex case, the isomorphism is given by $a \mapsto (\gamma - 1) \otimes a)$ (notation to be defined).*

To elucidate this isomorphism, we will deduce it in the following way; we focus on the complex case. We can define a natural product

$$K(X) \otimes K(Y) \to K(X \times Y), \qquad a \otimes b \mapsto p_1^*(a)p_2^*(b).$$

Next, note that $\tilde{KU}(S^1) = 0$ (unlike the real case). For S^2, let γ denote the canonical line bundle (considered over $\mathbb{CP}^1$); then by computing clutching functions we have $\gamma^2 + 1 = 2\gamma$. Moreover, we can show that this generates (with $\mathbb{Z}$) all of $KU(S^2)$, so we have $KU(S^2) \cong \mathbb{Z}[\gamma]/(\gamma-1)^2$.

Theorem 1.3 (Bott periodicity, version 2B)**.** *For compact X, the product map*

$$KU(X) \otimes KU(S^2) \to KU(X \times S^2)$$

is an isomorphism.

Note that in this map, we have

$$\alpha_1 + \alpha_2\gamma \mapsto \alpha_1 \otimes 1 + (\alpha_2 \otimes 1)(1 \otimes \gamma) = \alpha_1 \otimes 1 + \alpha_2 \otimes \gamma.$$

Finally, we have a homotopical interpretation of Bott periodicity.

Theorem 1.4 (Bott periodicity, version 3)**.** *We have homotopy equivalences*

$$BU \times \mathbb{Z} \cong \Omega^2(BU \times \mathbb{Z}), \qquad BO \times \mathbb{Z} \cong \Omega^8(BO \times \mathbb{Z}).$$

In other words, we have a periodic spectrum given by $KU_{2n} = BU \times \mathbb{Z}$ and $KU_{2n+1} = U$. One can work out something similar for KO that is 8=periodic, though the details are more involved. By the (Σ, Ω) adjunction this version implies the periodicity in version 2A. There are a variety of proofs of this statement whose sketches may be found in [4].

2 Proofs

We have seen that, at least with respect to the bare periodicity statement, version 2A implies version 1. Now we will show how version 2B implies version 2A and give a very rough sketch of a proof of version 2B. We will only deal with the complex case, and the details of this approach can be found in [3].

Proposition 2.1. *Version 2B $\Rightarrow$ version 2A.*

Proof. Consider the long exact sequence in K-theory of $(X \times S^2, X \vee S^2)$. We get that

$$K(X \times S^2, X \vee S^2) = \tilde{K}(\Sigma^2 X) = \ker(K(X \times S^2) \to K(X \vee S^2)).$$

By version 2B, every element of $K(X \times S^2)$ can be written as $\alpha \otimes (\gamma - 1) + \beta \otimes \gamma$. By inspection, the kernel must indeed be of the form $\alpha \otimes (\gamma - 1)$. This shows that

$$\tilde{K}(X) \to \tilde{K}(\Sigma^2 X), \qquad \alpha \mapsto \alpha \otimes (\gamma - 1)$$

is an isomorphism, as desired. □

Sketch of proof of version 2B. Let ξ be a vector bundle over $X \times S^2$. Covering S^2 with two copies of D^2, define the clutching function $u(x, z) : \alpha_x \to \alpha_x$, acting on the fiber over x for every point $z \in S^1$. We will try to reduce $u(x, z)$ to simpler forms. We may set

$$u(x, z) = \sum_{k=-N}^{N} u_k(x) z^k.$$

Tensoring ξ with γ^n, we get a new bundle ξ' whose U' is polynomial in z. Now ξ' is stably equivalent to ξ'', where we add a bunch of copies of α. By some linear algebra magic we can put ξ'' in the form $\beta \otimes 1 \oplus \gamma \otimes \zeta^{-1}$. This allows us to write ξ as an element in the image of $K(X) \otimes K(S^2)$, so the product map is surjective. Injectivity can be shown in a similar way. □

3 Comments

Bott's original proof involved quite a bit of differential geometry and Morse theory [2]. Atiyah's version [1] went beyond a mere proof by setting it in the context of indices of elliptic operators, which led to various generalizations. As just one example, he deduced a K-theoretic Thom isomorphism.

In a completely different direction, one may wonder about analogues in algebraic K-theory. Things seem to be less well-understood, but there are certainly many interesting conjectures. For example, assuming Vandiver's conjecture, the Quillen-Lichtenbaum conjecture (an extremely deep statement proven by Voevodsky and others) gives that the algebraic K-theory of the integers are 8-periodic and computes them in terms of the Bernoulli numbers.

For a wide-ranging overview of Bott periodicity in its many incarnations, see [5].

References

[1] Michael Atiyah. *Bott Periodicity and the Index of Elliptic Operators*. Quart. J. Math. Oxford (2), **19** (1968), 113–140.

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[4] Peter May. *A Concise Course in Algebraic Topology*. Retrieved online.
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