Distance & Midpoint Formulas:

Distance Formula:

We now find a formula for the distance $d(A, B)$ between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane. Recall from Section 1.1 that the distance between points $a$ and $b$ on a number line is $d(a, b) = |b - a|$. So from Figure 4 we see that the distance between the points $A(x_1, y_1)$ and $C(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$, and the distance between $B(x_2, y_2)$ and $C(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$.

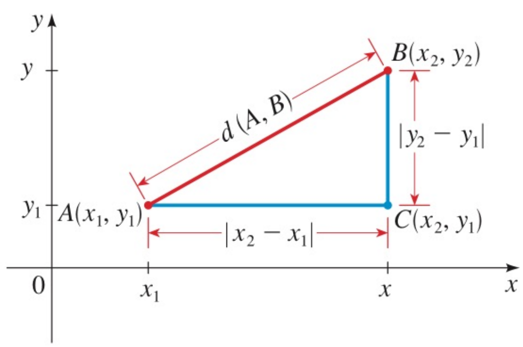



FIGURE 4

Since triangle $ABC$ is a right triangle, the Pythagorean Theorem gives

$$d(A, B) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**DISTANCE FORMULA**

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

## EXAMPLE 2 ■ Applying the Distance Formula

Which of the points $P(1, -2)$ or $Q(8, 9)$ is closer to the point $A(5, 3)$?

**SOLUTION** By the Distance Formula we have

$$d(P, A) = \sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

$$d(Q, A) = \sqrt{(5 - 8)^2 + (3 - 9)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

This shows that $d(P, A) < d(Q, A)$, so $P$ is closer to $A$ (see Figure 5).

Midpoint Formula:

Now let's find the coordinates $(x, y)$ of the midpoint $M$ of the line segment that joins the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$. In Figure 6 notice that triangles $APM$ and $MQB$ are congruent because $d(A, M) = d(M, B)$ and the corresponding angles are equal. It follows that $d(A, P) = d(M, Q)$, so

$$x - x_1 = x_2 - x$$

Solving this equation for $x$, we get $2x = x_1 + x_2$, so $x = \dfrac{x_1 + x_2}{2}$. Similarly, $y = \dfrac{y_1 + y_2}{2}$.

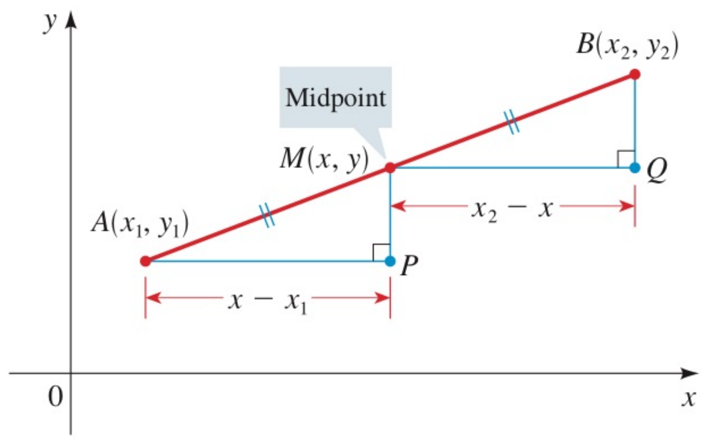



**FIGURE 6**

## MIDPOINT FORMULA

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

## EXAMPLE 3 ■ Applying the Midpoint Formula

Show that the quadrilateral with vertices $P(1, 2)$, $Q(4, 4)$, $R(5, 9)$, and $S(2, 7)$ is a parallelogram by proving that its two diagonals bisect each other.

**SOLUTION** If the two diagonals have the same midpoint, then they must bisect each other. The midpoint of the diagonal $PR$ is

$$\left(\frac{1 + 5}{2}, \frac{2 + 9}{2}\right) = \left(3, \frac{11}{2}\right)$$

# Graphing equations in two variables

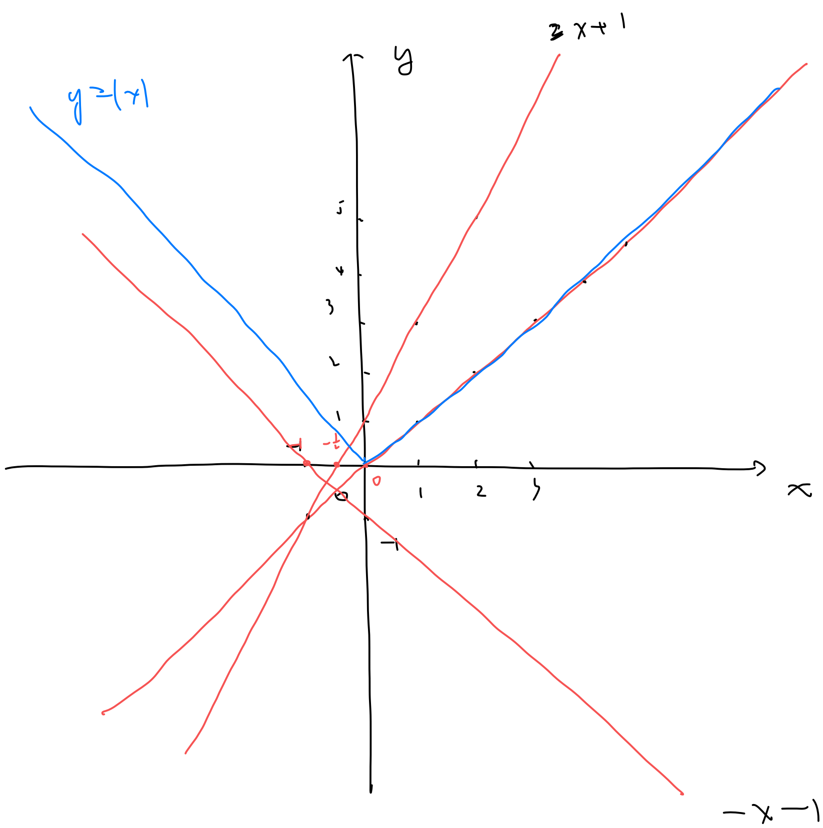
We have coordinate plane with x-axis & y-axis

Sometime we will get equations in two variables, e.g

e.g. $x - y = 0$ $\qquad\qquad$ $y - |x| = 0$

$2x + 1 - y = 0$

$x + 1 + y = 0$

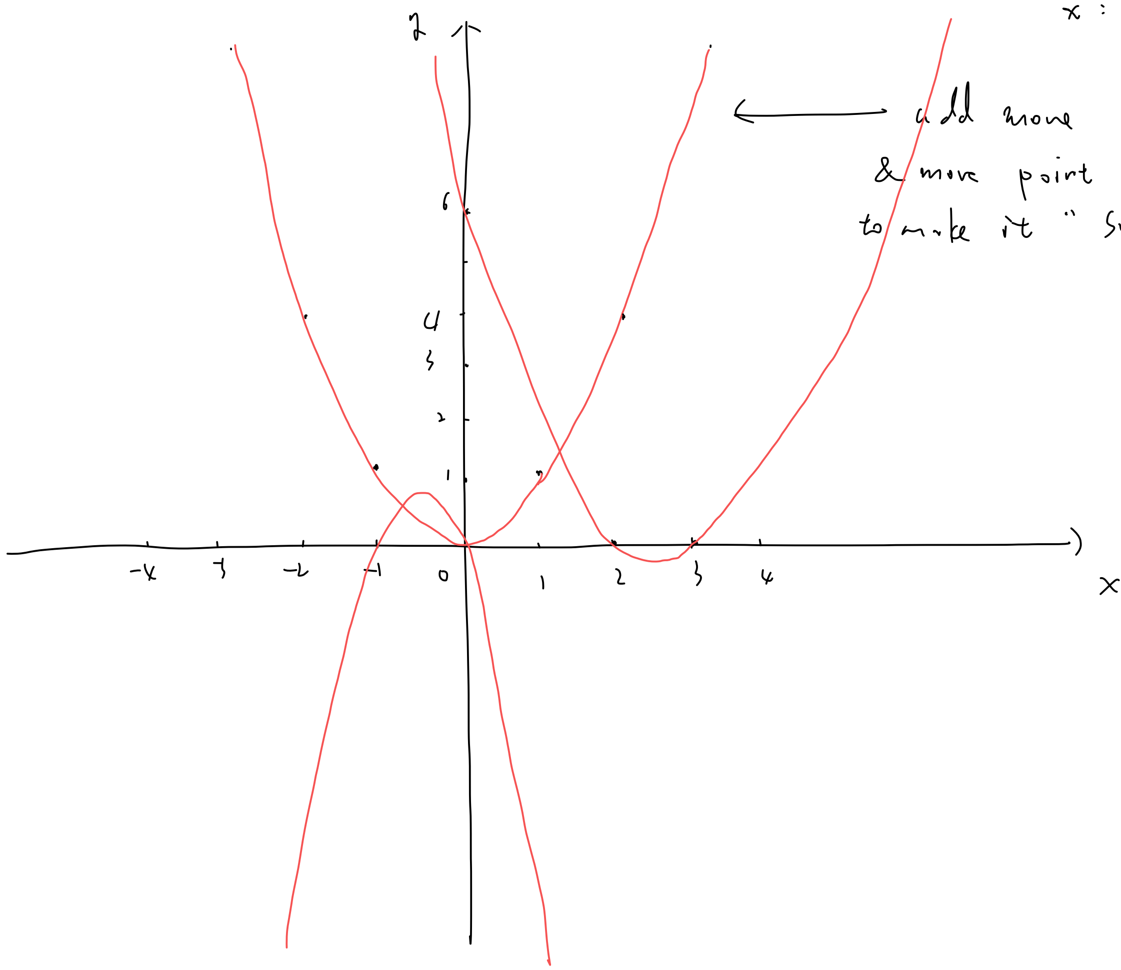


Quadratic polynomial : $ax^2+bx+c$ $\quad -\frac{1}{2}+1$

$x^2$, $\quad x^2-5x+6$, $\quad -2x^2-2x$

$x^2$: 0 ±1 ±2 ±3

0 1 4 9



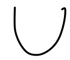

add more & more point to make it "smooth"

- graphs are smooth curves:
  - intersection with x-axis is solutions to the original equations

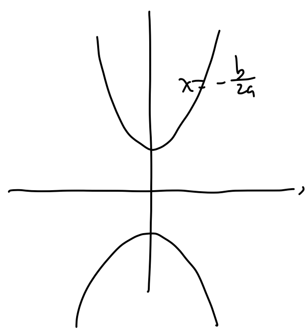
| | int | solution |
|---|---|---|
| $x^2$: | 0 | 0 |
| $x^2-5x+6$: | 2, 3 | 2, 3 |
| $-2x^2-2x$: | -1, 0 | -1, 0 |

Graphing quadratic equation: principle:

$$y = ax^2 + bx + c$$

- $a > 0$: open to the top 
- $a < 0$: open to the bottom 

- Find the symmetric axis: $x = -\frac{b}{2a}$ $\qquad y = a\left(x - \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$



## Geometric meaning of Intersection points

If we are give two equation

$$f(x,y)=0$$

$$g(x,y)=0$$

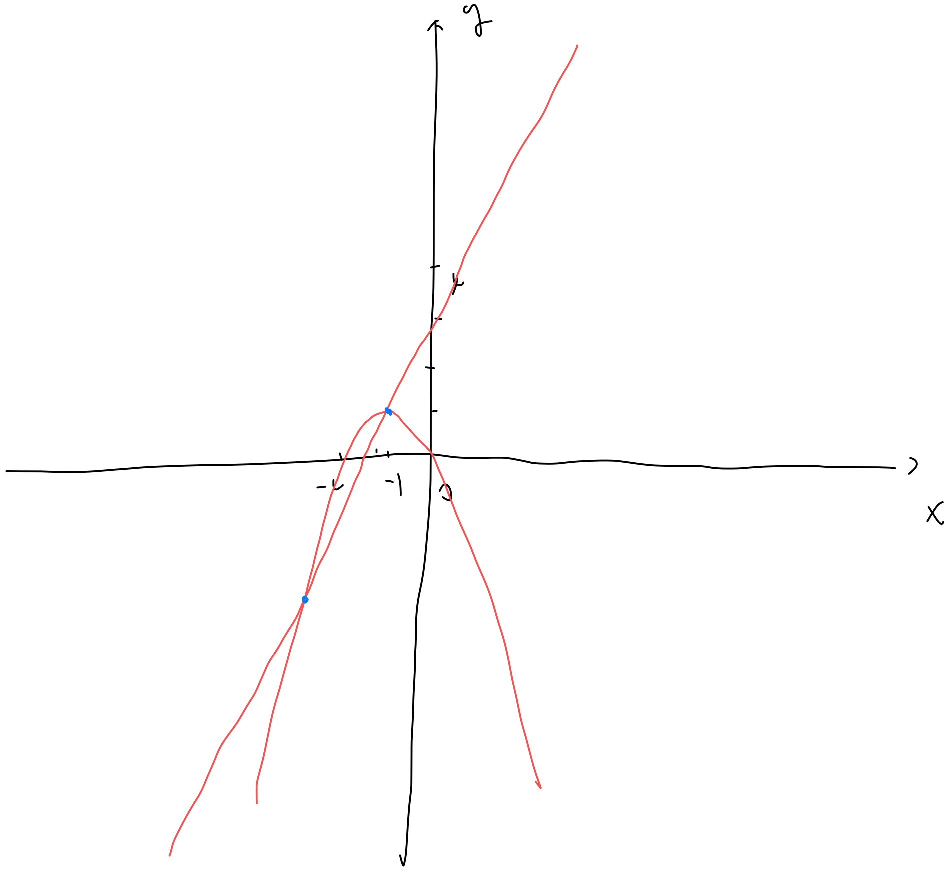
$x^2+4x+3$

e.g.

$$2x-y+3=0 \qquad y=2x+3$$

$$x^2+2x+y=0 \qquad y=-x^2-2x=-(x+1)^2+1$$

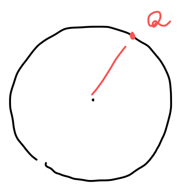
we get two graphs on the coordinate plane



# Circles & Lines (continued)

Recall: equation of a circle:

Q: What the equation for a circle with center $P=(x_1,y_1)$ & radius $r>0$ ?




Q lines on the circle $\Leftrightarrow$ $d(P,Q)=r$

$$d(P,Q)=\sqrt{(x-x_1)^2+(y-y_1)^2}=r$$

$$\Leftrightarrow \quad (x-x_1)^2+(y-y_1)^2=r^2$$

meaning of the equation

$$(x-x_1)^2+(y-y_1)^2=r^2$$

any solution $(x,y)$ of this equation $\rightleftarrows$ a point $Q=(x,y)$ on the circle

e.g.

## EXAMPLE 9 ■ Graphing a Circle

Graph each equation.

**(a)** $x^2+y^2=25$ **(b)** $(x-2)^2+(y+1)^2=25$

**SOLUTION**

**(a)** Rewriting the equation as $x^2+y^2=5^2$, we see that this is an equation of the circle of radius 5 centered at the origin. Its graph is shown in Figure 14.

**(b)** Rewriting the equation as $(x-2)^2+(y+1)^2=5^2$, we see that this is an equation of the circle of radius 5 centered at $(2,-1)$. Its graph is shown in Figure 15.

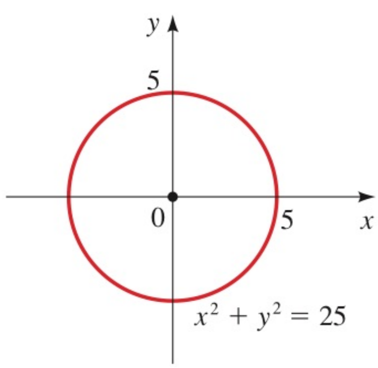



**FIGURE 14**

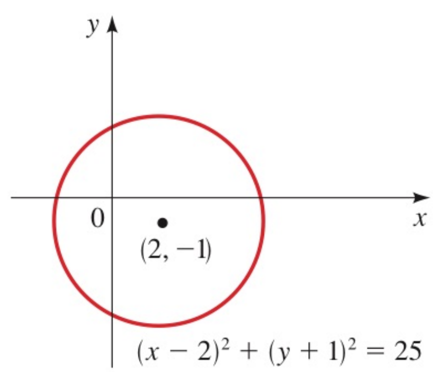



**FIGURE 15**

## EXAMPLE 11 ■ Identifying an Equation of a Circle

Show that the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ represents a circle, and find the center and radius of the circle.

**SOLUTION** We first group the $x$-terms and $y$-terms. Then we complete the square within each grouping. That is, we complete the square for $x^2 + 2x$ by adding $\left(\frac{1}{2}\cdot 2\right)^2 = 1$, and we complete the square for $y^2 - 6y$ by adding $\left[\frac{1}{2}\cdot(-6)\right]^2 = 9$.

$$(x^2 + 2x \qquad) + (y^2 - 6y \qquad) = -7 \qquad \text{Group terms}$$

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9 \qquad \text{Complete the square by adding 1 and 9 to each side}$$

$$(x + 1)^2 + (y - 3)^2 = 3 \qquad \text{Factor and simplify}$$

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$.

In general, if we have:

$$x^2 + y^2 + ax \quad by + c = 0$$

$$(x^2 + ax \qquad) + (y^2 + by \qquad) = -c$$

$$\downarrow$$

$$\left(x^2 + ax + \frac{a^2}{4}\right) + \left(y^2 + by + \frac{b^2}{4}\right) = -c + \frac{a^2 + b^2}{4}$$

$$\downarrow$$

$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4} - c$$

Two cases: if $\frac{a^2+b^2}{4} - c < 0$, no such points, $\varnothing$

if $\frac{a^2+b^2}{4} - c \geq 0$: circle with center $\left(-\frac{a}{2}, -\frac{b}{2}\right)$

with radius $\sqrt{\frac{a^2+b^2}{4} - c}$

Lines

Slope:

**SLOPE OF A LINE**

The **slope** $m$ of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

The slope is independent of which two points are chosen on the line. We can see that this is true from the similar triangles in Figure 3.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$

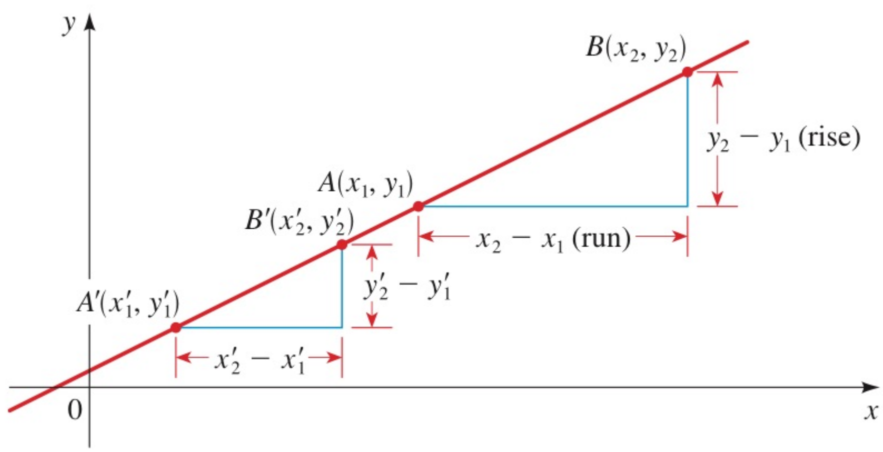


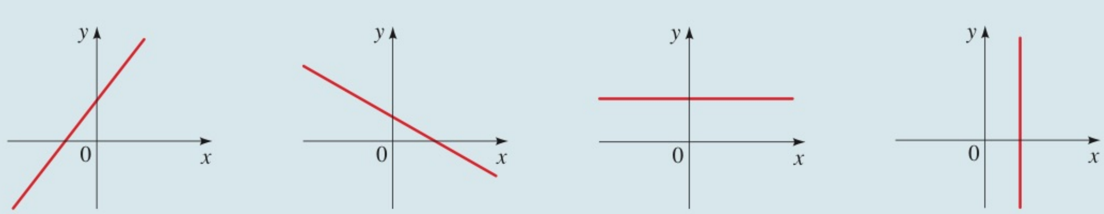

**FIGURE 3**

Q: If you pick other two points, will the result be different?
we always get the same answer

The figures in the box below show several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. The steepest lines are those for which the absolute value of the slope is the largest; a horizontal line has slope 0. The slope of a vertical line is undefined (it has a 0 denominator), so we say that a vertical line has no slope.

**SLOPE OF A LINE**

**Positive Slope** **Negative Slope** **Zero Slope** **No Slope**



## EXAMPLE 1 ■ Finding the Slope of a Line Through Two Points

Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

**SOLUTION** Since any two different points determine a line, only one line passes through these two points. From the definition the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

· Point-Slope form equation

Now let's find the equation of the line that passes through a given point $P(x_1, y_1)$ and has slope $m$. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through $P_1$ and $P$ is equal to $m$ (see Figure 5), that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form $y - y_1 = m(x - x_1)$; note that the equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

### POINT-SLOPE FORM OF THE EQUATION OF A LINE

An equation of the line that passes through the point $(x_1, y_1)$ and has slope $m$ is

$$y - y_1 = m(x - x_1)$$

## EXAMPLE 2 ■ Finding an Equation of a Line with Given Point and Slope

**(a)** Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$.

**(b)** Sketch the line.

## EXAMPLE 3 ■ Finding an Equation of a Line Through Two Given Points

Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

**SOLUTION** The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{6}{4} = -\frac{3}{2}$$

We can use *either* point, $(-1, 2)$ *or* $(3, -4)$, in the point-slope equation. We will end up with the same final answer.

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1) \qquad \text{Slope } m = -\tfrac{3}{2}, \text{ point } (-1, 2)$$

$$2y - 4 = -3x - 3 \qquad \text{Multiply by 2}$$

$$3x + 2y - 1 = 0 \qquad \text{Rearrange}$$

here we pick the point $(-1, 2)$, will it be different if we pick the point $(3, -4)$? No!

· Slope - Intercept form equation

Suppose a nonvertical line has slope $m$ and $y$-intercept $b$ (see Figure 7). This means that the line intersects the $y$-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x = 0$ and $y = b$, becomes

$$y - b = m(x - 0)$$

This simplifies to $y = mx + b$, which is called the **slope-intercept form** of the equation of a line.

### SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE

An equation of the line that has slope $m$ and $y$-intercept $b$ is

$$y = mx + b$$

## EXAMPLE 4 ■ Lines in Slope-Intercept Form

**(a)** Find an equation of the line with slope 3 and $y$-intercept $-2$.

**(b)** Find the slope and $y$-intercept of the line $3y - 2x = 1$.

### SOLUTION

**(a)** Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = 3x - 2$$

**(b)** We first write the equation in the form $y = mx + b$.

$$\begin{aligned} 3y - 2x &= 1 \\ 3y &= 2x + 1 && \text{Add } 2x \\ y &= \tfrac{2}{3}x + \tfrac{1}{3} && \text{Divide by 3} \end{aligned}$$

From the slope-intercept form of the equation of a line, we see that the slope is $m = \frac{2}{3}$ and the $y$-intercept is $b = \frac{1}{3}$.

Vertical & Horizontal lines :

point-slope form / slope-intercept form $\Longrightarrow$ lines

Question : Does every line can be written in point-slope, slope-intercept form ?

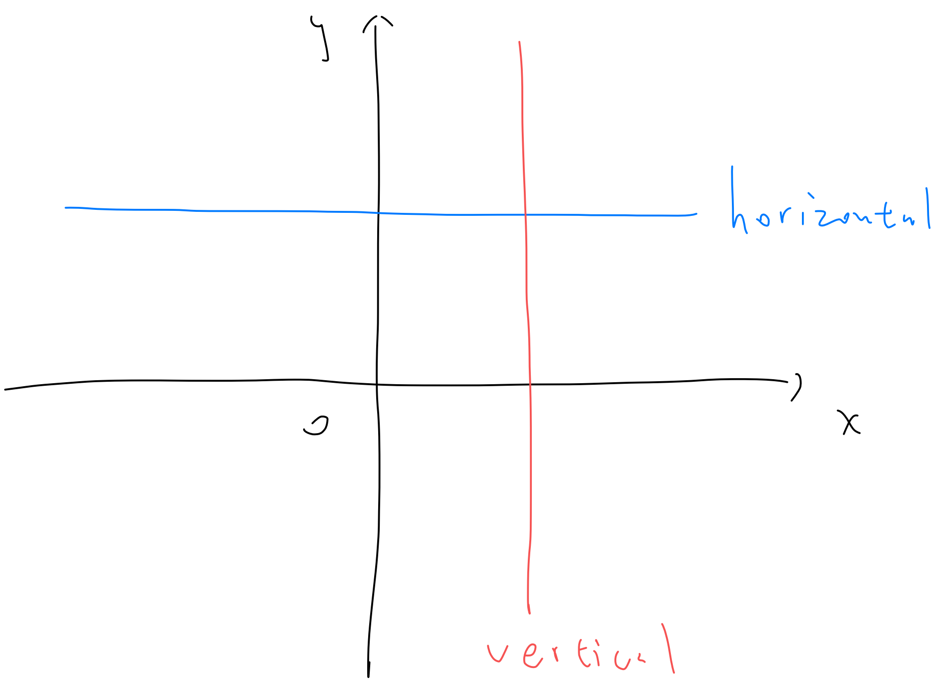
Answer : No !

Example : $x = 0$ : or, all the vertical lines

If a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where $b$ is the $y$-intercept (see Figure 8). A vertical line does not have a slope, but we can write its equation as $x = a$, where $a$ is the $x$-intercept, because the $x$-coordinate of every point on the line is $a$.

**VERTICAL AND HORIZONTAL LINES**

- An equation of the vertical line through $(a, b)$ is $x = a$.
- An equation of the horizontal line through $(a, b)$ is $y = b$.



General equation of a line

A **linear equation** in the variables $x$ and $y$ is an equation of the form

$$Ax + By + C = 0$$

where $A$, $B$, and $C$ are constants and $A$ and $B$ are not both 0. An equation of a line is a linear equation:

- A nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$, which is a linear equation with $A = -m$, $B = 1$, and $C = -b$.
- A vertical line has the equation $x = a$ or $x - a = 0$, which is a linear equation with $A = 1$, $B = 0$, and $C = -a$.

  Conversely, the graph of a linear equation is a line.
- If $B \neq 0$, the equation becomes

$$y = -\frac{A}{B}x - \frac{C}{B} \qquad \text{Divide by } B$$

  and this is the slope-intercept form of the equation of a line (with $m = -A/B$ and $b = -C/B$).
- If $B = 0$, the equation becomes

$$Ax + C = 0 \qquad \text{Set } B = 0$$

  or $x = -C/A$, which represents a vertical line.

We have proved the following.

**GENERAL EQUATION OF A LINE**

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

Parallel & Perpendicular

Since slope measures the steepness of a line, it seems reasonable that parallel lines should have the same slope. In fact, we can prove this.

**PARALLEL LINES**

Two nonvertical lines are parallel if and only if they have the same slope.

**Proof** Let the lines $l_1$ and $l_2$ in Figure 12 have slopes $m_1$ and $m_2$. If the lines are parallel, then the right triangles $ABC$ and $DEF$ are similar, so

$$m_1 = \frac{d(B, C)}{d(A, C)} = \frac{d(E, F)}{d(D, F)} = m_2$$

Conversely, if the slopes are equal, then the triangles will be similar, so $\angle BAC = \angle EDF$ and the lines are parallel. ■

## EXAMPLE 7 ■ Finding an Equation of a Line Parallel to a Given Line

Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

**SOLUTION** First we write the equation of the given line in slope-intercept form.

$$4x + 6y + 5 = 0$$

$$6y = -4x - 5 \qquad \text{Subtract } 4x + 5$$

$$y = -\tfrac{2}{3}x - \tfrac{5}{6} \qquad \text{Divide by 6}$$

So the line has slope $m = -\frac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\frac{2}{3}$. From the point-slope form of the equation of a line we get

$$y - 2 = -\tfrac{2}{3}(x - 5) \qquad \text{Slope } m = -\tfrac{2}{3}, \text{ point } (5, 2)$$

$$3y - 6 = -2x + 10 \qquad \text{Multiply by 3}$$

$$2x + 3y - 16 = 0 \qquad \text{Rearrange}$$

Thus an equation of the required line is $2x + 3y - 16 = 0$.

The condition for perpendicular lines is not as obvious as that for parallel lines.

### PERPENDICULAR LINES

Two lines with slopes $m_1$ and $m_2$ are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

**Proof** In Figure 13 we show two lines intersecting at the origin. (If the lines intersect at some other point, we consider lines parallel to these that intersect at the origin. These lines have the same slopes as the original lines.)

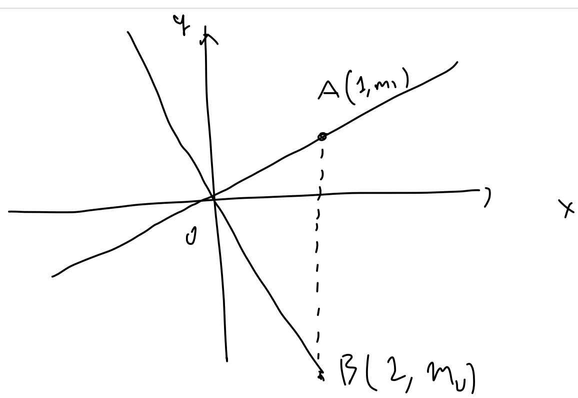
If the lines $l_1$ and $l_2$ have slopes $m_1$ and $m_2$, then their equations are $y = m_1 x$ and $y = m_2 x$. Notice that $A(1, m_1)$ lies on $l_1$ and $B(1, m_2)$ lies on $l_2$. By the Pythagorean Theorem and its converse (see Appendix A) $OA \perp OB$ if and only if

$$[d(O, A)]^2 + [d(O, B)]^2 = [d(A, B)]^2$$

By the Distance Formula this becomes

$$\begin{aligned} (1^2 + m_1^2) + (1^2 + m_2^2) &= (1 - 1)^2 + (m_2 - m_1)^2 \\ 2 + m_1^2 + m_2^2 &= m_2^2 - 2m_1 m_2 + m_1^2 \\ 2 &= -2m_1 m_2 \\ m_1 m_2 &= -1 \end{aligned}$$

■




## EXAMPLE 8 ■ Perpendicular Lines

Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

## EXAMPLE 9 ■ Finding an Equation of a Line Perpendicular to a Given Line

Find an equation of the line that is perpendicular to the line $4x + 6y + 5 = 0$ and passes through the origin.

**SOLUTION** In Example 7 we found that the slope of the line $4x + 6y + 5 = 0$ is $-\frac{2}{3}$. Thus the slope of a perpendicular line is the negative reciprocal, that is, $\frac{3}{2}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$y - 0 = \tfrac{3}{2}(x - 0) \qquad \text{Slope } m = \tfrac{3}{2}, \text{ point } (0, 0)$$

$$y = \tfrac{3}{2}x \qquad \text{Simplify}$$

**Now Try Exercise 47** ■

Look back: Algebra $\longleftrightarrow$ Geometry

solving equations & inequalities

To solve a one-variable equation such as $3x - 5 = 0$ graphically, we first draw a graph of the two-variable equation $y = 3x - 5$ obtained by setting the nonzero side of the equation equal to a variable $y$. The solutions of the given equation are the values of $x$ for which $y$ is equal to zero. That is, the solutions are the $x$-intercepts of the graph. The following describes the method.

## SOLVING AN EQUATION

**Algebraic Method**

Use the rules of algebra to isolate the unknown $x$ on one side of the equation.

**Example:** $3x - 4 = 1$

$3x = 5$     Add 4

$x = \frac{5}{3}$     Divide by 3

The solution is $x = \frac{5}{3}$.

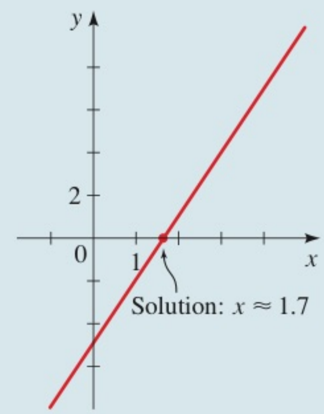
**Graphical Method**

Move all terms to one side, and set equal to $y$. Graph the resulting equation, and find the $x$-intercepts.

**Example:** $3x - 4 = 1$

$3x - 5 = 0$

Set $y = 3x - 5$ and graph. From the graph we see that the solution is $x \approx 1.7$




Graphical Method: Explaination

the line has slope = 3

and passes through (0, −5)     $y = 3x - 5$

· intercepts with x-axi: $(x, 0) \iff$ solving $3x - 5 = 0$

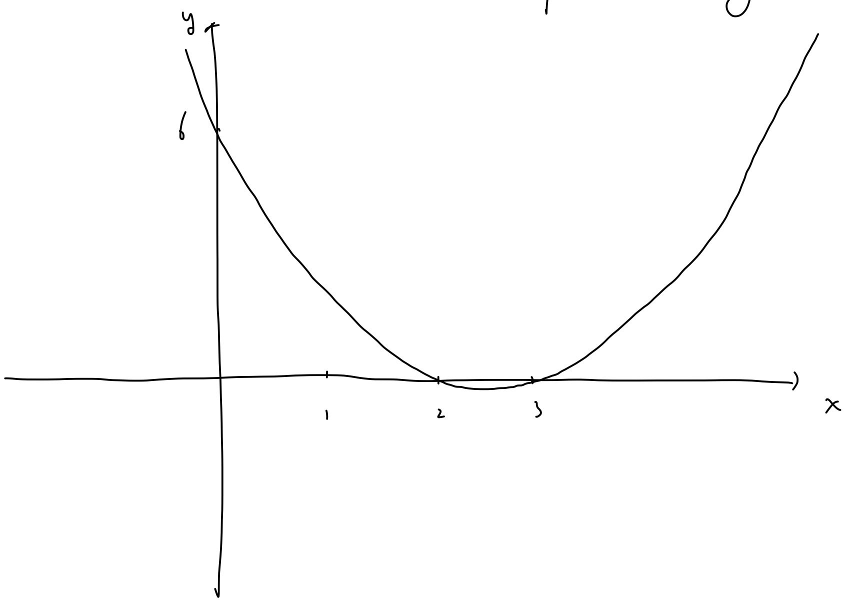
$$3 = \frac{\text{rise}}{\text{run}} = \frac{0-(-5)}{x-0} = \frac{5}{x} \Rightarrow x = \frac{5}{3}$$

---

| | |
|---|---|
| $3x - 5 \geq 0$<br>$x \geq \frac{5}{3}$ | Graphically: $x \geq \frac{5}{3}$ |
| $3x - 5 < 0$ | Graphically: $x < \frac{5}{3}$ |

Example: Consider the equation: $x^2 - 5x + 6$

and the two variable equation: $y = x^2 - 5x + 6$




If we want to solve:

$x^2 - 5x + 6 = 0 \iff$ finding the x-intercepts $\Rightarrow x = 2, 3$

$x^2 - 5x + 6 \geq 0 \iff x \leq 2,\ x \geq 3$

$x^2 - 5x + 6 < 0 \iff 2 < x < 3$