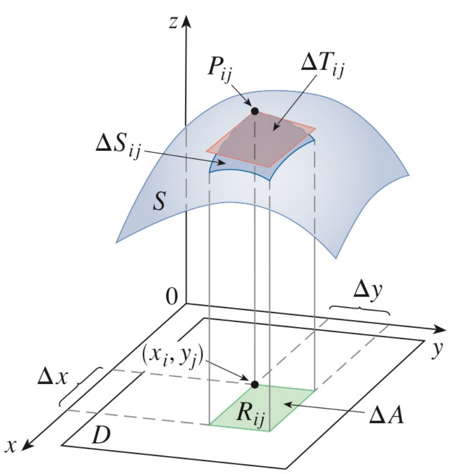
Tu 5/30

Surface Area

Goal: given a surface by the equation $z = f(x,y)$, computes its area

Set-up: Let S be a surface defined by the equation $z = f(x,y)$, here f has continuous partial derivatives, and f is defined on a region $D \subset \mathbb{R}^2$

Let's first assume that $D = [a,b] \times [c,d]$ is a rectangle. for simiplicity, sketch S:



Step 1: We divided D into mn parts Δ_{ij}
pick $P_{ij}^* \in \Delta_{ij}$ arbitraryly.

Step 2: form the tangent space T_{ij} of the surface S at the point P_{ij}^*.

Step 3: ΔT_{ij} = area of $T_{ij} \cap$ the cuboid formed by $\Delta_{ij} \times z$

ΔT_{ij} can be viewed as the approximate of the area ΔS_{ij}, and become more precise when $m,n \to \infty$

Step 4: form the following Riemann sum:

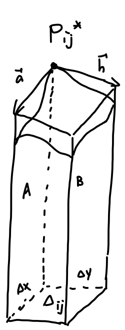
$$\sum_{i,j} \Delta T_{ij}$$

and taking limits:

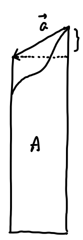
$$A(S) = \lim_{m,n \to \infty} \sum_{i,j} \Delta T_{ij}$$

Remaining problem:

- how to express ΔT_{ij} in terms of f ?

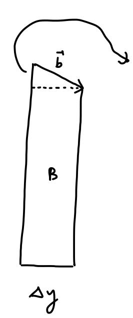


$$\Delta T_{ij} = |\vec{a} \times \vec{b}|$$



the function section is $f_j(x) = f(x, y_j)$
then we get the height is

$$\frac{df_j}{dx}(P_{ij}^*) \cdot \Delta x = \frac{\partial f}{\partial x}(P_{ij}^*) \cdot \Delta x$$



the function section is $f_i(y) = f(x_i, y)$
then we get the height is

$$\frac{df_i}{dy}(P_{ij}^*) \cdot \Delta y = \frac{\partial f}{\partial y}(P_{ij}^*) \cdot \Delta y$$

the we have: $\vec{a} = \left(\Delta x,\ 0,\ \frac{\partial f}{\partial x}(P_{ij}^*)\,\Delta x\right)$, $\vec{b} = \left(0,\ \Delta y,\ \frac{\partial f}{\partial y}(P_{ij}^*)\,\Delta y\right)$

and

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ \Delta x & 0 & \frac{\partial f}{\partial x}(P_{ij}^*)\,\Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y}(P_{ij}^*)\,\Delta y \end{vmatrix} = i\left(-\frac{\partial f}{\partial x}(P_{ij}^*)\,\Delta x\Delta y\right) - j\left(\frac{\partial f}{\partial y}(P_{ij}^*)\,\Delta x\Delta y\right) + k\left(\Delta x\Delta y\right)$$

$$= \left(-\frac{\partial f}{\partial x}(P_{ij}^*)\,\Delta x\Delta y,\ -\frac{\partial f}{\partial y}(P_{ij}^*)\,\Delta x\Delta y,\ \Delta x\Delta y\right)$$

hence

$$\Delta T_{ij} = \left|\vec{a} \times \vec{b}\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}(P_{ij}^*)\right)^2 + \left(\frac{\partial f}{\partial y}(P_{ij}^*)\right)^2} \cdot \Delta x\,\Delta y$$

then

$$A(S) = \lim_{m,n\to\infty} \sum_{i,j} \sqrt{1 + \left(\frac{\partial f}{\partial x}(P_{ij}^*)\right)^2 + \left(\frac{\partial f}{\partial y}(P_{ij}^*)\right)^2} \cdot \Delta x\,\Delta y$$

$$= \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

Although we derive this formula under the assumption that D is a rectangle, very similar proof applies to the general region D

Examples

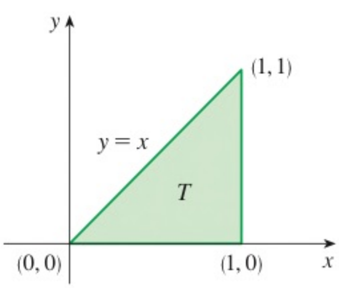


FIGURE 3

EXAMPLE 1 Find the surface area of the part of the surface $z = x^2 + 2y + 2$ that lies above the triangular region T in the xy-plane with vertices (0, 0), (1, 0), and (1, 1).

SOLUTION The region T is shown in Figure 3 and is described by

$$T = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x\}$$

Using Formula 2 with $f(x, y) = x^2 + 2y + 2$, we get

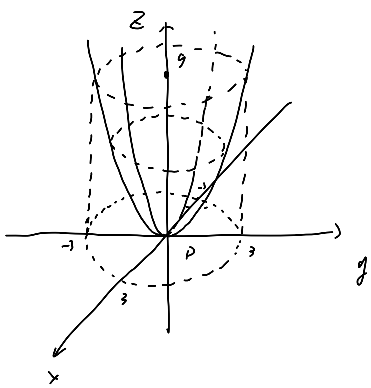
$$A = \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx$$

$$= \int_0^1 x\sqrt{4x^2 + 5}\, dx = \tfrac{1}{8} \cdot \tfrac{2}{3}(4x^2 + 5)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(27 - 5\sqrt{5}\right)$$

Figure 4 shows the portion of the surface whose area we have just computed. ■

EXAMPLE 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Sketch the surface & the integral region



then:

$$A = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\; dA$$

$$= \iint_D \sqrt{1 + 4(x^2+y^2)}\; dA$$

$$= \int_0^3 \int_0^{2\pi} \sqrt{1+4r^2}\; r\, d\theta\, dr$$

$$= 2\pi \int_0^3 r\sqrt{1+4r^2}\; dr$$

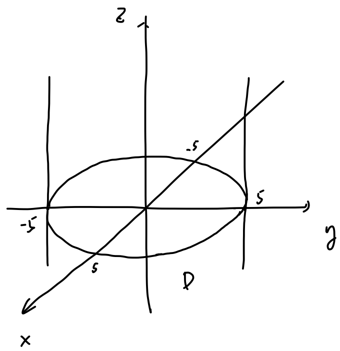
$u = r^2$

$$= \pi \int_0^9 \sqrt{1+4u}\; du$$

$$= \pi \times \frac{2}{3} \times \frac{1}{4} (1+4u)^{\frac{3}{2}} \Big|_0^9$$

Exercise:

4. The part of the plane $6x + 4y + 2z = 1$ that lies inside the cylinder $x^2 + y^2 = 25$



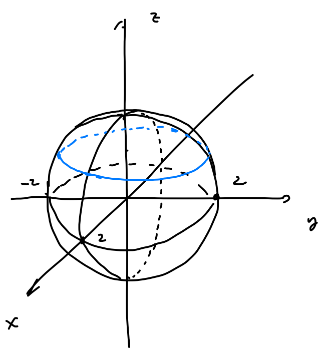
$$z = -3x - 2y + \frac{1}{2}$$

then

$$A(S) = \iint_D \sqrt{9+4+1}\, dA$$

$$= 25\sqrt{14}\,\pi$$

12. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$



$$D: x^2 + y^2 \leq 3, \quad z = \sqrt{4 - (x^2+y^2)}$$

$$z_x = \frac{-x}{\sqrt{4-(x^2+y^2)}}, \quad z_y = \frac{-y}{\sqrt{4-(x^2+y^2)}}$$

$$A(S) = \iint_D \sqrt{\frac{x^2}{4-(x^2+y^2)} + \frac{y^2}{4-(x^2+y^2)} + 1}\ dA$$

$$= \iint_D \frac{2}{\sqrt{4-(x^2+y^2)}}\, dA$$

$$= \int_0^{\sqrt{3}} \int_0^{2\pi} \frac{2r}{\sqrt{4-r^2}}\, d\theta\, dr \qquad -1-(-2)$$

$$= 2\pi \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\, dr = 4\pi \cdot \left(-\sqrt{4-r^2}\right)\Big|_0^{\sqrt{3}}$$

$$= 4\pi \cdot (2-1) = 4\pi$$

Triple Integrals

Now we go one-dimension higher – let's consider integrating over a 3-dim'l region.

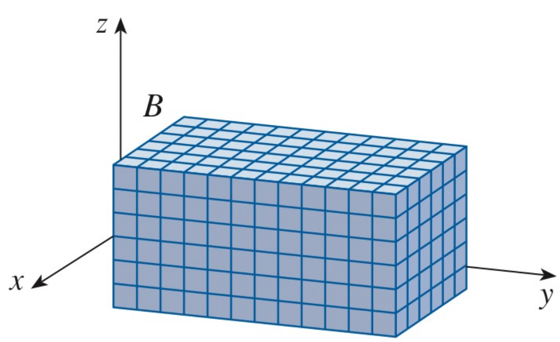
- Triple integral over Rectangular Boxes ~ double integral over Rectangle

$$B = [a,b] \times [c,d] \times [r,s] = \{ (x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s \}$$

f : a continuous function defined over B, i.e. $f : B \to \mathbb{R}$

Note: the graph of f lives in $3+1 = 4$-dimensional world
you may imagine f as the density function, and we are aiming to finding
the mass $\qquad m(B) = \iiint\limits_B f(x,y,z)\, dV$

Step 1: divide the box B into sub-boxes:
divide $[a,b]$ into l parts, $[c,d]$ into m parts, $[r,s]$ into n parts
then we get lmn sub-boxes B_{ijk}, $1 \le i \le l$, $1 \le j \le m$, $1 \le k \le n$



Step 2: pick an arbitrary point $P^*_{ijk} = (x_i^*, y_j^*, z_k^*) \in B_{ijk}$, then form the Riemann sum

$$\sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\, \mathrm{Vol}(B_{ijk})$$

this serves an approximation of the "mass"

Step 3: taking limits $l, m, n \to +\infty$, then $\sum\limits_{i,j,k} f(x_i^*, y_j^*, z_k^*)\, \mathrm{Vol}(B_{ijk})$ converges to a number which is independent of the choice of the point $P^*_{ijk} = (x_i^*, y_j^*, z_k^*)$.

$$\iiint\limits_B f(x,y,z)\, dV$$

· How to compute $\iiint_B f(x,y,z)\,dV$ effectively ?

Good news: Fubini's theorem still holds, i.e. we can compute by iterated integral

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \underbrace{\int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz}_{\text{can be arranged into any order!}}$$

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\bigg]_0^3 = \frac{27}{4}$$

■

· Triple integrals over general regions

Now suppose we have a region E in $\mathbb{R}^3$ which is bounded, and a continuous function $f: E \to \mathbb{R}$, we want to define $\iiint_E f(x,y,z)\,dV$

the strategy is much similar to the 2-dimensional case, we choose a box B containing E then define:

$$F(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z) \in E, \\ 0, & \text{if } (x,y,z) \notin E. \end{cases}$$

and

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

we can show that this definition is independent of the choice of the big box B containing E

· some special types of the region E:

type 1:

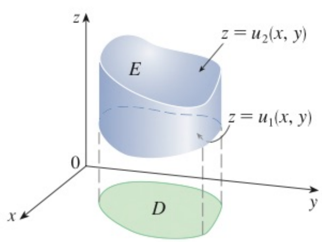


FIGURE 2
A type 1 solid region

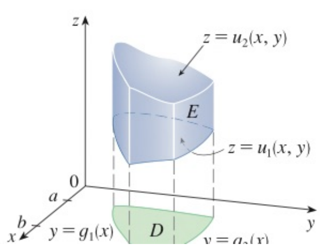
This integral exists if f is continuous and the boundary of E is "reasonably smooth." The triple integral has essentially the same properties as the double integral (Properties 5–8 in Section 15.2).

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

5 $$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x,y)$, while the lower boundary is the surface $z = u_1(x,y)$.

By the same sort of argument that led to (15.2.3), it can be shown that if E is a type 1 region given by Equation 5, then



6 $$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA}$$

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x,y)$ and $u_2(x,y)$ are regarded as constants, while $f(x,y,z)$ is integrated with respect to z

In particular, if D is also type I or II, we have

type I:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

type II:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy}$$

type 2 :

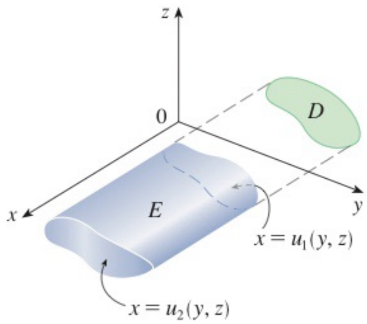


FIGURE 8
A type 2 region

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 8). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

10
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$

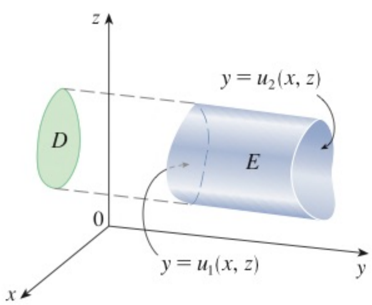
type 3 :

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 9). For this type of region we have

11
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$



In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

Rmk : the key point is not to remember the number 1, 2, 3
the key point is : observe the shape of the region E,
project it to xy, yz, xz-plane to see which
projection gives you the simplest description
of the region E

Examples:

EXAMPLE 2 Evaluate $\iiint_E z\,dV$ where E is the solid in the first octant bounded by the surface $z = 12xy$ and the planes $y = x$, $x = 1$.

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (Figure 5) and, for a type 1 region, one of its projection D onto the xy-plane (Figure 6). The lower boundary of the solid E is the plane $z = 0$ and the upper boundary is the surface $z = 12xy$, so we use $u_1(x, y) = 0$ and $u_2(x, y) = 12xy$ in Formula 7. Notice that the projection of E onto the xy-plane is the triangular region shown in Figure 6, and we have

9 $$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x, 0 \leqslant z \leqslant 12xy\}$$

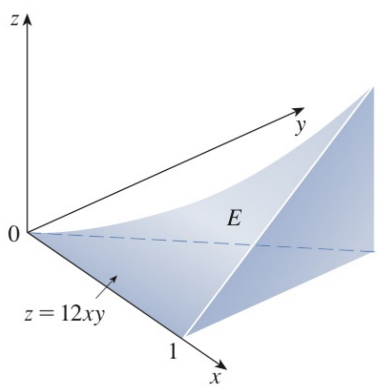


FIGURE 5

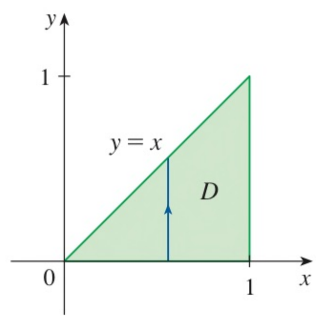


FIGURE 6

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint_E z\,dV = \int_0^1 \int_0^x \int_0^{12xy} z\,dz\,dy\,dx = \int_0^1 \int_0^x \left[\frac{z^2}{2}\right]_{z=0}^{z=12xy} dy\,dx$$

$$= \tfrac{1}{2}\int_0^1 \int_0^x (12xy)^2\,dy\,dx = 72\int_0^1 \int_0^x x^2y^2\,dy\,dx$$

$$= 72\int_0^1 \left[x^2\frac{y^3}{3}\right]_{y=0}^{y=x} dx = 24\int_0^1 x^5\,dx = 24\left[\frac{x^6}{6}\right]_{x=0}^{x=1} = 4$$

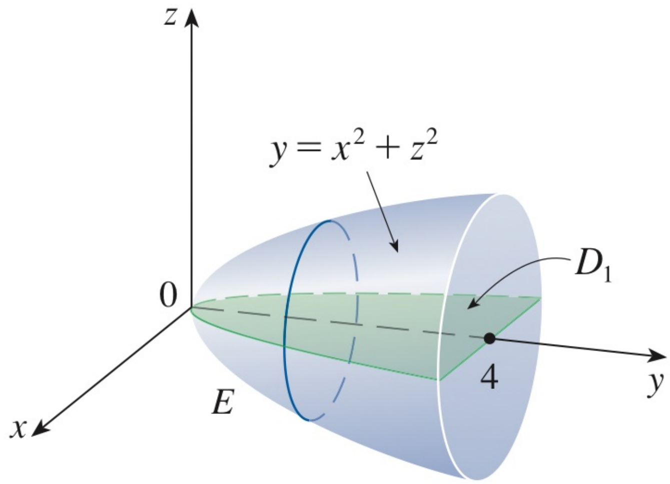
Sketch the integral region; Sketch its projection region D

Set up the integral

Calculation

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Sketch the region E :



Observe it: type 3, project to xz-plane, E is easy to describe:

$$\{ (x, y, z) \mid \underbrace{x^2 + z^2 \le 4}_{D}, \quad x^2 + z^2 \le y \le 4 \}$$

Set-up the integral: $$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dz \right) dA$$

$$= \iint_D \left(4 - (x^2+z^2)\right) \cdot \sqrt{x^2+z^2}\, dA$$

$$= \int_0^2 \int_0^{2\pi} (4 - r^2) \cdot r \cdot r\, d\theta\, dr$$

$$= 2\pi \int_0^2 (4 - r^2) r^2\, dr$$

$$= 2\pi \left(\frac{4}{3} r^3 - \frac{1}{5} r^5 \right) \Big|_0^2$$

$$= 2\pi \left(\frac{4}{3} \times 8 - \frac{1}{5} \times 32 \right)$$

$$= \frac{128}{15} \pi$$

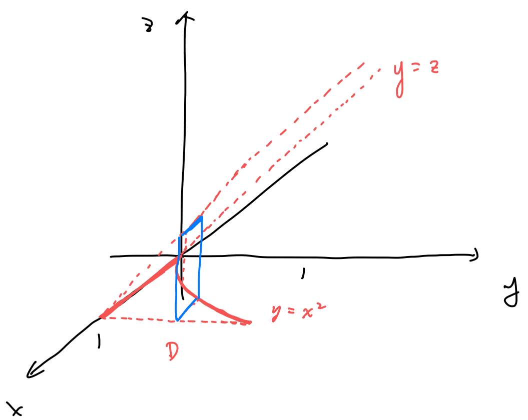
Changing the order of integration :

For Rectangular Box, when we compute the triple integral by iterated integrals, we can change the order of integration arbitrarily. However, this is not true for triple integral over general regions.

EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in the following orders.
(a) Integrate first with respect to x, then z, and then y.
(b) Integrate first with respect to y, then x, and then z.

Sketch the region

$$E = \{ (x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y \}$$



E is some solid above D

(a) fix y, where $0 \le y \le 1$, then

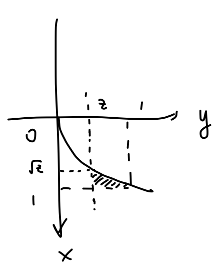
$$y \le x^2 \le 1 \iff \sqrt{y} \le x \le 1$$
$$0 \le z \le y$$

we get

$$\iiint_E f(x, y, z)\, dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x, y, z)\, dx\, dz\, dy$$

(b) fix z, where $0 \le z \le 1$, then

$$0 \le x \le 1 \quad \& \quad z \le y \le x^2$$



$$\iiint_E f(x, y, z)\, dV = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x, y, z)\, dy\, dx\, dz$$

Applications of Triple Integrals

Volume: Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

12
$$\boxed{V(E) = \iiint_E dV}$$

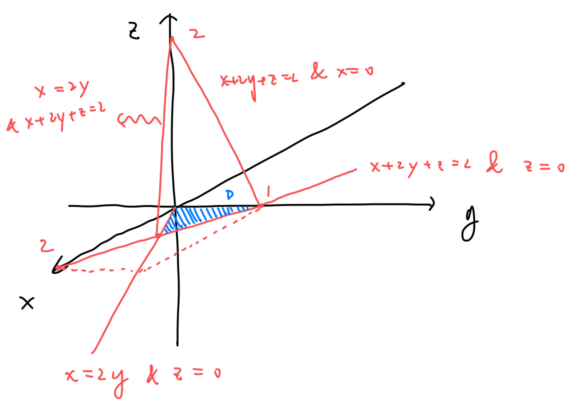
For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\, dA$$

and from Section 15.2 we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Sketch the region



$y = 1 - \frac{x}{2}$

$$V = \iiint_E 1\, dV$$

$$= \iint_D \left(\int_0^{2-x-2y} dz \right) dA$$

$$= \iint_D (2 - x - 2y)\, dA$$

$$= \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} (2 - x - 2y)\, dy\, dx$$

$$= \int_0^1 \left(2y - xy - y^2 \Big|_{x}^{1-\frac{x}{2}} \right) dx$$

Mass & Moments

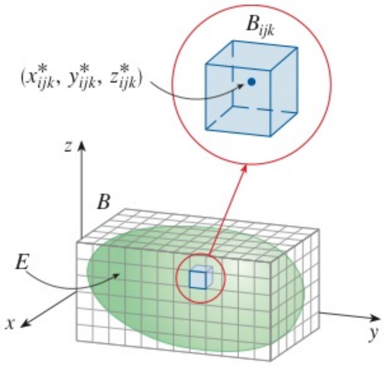


FIGURE 18
The mass of each sub-box B_{ijk} is approximated by $\rho(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$

All the applications of double integrals in Section 15.4 can be extended to triple integrals using analogous reasoning. For example, suppose that a solid object occupying a region E has density $\rho(x, y, z)$, in units of mass per unit volume, at each point (x, y, z) in E. To find the total mass m of E we divide a rectangular box B containing E into sub-boxes B_{ijk} of the same size (as in Figure 18), and consider $\rho(x, y, z)$ to be 0 outside E. If we choose a point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ in B_{ijk}, then the mass of the part of E that occupies B_{ijk} is approximately $\rho(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$, where ΔV is the volume of B_{ijk}. We get an approximation to the total mass by adding the (approximate) masses of all the sub-boxes, and if we increase the number of sub-boxes, we obtain the total mass m of E as the limiting value of the approximations:

13
$$m = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \rho(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V = \iiint_E \rho(x, y, z)\, dV$$

Similarly, the **moments** of E about the three coordinate planes are

14
$$M_{yz} = \iiint_E x\,\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\,\rho(x, y, z)\, dV$$
$$M_{xy} = \iiint_E z\,\rho(x, y, z)\, dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

15
$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of E. The **moments of inertia** about the three coordinate axes are

16
$$I_x = \iiint_E (y^2 + z^2)\,\rho(x, y, z)\, dV \qquad I_y = \iiint_E (x^2 + z^2)\,\rho(x, y, z)\, dV$$
$$I_z = \iiint_E (x^2 + y^2)\,\rho(x, y, z)\, dV$$

Probability density function

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\, dV$$

In particular,

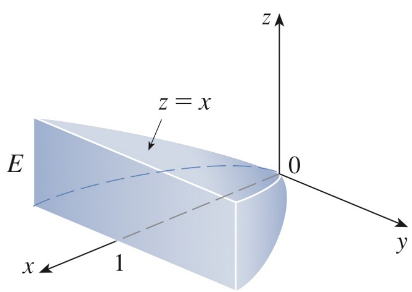
$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d,\ r \leqslant Z \leqslant s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

The joint density function satisfies

$$f(x, y, z) \geqslant 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1$$

EXAMPLE 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

Sketch:



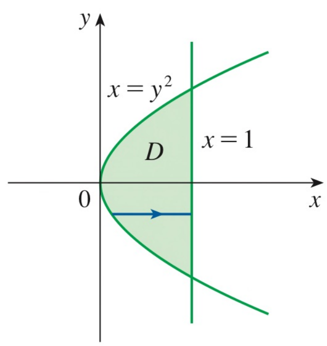


FIGURE 19

SOLUTION The solid E and its projection onto the xy-plane are shown in Figure 19. The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant x\}$$

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$\begin{aligned} m &= \iiint_E \rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho\, dz\, dx\, dy \\ &= \rho \int_{-1}^{1} \int_{y^2}^{1} x\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2} \right]_{x=y^2}^{x=1} dy \\ &= \frac{\rho}{2} \int_{-1}^{1} (1 - y^4)\, dy = \rho \int_0^1 (1 - y^4)\, dy \\ &= \rho \left[y - \frac{y^5}{5} \right]_0^1 = \frac{4\rho}{5} \end{aligned}$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that $M_{xz} = 0$ and therefore $\bar{y} = 0$. The other moments are

$$\begin{aligned} M_{yz} &= \iiint_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\rho\, dz\, dx\, dy \\ &= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3} \right]_{x=y^2}^{x=1} dy \\ &= \frac{2\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{3} \left[y - \frac{y^7}{7} \right]_0^1 = \frac{4\rho}{7} \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iiint_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho\, dz\, dx\, dy \\ &= \rho \int_{-1}^{1} \int_{y^2}^{1} \left[\frac{z^2}{2} \right]_{z=0}^{z=x} dx\, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy \\ &= \frac{\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{7} \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\tfrac{5}{7}, 0, \tfrac{5}{14} \right)$$ ■

Exercise

14. $\iiint_E e^{z/y}\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1, 0 \leqslant z \leqslant xy\}$

$$\iiint_E e^{z/y}\, dV = \int_0^1 \int_y^1 \int_0^{xy} e^{z/y}\, dz\, dx\, dy$$

$$= \int_0^1 \int_y^1 (ye^x - y)\, dx\, dy$$

$$= \int_0^1 (ey - y) - (ye^y - y^2)\, dy$$

32. $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$

$E = \{(x, y, z) \mid 0 \leq y \leq 2,\ 0 \leq z \leq 2-y,\ 0 \leq x \leq 4-y^2\}$