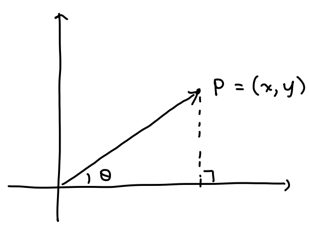
Thu 5/25

Double Integrals in Polar coordinates

- Review: Polar coordinate on plane

  Given a point P on $\mathbb{R}^2$, with classical coordinate (x,y)



consider $r = \sqrt{x^2+y^2}$: the distance from P to O

then $x = r\cos\theta$

$y = r\sin\theta$

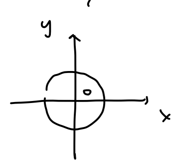
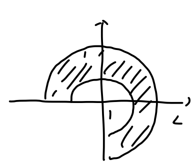
so $(x,y) \rightsquigarrow (r,\theta)$

Now given (r,θ), we can recover (x,y) by

$$x = r\cos\theta, \quad y = r\sin\theta$$

- Some figures are <u>much more easier</u> to express in Polar coordinate

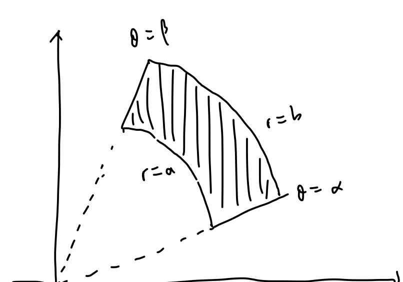
<u>Example</u> :

	xy - coordinate	Polar coordinate
unit cycle 	$x^2+y^2=1$	$r=1, \ 0 \le \theta \le 2\pi$
	$1 \le x^2+y^2 \le 4$ $x \ge 0$ or $y \ge 0$	$1 \le r \le 2, \ -\frac{\pi}{2} \le \theta \le \pi$

- <u>Double Integrals in Polar coordinates</u>

"rectangle" in Polar coordinate : $a \le r \le b, \quad \alpha \le \theta \le \beta \quad = D$

which is not a rectangle in xy-coordinates !



If given a continuous function f on the region D, we want to know the double integral:

$$\iint_D f(x,y)\, dA$$

you may think that in this way:

$$\iint_D f(x,y)\, dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, d\theta\, dr$$

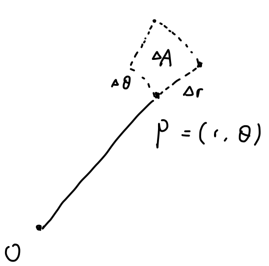
but this is wrong! because

$$dA = dx\, dy \neq dr\, d\theta$$

or to be more specific: $\Delta A = \Delta x \cdot \Delta y \neq \Delta r \cdot \Delta\theta$

<u>Question</u>: What's the relation between ΔA & $\Delta r \cdot \Delta\theta$?

<u>Ans</u>: Given small quantity Δr & $\Delta\theta$



$$\Delta A = \frac{(r+\Delta r)^2}{2}\Delta\theta - \frac{r^2}{2}\Delta\theta$$

$$= r\,\Delta r \cdot \Delta\theta + \frac{1}{2}(\Delta r)^2 \Delta\theta$$

since Δr is "super small", $(\Delta r)^2 \ll \Delta r$

$$\Rightarrow \Delta A \approx r \cdot \Delta r \cdot \Delta\theta$$

$$\Updownarrow$$

$$dA = r\, dr\, d\theta$$

i.e. the correct formula is:

$$\iint_D f(x,y)\, dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

2 Change to Polar Coodinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example:

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Step 1. change to polar coordinate: $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$

Step 2:

$$\iint_R (3x + 4y^2)\, dA = \int_1^2 \int_0^\pi (3r\cos\theta + 4r^2\sin^2\theta) r\, d\theta\, dr$$

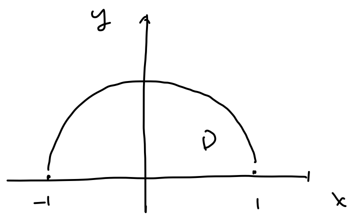
$$= \int_1^2 \int_0^\pi 2r^3 (1 - \cos 2\theta)\, d\theta\, dr$$

$$= 2\pi \int_1^2 r^3\, dr = 2\pi \times \frac{r^4}{4}\Big|_1^2 = \frac{15}{2}\pi$$

EXAMPLE 2 Evaluate the double integral

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

Step 1: Sketch the integral region



change to Polar coordinate: $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$

Step 2:

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^1 \int_0^\pi r^2 \cdot r\, d\theta\, dr$$

$$= \pi \cdot \frac{1}{4} = \frac{\pi}{4}$$

Classical:

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

$$= \int_{-1}^{1} \left(x^2 y + \frac{1}{3} y^3 \Big|_0^{\sqrt{1-x^2}} \right) dx$$

$$= \int_{-1}^{1} x^2\sqrt{1-x^2} + \frac{1}{3}(1-x^2)\sqrt{1-x^2}\, dx$$

$$= \int_{-1}^{1} \frac{1+2x^2}{3} \cdot \sqrt{1-x^2}\, dx$$

$$\underline{\underline{x = \cos\theta}} \; - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+2\cos^2\theta}{3} \cdot \sin^2\theta\, d\theta$$

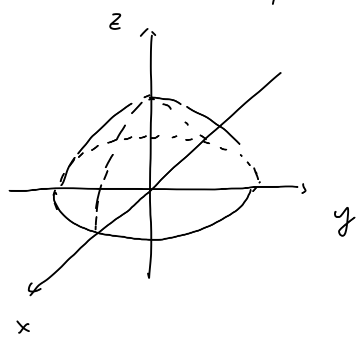
$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2+\cos 2\theta}{3} \cdot \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \cdots$$

EXAMPLE 3 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Step 1 : Sketch the region & the paraboloid

let $z = 0 \Rightarrow x^2 + y^2 = 1$



D: $x^2 + y^2 = 1$

in Polar: $0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

Step 2 : Set up the integral:

$$\text{Vol} = \iint_D (1 - x^2 - y^2)\, dA = \int_0^1 \int_0^{2\pi} (1 - r^2)\, r\, d\theta\, dr$$

$$= 2\pi \int_0^1 (r - r^3)\, dr = 2\pi \times \frac{1}{4} = \frac{\pi}{2}$$

$$\frac{1}{2}r^2 - \frac{1}{4}r^4$$

Example : Volume of a sphere :

Step 1 : the region: $0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

Step 2 : $\frac{\text{Vol}}{2} = \iint \sqrt{1 - x^2 - y^2}\, dA = \int_0^1 \int_0^{2\pi} \sqrt{1 - r^2}\, r\, d\theta\, dr$

$$= 2\pi \int_0^1 r\sqrt{1 - r^2}\, dr$$

$$= \pi \int_0^1 \sqrt{1 - r^2}\, dr^2 \overset{t = r^2}{=\!=} \pi \int_0^1 \sqrt{1 - t}\, dt$$

$$= -\frac{2\pi}{3} (1 - t)^{\frac{3}{2}} \Big|_0^1 = \frac{2\pi}{3}$$

$$\text{Vol} = \frac{4\pi}{3}$$

General Shape

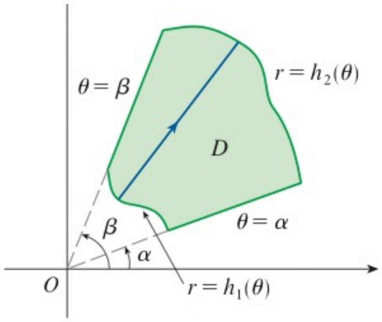


FIGURE 8
$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

What we have done so far can be extended to the more complicated type of region shown in Figure 8. It's similar to the type II rectangular regions we considered in Section 15.2. In fact, by combining Formula 2 in this section with Formula 15.2.4, we obtain the following formula.

3 If f is continuous on a polar region of the form

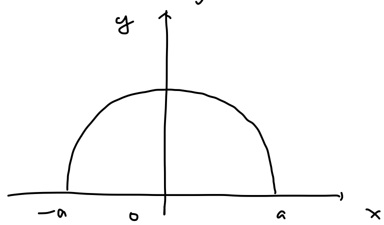
$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then $$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Exercise :

40. $\displaystyle\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x + y)\, dx\, dy$

Step 1 : Sketch the region



change to polor coordinate

$0 \leq r \leq a$

$0 \leq \theta \leq \pi$

then $\displaystyle\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\, dx\, dy = \int_0^a \int_0^{\pi} \left(2r\cos\theta + r\sin\theta\right) r \, d\theta\, dr$

$$= \int_0^a r^2 dr \cdot \int_0^{\pi} (2\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{a^3}{3} \cdot \left(2\sin\theta - \cos\theta \Big|_0^{\pi} \right)$$

$$= \frac{2a^3}{3}$$

50. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this t show that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

(d) By making the change of variable $t = \sqrt{2}\, x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

(a) use Polar coordinate

$D_a \sim 0 \le r \le a, \quad 0 \le \theta \le 2\pi$

$$\iint_{D_a} e^{-(x^2+y^2)}\, dA = \int_0^a \int_0^{2\pi} e^{-r^2} r\, d\theta\, dr$$

$$= 2\pi \int_0^a r e^{-r^2}\, dr$$

$$= 2\pi \cdot -\frac{1}{2} e^{-r^2}\Big|_0^a = \pi(1 - e^{-a^2})$$

then $I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \pi$

(b) $$\iint_{S_a} e^{-(x^2+y^2)}\, dA$$

$$= \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\, dx\, dy$$

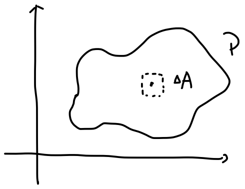
$$= \int_{-a}^{a} e^{-x^2} dx \cdot \int_{-a}^{a} e^{-y^2} dy$$

$$= \left(\int_{-a}^{a} e^{-x^2} dx\right)^2$$

Application of Double integrals

- Density and Mass

Given a lamina with mass,, define its density function to be



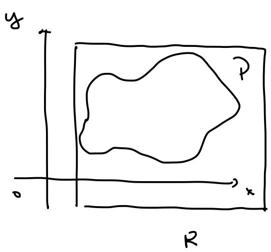
$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

suppose ρ is continuous on the region D

Goal: compute the mass

divide the region D into smaller part, Δ_{ij}, then choose $(x_{ij}^*, y_{ij}^*) \in \Delta_{ij}$

form the Riemann sum:



$$\sum_{i,j} \rho(x_{ij}^*, y_{ij}^*)\,|\Delta_{ij}|$$

taking limits, we will get: $m = \lim_{m,n\to\infty} \sum_{i,j} \rho(x_{ij}^*, y_{ij}^*)\,|\Delta_{ij}| = \iint_D \rho(x,y)\,dA$

- Moments and center of mass

Definitions: If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

3 $$\boxed{M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D y\,\rho(x,y)\,dA}$$

Similarly, the **moment about the *y*-axis** is

4 $$\boxed{M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D x\,\rho(x,y)\,dA}$$

As before, we define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus the lamina balances horizontally when supported at its center of mass (see Figure 4).

5 The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

<u>Example</u>: see Exercise 16

· Moment of inertia

Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments. We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large. The result is the **moment of inertia** of the lamina **about the x-axis**:

6
$$I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D y^2 \rho(x, y) \, dA$$

Similarly, the **moment of inertia about the y-axis** is

7
$$I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D x^2 \rho(x, y) \, dA$$

We also consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

8
$$I_0 = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D (x^2 + y^2) \rho(x, y) \, dA$$

Note that $I_0 = I_x + I_y$.

· Probability

Let X be a continuous random variable, the probability density function of X is a continuous function $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$, s.t.

$$\cdot \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\cdot P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Now we consider a pair of continuous random variables X & Y,

at random. The **joint density function** of X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is

$$P((X, Y) \in D) = \iint_D f(x, y)\,dA$$

In particular, if the region is a rectangle, then the probability that X lies between a and b and Y lies between c and d is

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d) = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Question: Suppose X has probability density function f_1,
Y has probability density function f_2.

What the relation of f_1 & f_2 and their joint density function f?

Ans: the relation is essentially the relation of these two variables X & Y

Some cases: $Y = 2X$, then

$$f_2(x) = \frac{1}{2} f_1(2x)$$

because $\int_a^b f_2(x)\,dx = \int_{\frac{a}{2}}^{\frac{b}{2}} f_1(x)\,dx$

If X & Y are independent!, which can be expressed as:

$$f(x,y) = f_1(x) f_2(y)$$

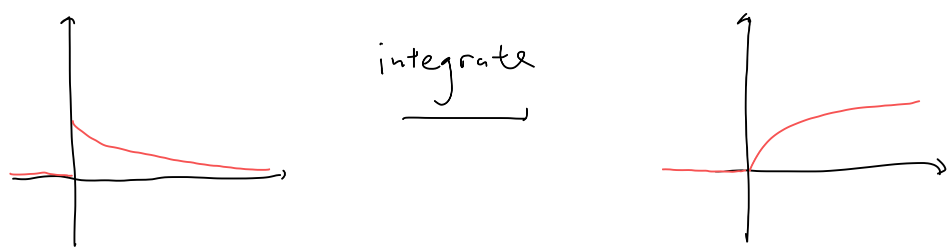
then X & Y has no relations at all!

Example: Waiting time model

In Section 8.5 we modeled waiting times by using exponential density functions

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \geqslant 0 \end{cases}$$

where μ is the mean waiting time. In the next example we consider a situation with two independent waiting times.



EXAMPLE 7 The manager of a movie theater determines that the average time moviegoers wait in line to buy a ticket for a film is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before taking his or her seat.

SOLUTION Assuming that both the waiting time X for the ticket purchase and the waiting time Y in the refreshment line are modeled by exponential probability density functions, we can write the individual density functions as

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10}e^{-x/10} & \text{if } x \geqslant 0 \end{cases} \qquad f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5}e^{-y/5} & \text{if } y \geqslant 0 \end{cases}$$

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x/10}e^{-y/5} & \text{if } x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

We are asked for the probability that $X + Y < 20$:

$$P(X + Y < 20) = P((X, Y) \in D)$$

where D is the triangular region shown in Figure 8. Thus

$$\begin{aligned} P(X + Y < 20) &= \iint_D f(x, y)\, dA = \int_0^{20}\int_0^{20-x} \frac{1}{50}e^{-x/10}e^{-y/5}\, dy\, dx \\ &= \frac{1}{50}\int_0^{20} \Big[e^{-x/10}(-5)e^{-y/5}\Big]_{y=0}^{y=20-x} dx = \frac{1}{10}\int_0^{20} e^{-x/10}(1 - e^{(x-20)/5})\, dx \\ &= \frac{1}{10}\int_0^{20} (e^{-x/10} - e^{-4}e^{x/10})\, dx = 1 + e^{-4} - 2e^{-2} \approx 0.7476 \end{aligned}$$

This means that about 75% of the moviegoers wait less than 20 minutes before taking their seats. ■

· Normal distribution

In the next example we deal with normal distributions. As in Section 8.5, a single random variable is *normally distributed* if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and σ is the standard deviation.

EXAMPLE 8 A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it. Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

SOLUTION We are given that X and Y are normally distributed with $\mu_1 = 4.0$, $\mu_2 = 6.0$, and $\sigma_1 = \sigma_2 = 0.01$. So the individual density functions for X and Y are

$$f_1(x) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(x-4)^2/0.0002} \qquad f_2(y) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(y-6)^2/0.0002}$$

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x)f_2(y) = \frac{1}{0.0002\pi}\, e^{-(x-4)^2/0.0002}e^{-(y-6)^2/0.0002}$$

5

$$= \frac{5000}{\pi}\, e^{-5000[(x-4)^2+(y-6)^2]}$$

A graph of this function is shown in Figure 9.

Let's first calculate the probability that both X and Y differ from their means by less than 0.02 cm. Using a calculator or computer to estimate the integral, we have

$$P(3.98 < X < 4.02,\, 5.98 < Y < 6.02) = \int_{3.98}^{4.02}\int_{5.98}^{6.02} f(x, y)\, dy\, dx$$

$$= \frac{5000}{\pi}\int_{3.98}^{4.02}\int_{5.98}^{6.02} e^{-5000[(x-4)^2+(y-6)^2]}\, dy\, dx$$

$$\approx 0.91$$

Then the probability that either X or Y differs from its mean by more than 0.02 cm is approximately

$$1 - 0.91 = 0.09$$ ■

· Expected values

Recall from Section 8.5 that if X is a random variable with probability density function f, then its *mean* is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Now if X and Y are random variables with joint density function f, we define the ***X*-mean** and ***Y*-mean**, also called the **expected values** of X and Y, to be

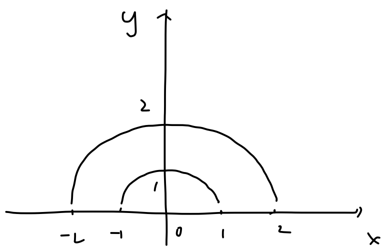
11 $$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$$

Exercise

15. The boundary of a lamina consists of the semicircles $y = \sqrt{1 - x^2}$ and $y = \sqrt{4 - x^2}$ together with the portions of the x-axis that join them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

16. Find the center of mass of the lamina in Exercise 15 if the density at any point is inversely proportional to its distance from the origin.

Sketch the region:



$$\rho(x,y) = K \cdot \frac{1}{\sqrt{x^2+y^2}}$$

then

$$m = \iint_D \rho(x,y)\, dA$$

Set up the integral: $M_x = \iint_D y\, \rho(x,y)\, dA$, $\quad M_y = \iint_D x\, \rho(x,y)\, dA$

Change to polar coordinate: $1 \le r \le 2, \quad 0 \le \theta \le \pi$

$$M_x = \int_1^2 \int_0^\pi r\sin\theta \cdot \frac{K}{r} \cdot r\, d\theta\, dr$$

$$= \int_1^2 \int_0^\pi K r \sin\theta\, d\theta\, dr$$

30. (a) Verify that

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint density function.

(b) If X and Y are random variables whose joint density function is the function f in part (a), find

(i) $P(X \geqslant \frac{1}{2})$ (ii) $P(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2})$

(c) Find the expected values of X and Y.

(a)

- $f(x, y) \geqslant 0$
- $\iint_{\mathbb{R}^2} f(x,y) = \int_0^1 \int_0^1 4xy \, dx \, dy = 1$

(b) (i) $P(X \geqslant \frac{1}{2}) = \int_0^1 \int_{\frac{1}{2}}^1 4xy \, dx \, dy$

(ii) $P(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 4xy \, dx \, dy$

(c)

X: $\int_0^1 \int_0^1 x \cdot 4xy \, dx \, dy$

Y: $\int_0^1 \int_0^1 y \cdot 4xy \, dy \, dx$