① Mon 5

Introductory lecture, MSRI Summer Grad Program

Dave Bayer

Welcome! Algorithms in algebra and geometry (digress on fog)

What is this about?

linear algebra & Gaussian elimination

- essential throughout pure & applied math

algebraic geometry & Gröbner bases

- higher degree analogue
- algorithms relatively new, 70's →
- becoming essential to applications of commutative algebra to other fields

Basic setup: k a field

$S = k[x_1, \dots, x_n]$ a polynomial ring

given an ideal $I \subset S$, Gröbner bases give deformations $I \rightsquigarrow \text{in}(I)$ to "leading term" ideals generated by monomials

$$x^a = x_1^{a_1} \dots x_n^{a_n} \in S \quad \left((a_1, \dots, a_n) \in \mathbb{N}^n \right)$$

Can think of $\text{in}(I) = (x^a, x^b, \dots) \subseteq S$ either as an ideal, or as the set $J \subseteq \mathbb{N}^n$, [illegible]

$$J := \{ c \in \mathbb{N}^n \mid c \geq a \text{ for some generator } x^a \text{ of } \text{in}(I) \}$$

↖ partial order corresponds to divisibility $x^a \mid x^c$.

J is a union of "orthants" $\mathbb{N}^n$ translated by a for generators ~~a~~ x^a of $\text{in}(I)$.

Construction reduces geometric, algebraic questions to combinatorics.

<u>Example</u> twisted cubic curve is image of map

$\mathbb{A}^2_k \hookrightarrow \mathbb{A}^4_k$ affine spaces over k

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ taking projective spaces of lines through origins

$(s,t) \to (s^3, s^2t, st^2, t^3)$

pull polynomial functions on $\mathbb{P}^3$ back to $\mathbb{P}^1$ by restriction, using this parametrization:

$$k[s,t] \longleftarrow k[a,b,c,d] \quad (a,b,c,d) \text{ coords on } \mathbb{P}^3$$

$$\rotatebox{90}{=} S$$

$$\begin{aligned} s^3 &\longleftarrow a \\ s^2t &\longleftarrow b \\ st^2 &\longleftarrow c \\ t^3 &\longleftarrow d \end{aligned}$$

<u>kernel</u> of this ring homomorphism

$$I = (\underline{\underline{b}}^2 - ac, \underline{\underline{bc}} - ad, \underline{\underline{c}}^2 - bd) \subseteq S$$

is ideal of polynomials which vanish on twisted cubic.

Take <u>term ordering</u> "reverse lex" (avoid d, then avoid c, ..) then underlined terms $\underline{\underline{b}}^2$, $\underline{\underline{bc}}$, $\underline{\underline{c}}^2$ are leading terms, these generators are a Gröbner basis (won't always be so lucky), and

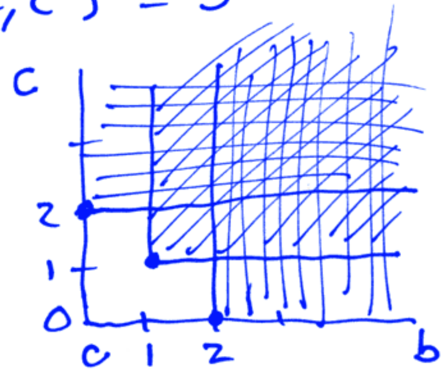
$$in(I) = (b^2, bc, c^2) \subseteq S$$

2D picture of J:

$b^2 \Rightarrow (2,0)$

$bc \Rightarrow (1,1)$

$c^2 \Rightarrow (0,2)$



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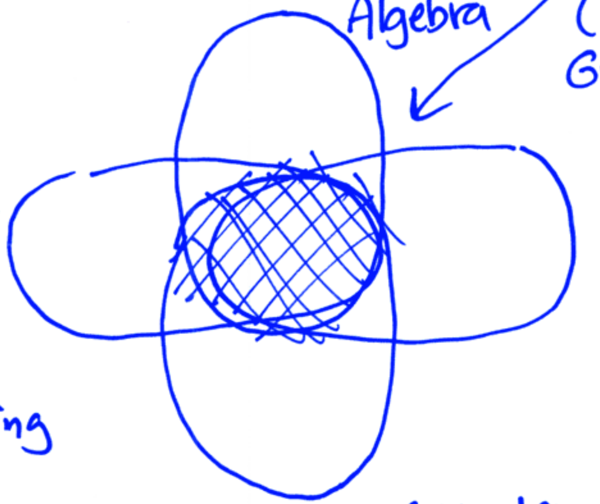
A dozen years ago, would have gone on very differently…

Focus on objects that have interpretations in several fields, and their interplay:



In turn, leads back to new ideas in algebra.

return to Example, introduce many actors we will see later…

Ideal of twisted cubic very special binomial ideal (toric ideal):

Two monomials in $k[a,b,c,d]$ are equiv mod I

$\Leftrightarrow$ they have same multidegree w.r.t. pair of gradings

$$\begin{array}{l|cccc} & a & b & c & d \\ \hline \text{grading } 1 & 3 & 2 & 1 & 0 \\ \text{grading } 2 & 0 & 1 & 2 & 3 \end{array} \longleftarrow \text{cols are exponents of } s^3, s^2t, st^2, t^3$$

$$\underbrace{b^2-ac}_{\deg=(4,2)}, \quad \underbrace{bc-ad}_{(3,3)}, \quad \underbrace{c^2-bd}_{(2,4)}$$

$\Leftrightarrow$ difference of their exponent vectors belongs to subgroup (lattice) $\ker(A) \subseteq \mathbb{Z}^4$

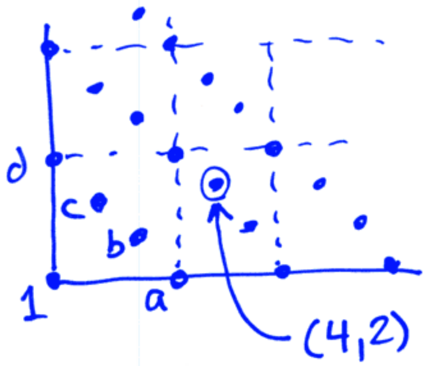
where $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$\left.\begin{array}{lcl} b^2 - ac & \Rightarrow & (-1,2,-1,0) \\ bc - ad & \Rightarrow & (-1,1,1,-1) \\ c^2 - bd & \Rightarrow & (0,-1,2,-1) \end{array}\right\} \text{all belong to } \ker(A)$$

$\Leftrightarrow$ they map to same monomial in subring

$$R = k[\underbrace{s^3, s^2t, st^2, t^3}_{\text{subring generators}}] \subseteq k[s,t]$$

corresponding to <u>semigroup</u>



} picture of monomials in subring

$b^2 = ac$

To alg geometer, the function

$f(n) = \dim (S/I)_n$ ← homog deg n part of quotient

tells a great deal about zero locus defined by I.

It is a polynomial that all similar sets share.

How do we find these dims?

(n now degree, not # vars)

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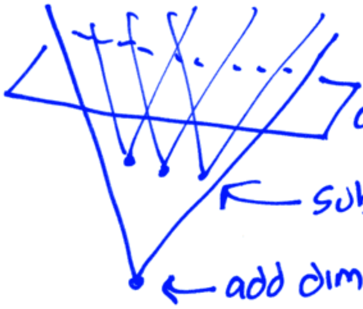
Syzygies:

$$\dim k[a,b,c,d]_n = \binom{n+3}{3}$$

n	0	1	2	3	4	…
$\binom{n+3}{3}$	1	4	10	20	35	…



Then what?
These spaces overlap!

1	1	4	10	20	35
b^2-ac			−1	−4	−10
$bc-ad$			−1	−4	−10
c^2-bd			−1	−4	−10
$0 = d(b^2-ac) - c(bc-ad) + b(c^2-bd)$				1	4
$0 = c(\;"\;) - b(\;"\;) + a(\;"\;)$				1	4
	1	4	7	10	13 …

$f(n) = 3n+1$

Most enlightening way to rewrite $f(n)$:

ring $k[a]$ for $\mathbb{P}^0$ has fn $\binom{n+0}{0} = 1$

" $k[a,b]$ " $\mathbb{P}^1$ " $\binom{n+1}{1} = n+1$

" $k[a,b,c]$ " $\mathbb{P}^2$ " $\binom{n+2}{2}$

$f(n) = 3\mathbb{P}^1 - 2\mathbb{P}^0$

twisted cubic

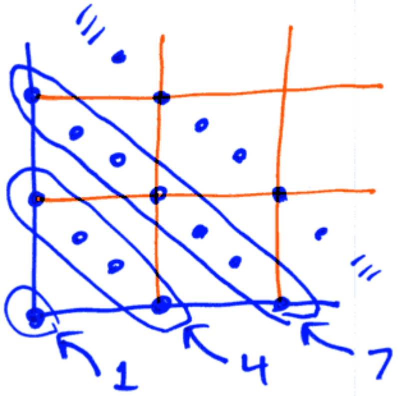
looks like

3 lines meeting along 2 points

How can we make same count combinatorally?

1) we did this the hard way!

$$S/I \cong k[s^3, s^2t, st^2, t^3]$$



Ehrhart polynomial of lattice points inside a cone.
(convex polyhedral cone, in general)

2) Theorem (Macaulay, 1927)

$$\dim (S/I)_n = \dim (S/\mathrm{in}(I))_n$$

$f(n)$ = # monomials, deg n in a, b, c, d, not a multiple of $\underset{①}{b^2}$, $\underset{②}{bc}$, or $\underset{③}{c^2}$.

inclusion/exclusion counting:

			0	1	2	3	4	n
+	∅	1	1	4	10	20	35	
−	{1}	b^2			−1	−4	−10	
−	{2}	bc			−1	−4	−10	
−	{3}	c^2			−1	−4	−10	
+	{1,2}	b^2c				+1	+4	
+	{1,3}	b^2c^2					+1	
+	{2,3}	bc^2				+1	+4	
−	{1,2,3}	b^2c^2					−1	
			1	4	7	10	13	…

$1 - b^2 - bc - c^2 + b^2c + bc^2$ encodes, even gets multigrading right.

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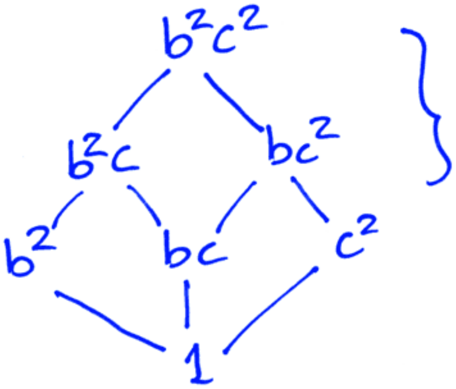
combinatorists distill this information for sharper ways to count:

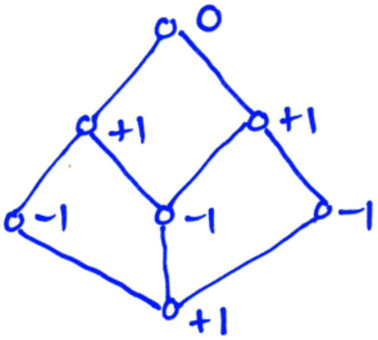
<u>Def</u> partially ordered set (poset)
set with $\prec$ sometimes defined, behaves like usual

our poset: lcm's of generators of in(I), $\prec \Leftrightarrow$ divisibility



} rising path whenever elements are comparable $\prec$

Contribution to count only depends on order relation:



related to <u>Möbius inversion</u> on poset

we form topological space (simplicial complex) from poset:

Given a pair $x < y$,

- automatically include the empty set $\emptyset$
- introduce a point for every z, $x < z < y$
- introduce a line segment $\overline{wz}$ for every $x < w < z < y$
- " " triangle " ...

and glue these all together to make a topological space,

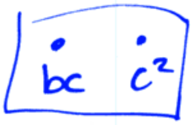
$$\Delta(x,y)$$

$\Delta(1,b^2) = \Delta(1,bc) = \Delta(1,c^2) = \{\emptyset\}$ $\tilde{\chi} = -1$

$\Delta(1,b^2c) = \{\emptyset, b^2, bc\} =$

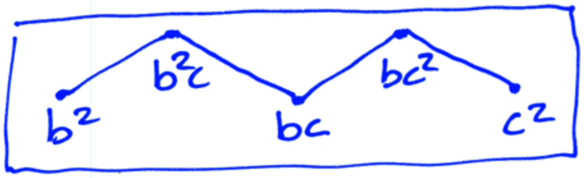
$\tilde{\chi} = 1$

$\Delta(1,bc^2) =$



$\tilde{\chi} = 1$

$\Delta(1,b^2c^2) =$



$\tilde{\chi} = 0$ contractible

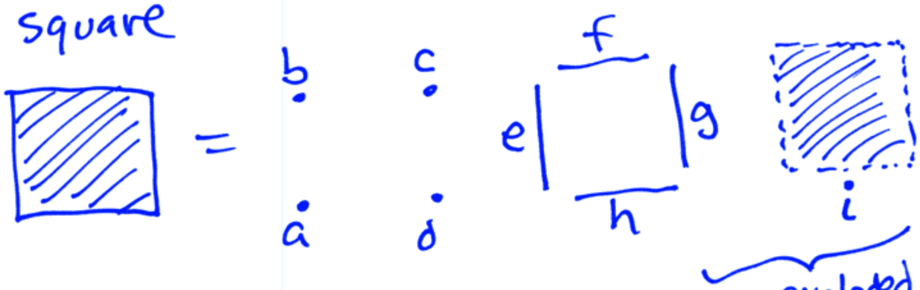
reduced Euler characteristic $\tilde{\chi} = -\#$ empty sets (1)
$+ \#$ points
$- \#$ line segments
$+ \#$ triangles ...

↑ best choice for combinatorics, algebra

So our contributions were $-\tilde{\chi}(x,y)$

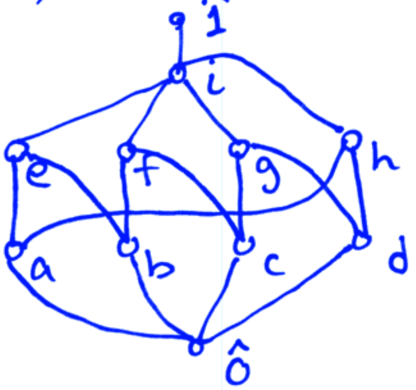
Can think of any poset with min elem $\hat{0}$, max elem $\hat{1}$ as standing for the topological space $\Delta(\hat{0},\hat{1})$

Example square

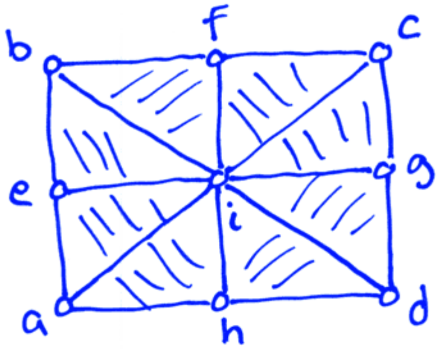


exploded view of faces

poset of faces, under inclusion



$\Delta(\hat{0}, \hat{1}) =$



homeomorphic to original square (barycentric subdivision)

we think of any poset, even not a face poset, as standing for a topological space

$\Rightarrow$ Can apply topology whenever we see an order relationship.

Many ways to get order relationships in combinatorial commutative algebra.

Get to recognize and apply duality, homology theories from topology to algebra.

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Face rings and Alexander duality I

Recall an abstract simplicial complex X is defined as follows:

$\Delta :=$ all subsets of $\{1, \dots, n\}$

$X \subseteq \Delta$ is a simplicial complex

$\Leftrightarrow$ X is closed under taking subsets, i.e.

$F \in X$ and $G \subset F \Rightarrow G \in X$

Any time we recognize such a subset relationship "in nature", we recognize a simplicial complex, and anticipate effects related to Alexander duality.

example: geometric realization (union of simplices)

Let $|X| \subseteq \mathbb{R}^n_{\geq 0}$ be the set of points $p \in \mathbb{R}^n_{\geq 0}$ so

1) $\sum p_i = 1$

2) $\mathrm{supp}(p) \in X$, as a subset of $\{1, \dots, n\}$

We recover X from $|X|$ by taking the vertex sets of standard simplices $\subseteq |X|$, and including the empty set $\emptyset$ whenever $|X|$ is nonempty. (When $|X|$ is empty we are puzzled.)

- Advantages: (co)homology of this topological space agrees under any usual theory with (co)homology in simplicial sense. Easy to draw, in low dimensions.
- Disadvantages: Belief this is only legitimate example impedes recognizing simplicial complexes in algebra.

Can't tell whether empty set $\emptyset$ belongs, when $|X|$ has no points.

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example: <u>linear</u> <u>realization</u> (union of coordinate subspaces)

Let $\|X\| \subseteq \mathbb{R}^n$ be the set of points $p \in \mathbb{R}^n$ so

1) $\mathrm{supp}(p) \in X$, as a subset of $\{1, \ldots, n\}$

We recover X from $\|X\|$ by taking the standard bases of coordinate subspaces $V \subseteq \|X\|$, and including the empty set $\emptyset$ whenever $\vec{0} \in \|X\|$.

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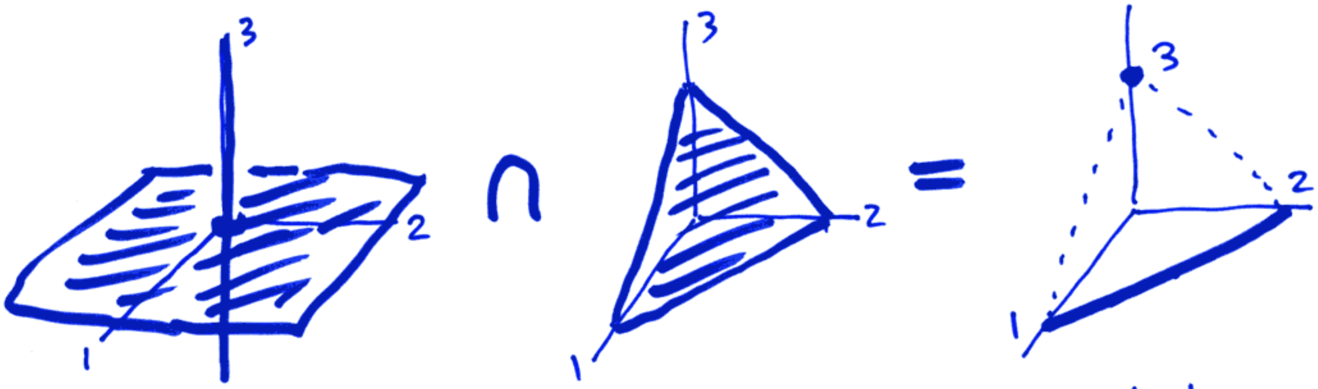
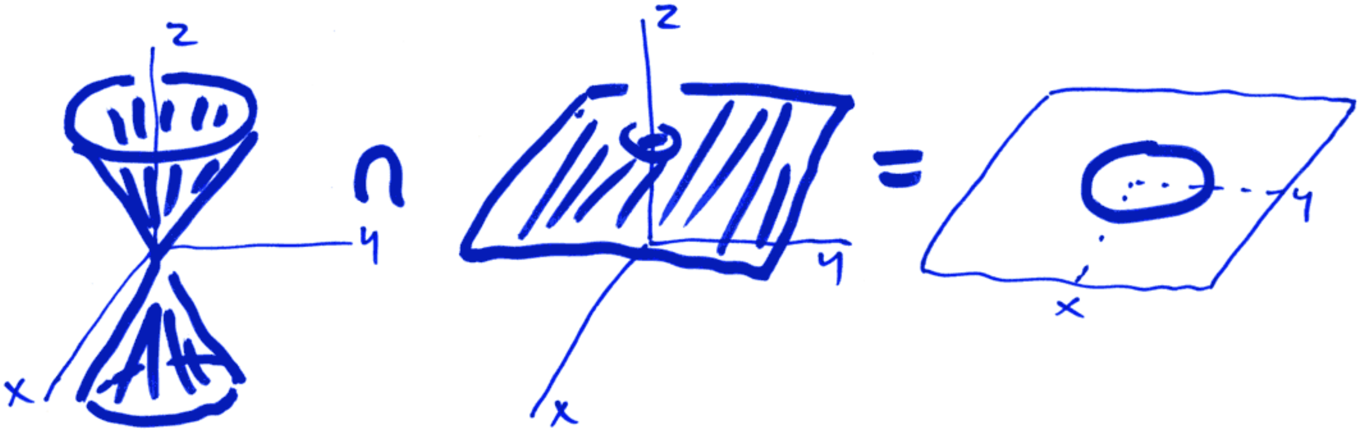
example: $X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\} \subseteq 2^{\{1,2,3\}}$



$\|X\|$ $|\Delta|$ $|X|$

compare with usual projective -vs- affine setup in geometry

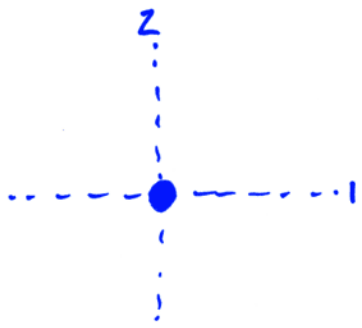


lines through $\vec{0}$ forming a projective curve

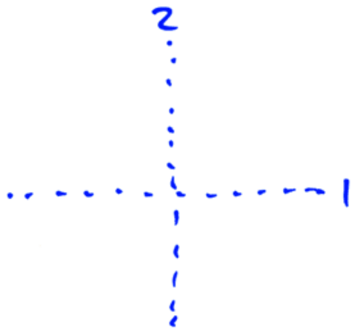
$\{z = 1\}$ hyperplane

affine curve

example: $$\left.\begin{array}{ll} X = \{\emptyset\} & \text{empty complex} \\ Y = \{\ \} & \text{void complex} \end{array}\right\} \text{ in } 2^{\{1,2\}}$$



$\|X\|$



$\|Y\|$

$|X|, |Y|$,
white cow in snowstorm
(indistinguishable)

Note that ideals $I, J \subseteq k[x_1, x_2]$ of polys vanishing on $\|X\|, \|Y\|$ are

$I = (x_1, x_2)$ "irrelevant" ideal
$J = (1)$ "unit" ideal

carefully distinguished in projective geometry.

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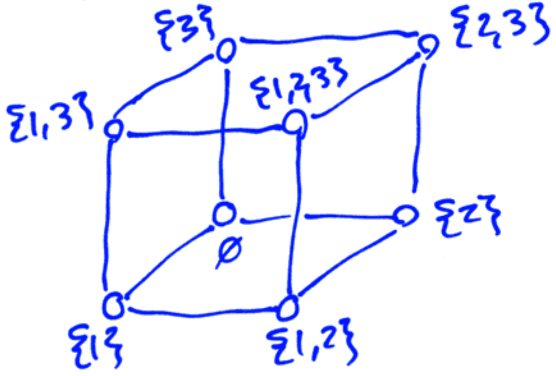
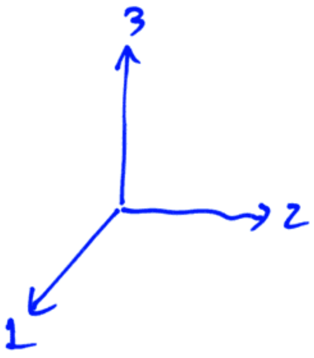
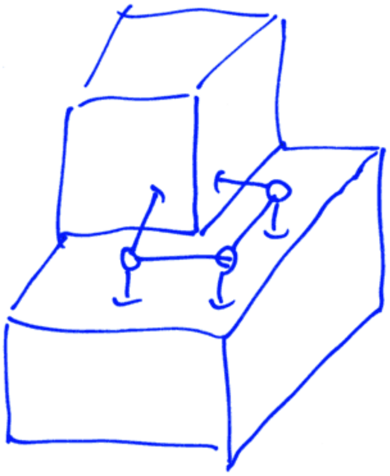
example: corners We live in too few dimensions for it to be obvious that corners are classified by simplicial complexes, but...

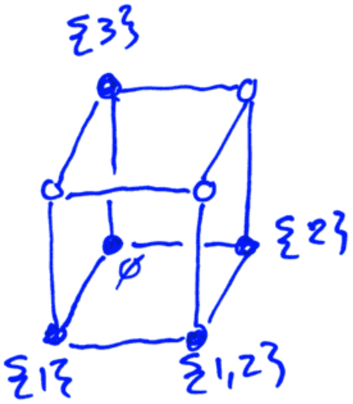
identify vertices of hypercube with subsets of coordinates





now center a hypercube (so labeled) around a corner, and read off the subsets on the ∅ side of the corner:





recovers

X = {∅, 1, 2, 3, 12}

This is a 1:1 correspondence corners :: simplicial complexes

Alexander duality

Theorem X $\overset{\text{huh?}}{\underbrace{\text{proper}}}$, $\overset{\text{you've got to be kidding!}}{\underbrace{\text{nonempty}}}$ subset of the sphere S^{n-2}. suppose pair (S^{n-2}, X) is triangulable. Then

$$\tilde{H}_i(X;G) \cong \tilde{H}^{n-i-3}(S^{n-2} \setminus X; G)$$

$$\tilde{H}^i(X;G) \cong \tilde{H}_{n-i-3}(S^{n-2} \setminus X; G)$$

ex:

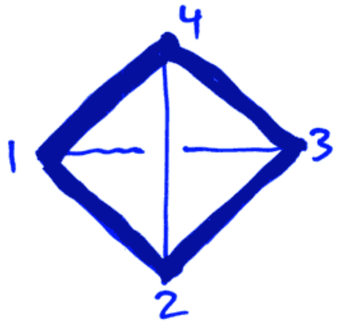
$X \subseteq S^2$ $(n=4)$

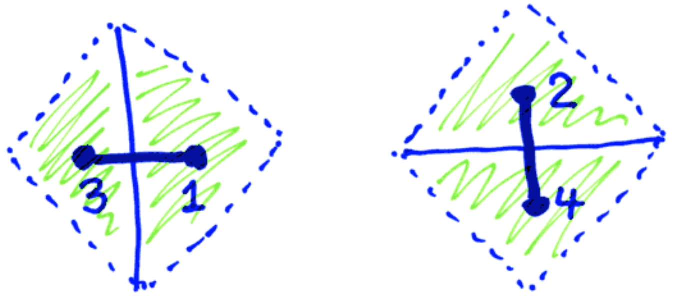
$\dim \tilde{H}_1(X) = 1$

$S^2 \setminus X$

$\dim \tilde{H}^0(x) = 1$

Same example, simplicially:





$X \subseteq \Delta \setminus \{1,2,3,4\}$
$\cong S^2$

$S^2 \setminus X$

$X = \{\emptyset, 1, 2, 3, 4,$
$12, 23, 34, 14\}$

$X^\vee = \{\emptyset, 1, 2, 3, 4, 13, 24\}$

$\Updownarrow$ complement in Δ

$\Updownarrow$ complement each subset in $\{1,\dots,n\}$

$X^c = \Delta \setminus X = \{1234, 234, 134, 124, 123, 24, 13\}$

$X^{\vee\vee} = X$, and this works for any simplicial complex:

(empty) $\{\emptyset\}^\vee = \Delta \setminus \{1,..,n\} \cong S^{n-2}$

$\dim \tilde{H}_{-1}(\{\emptyset\}) = 1 \quad \dim \tilde{H}^{n-2}(S^{n-2}) = 1$

(void) $\{\ \}^\vee = \Delta$, both acyclic

~~examples (corners)~~

Theorem (simplicial Alexander duality):

$X \subseteq \Delta$ simplicial complex, $X^\vee$ its dual. Then

$$\tilde{H}_i(X;G) \cong \tilde{H}^{n-i-3}(X^\vee;G)$$

$$\tilde{H}^i(X;G) \cong \tilde{H}_{n-i-3}(X^\vee;G)$$

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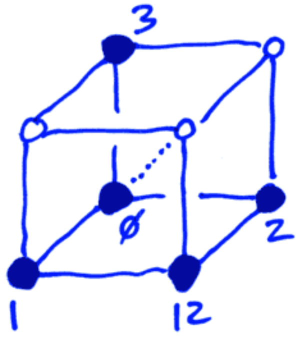
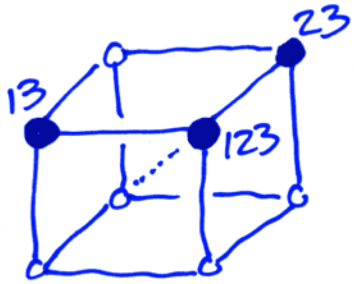
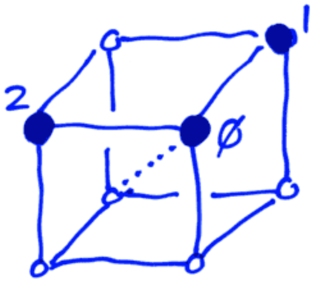
example (corners, continued





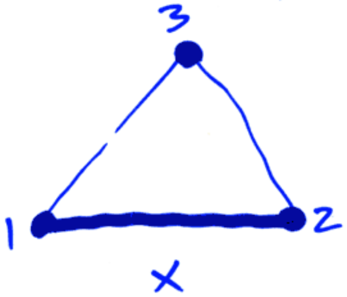


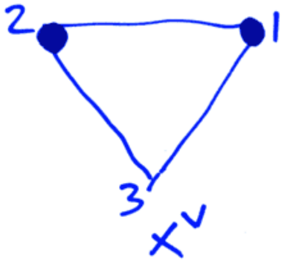
$X = \{\emptyset, 1, 2, 3, 12\}$

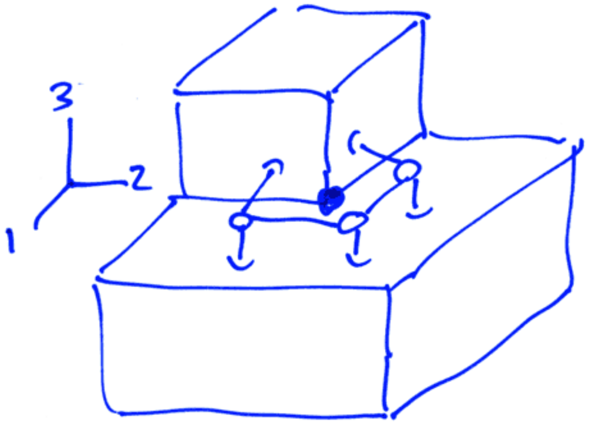
complement subsets in $\{1,2,3\}$

complement X in Δ

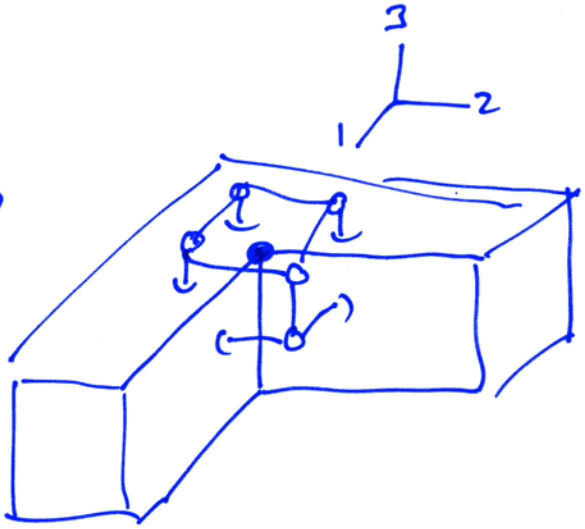
$X^\vee = \{\emptyset, 1, 2\}$



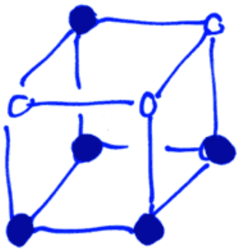




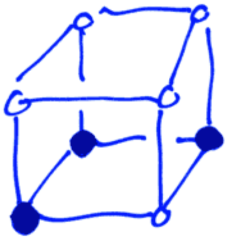
$\Longrightarrow$



dual corners
(viewed from opposite sides)



$X = \{\emptyset, 1, 2, 3, 12\}$



$X^\vee = \{\emptyset, 1, 2\}$

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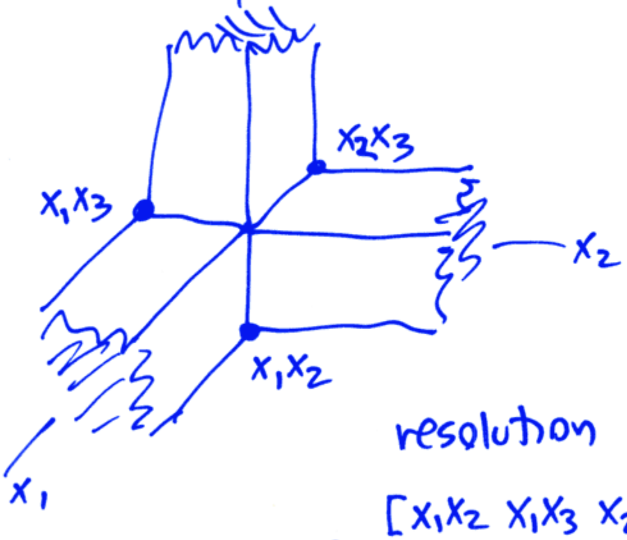
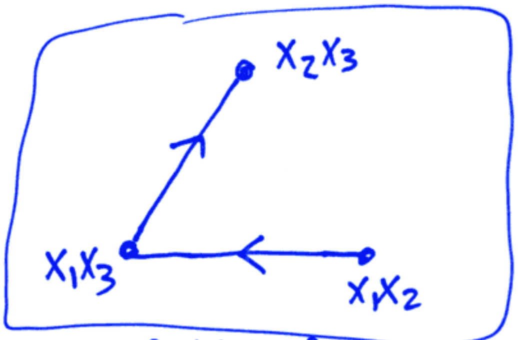
example monomial ideal $I = (x_1x_2, x_1x_3, x_2x_3) \subseteq k[x_1,x_2,x_3]$

draw complement of I (picture of S/I):





picture of resolution

resolution is

$$0 \leftarrow I \xleftarrow{[x_1x_2\ \ x_1x_3\ \ x_2x_3]} S^3 \xleftarrow{\begin{bmatrix}-x_2 & 0\\ x_3 & -x_2\\ 0 & x_1\end{bmatrix}} S^2 \leftarrow 0$$

$\beta_{0,(1,1,0)} = \beta_{0,(1,0,1)} = \beta_{0,(0,1,1)} = 1$

$\beta_{1,(1,1,1)} = 2$

now clear inside interior of I, $K_b = \Delta$, all $\beta_i = 0$

$K_{(1,1,0)} = \{\emptyset\}$ and $\dim H_{-1}(\{\emptyset\}) = 1$

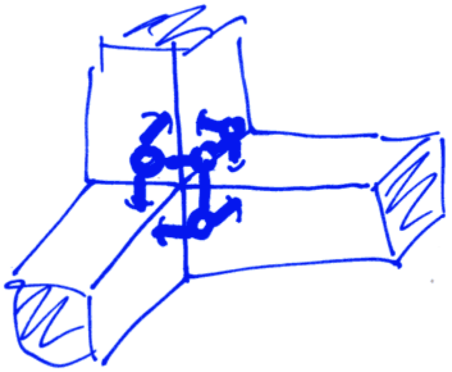
usual corner we are made to sit in.

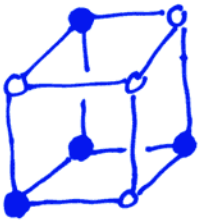
$\dim H_{-1}(K_b) = 1$

$X = \{\emptyset, 1, 2, 3, 12, 13, 23\}$

$X^\vee = \{\emptyset\} = K_b \quad b = (1,1,0)$

$K_{(1,1,1)} = \{\emptyset, 1, 2, 3\}$





$X = X^\vee = \{\emptyset, 1, 2, 3\}$

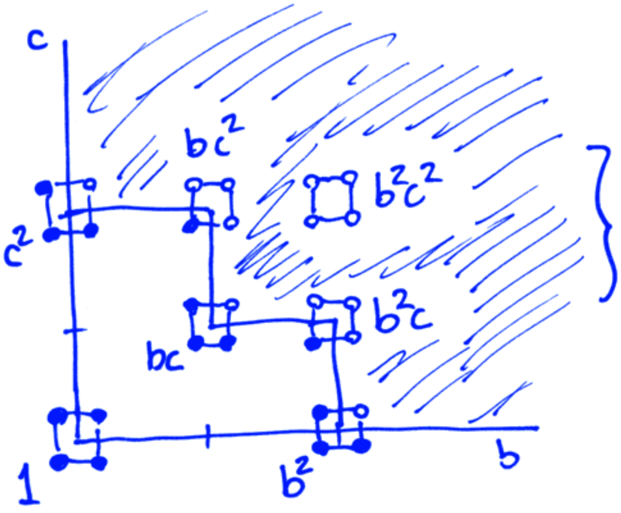
$X^\vee = K_b, \quad b = (1,1,1)$

$\dim H_0(K_b) = 2$

⑧ Tue 6, Bayer

Return to familiar example, derived from twisted cubic curve.

$I = (b^2, bc, c^2) \subseteq S = K[b,c]$ (we ignore a,d from twisted cubic, so we can draw pictures)

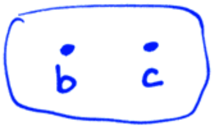


shaded region represents I as solid subset of $\mathbb{R}^n$

1  $K_{(0,0)} = \{\ \}$ acyclic

b^2  $K_{(2,0)} = \{\emptyset\}$, $\dim \tilde{H}_{-1}(\{\emptyset\}) = 1$, $\beta_{0,(2,0)} = 1$

b^2c  $K_{(2,1)} = \{\emptyset, 1, 2\}$, $\dim \tilde{H}_0(\cdot\ \cdot) = 1$, $\beta_{1,(2,1)} = 1$



2 points

b^2c^2  $K_{(2,1)} = \Delta$ acyclic

Betti numbers of a monomial ideal are supported on corners on boundary between I and complement.

To get characteristic dependence, build a corner whose corresponding simplicial complex exhibits characteristic dependence.

① Wed, Bayer ~~8~~7 July 98

Face rings and Alexander duality II

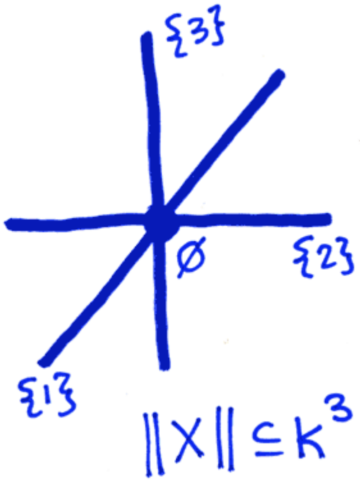
Construction: Given an abstract simplicial complex $X \subseteq \Delta = 2^{\{1,\dots,n\}}$, take its linear realization $\|X\|_k$ over a field k as the union of the corresponding coordinate subspaces, ~~and define~~ in k^n, and define

$$I_X \subseteq k[x_1,\dots,x_n] = S$$

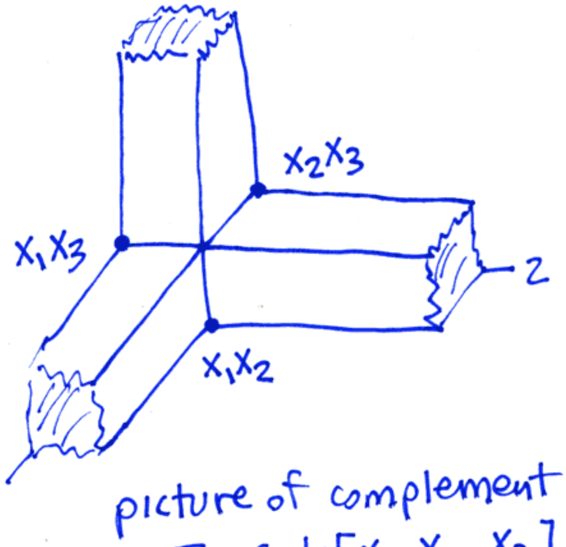
to be the ideal of polynomials vanishing on $\|X\|_k$. Then I_X is a squarefree monomial ideal generated by the minimal nonfaces of X:

$$x^a \text{ is a min gen of } I_X$$
$$\Leftrightarrow \text{supp}(a) \notin X, \text{ but any proper subset} \in X$$

Example: $X = \{\emptyset, \{1\}, \{2\}, \{3\}\} \subseteq \Delta = 2^{\{1,2,3\}}$



$\|X\| \subseteq k^3$



picture of complement of $I_X \subseteq k[x_1,x_2,x_3]$

Note we can see the simplicial complex formed by the pattern of monomials left out of I.

② Bayer, Wed 7

I_X is the Stanley-Reisner ideal of X, and S/I_X is the face ring of X.

- Every squarefree monomial ideal can be interpretted in this way.
- As we see from the picture, the nonzero monomials in S/I_X are precisely the monomials supported on faces of X:

$$x^a \notin I_X \iff \overset{\text{evaluate on } p}{x^a} \neq 0 \text{ for some } p \in \|X\|$$

(recall $\mathrm{supp}(p) \in X$)

$$\iff \mathrm{supp}(a) \subseteq \mathrm{supp}(p)$$

$$\iff \mathrm{supp}(a) \in X$$

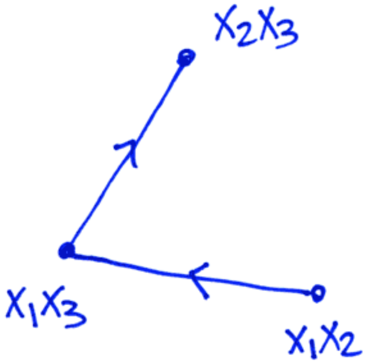
It follows that

x^a is a min generator of I_X

$\iff$ supp(a) is a minimal nonface of I_X

We want to understand this characterization in the language of combinatorial topology, and generalize it (due to Hochster) to the syzgies of I_X.

resolution is: (for our example)



$$0 \leftarrow I \xleftarrow{[x_1x_2 \;\; x_1x_3 \;\; x_2x_3]} S^3 \xleftarrow{\begin{bmatrix} -x_2 & 0 \\ x_3 & -x_2 \\ 0 & x_1 \end{bmatrix}} S^2 \leftarrow 0$$

1	–	–
–	3	2

③ Bayer, Wed 7

Recall $K_b = \{ F \in \Delta \mid x^{b-F} \in I \}$
(simplicial complex of directions <u>down</u> from x^b which stay in ideal I)

and that $\dim \widetilde{H}_{i-1}(K_b) = \beta_{i,b} = \#\ i^{th}$ syzygies of degree b

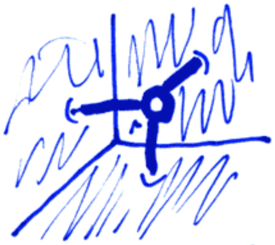
---

This is the simplicial complex one sees directly, studying a degree b multigraded strand of Tor

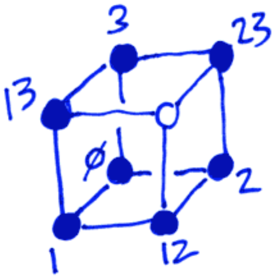
---

For our example,

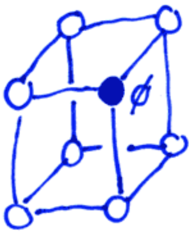
$K_{(1,1,0)} = \{\emptyset\}$ $\quad \dim \widetilde{H}_{-1}(\{\emptyset\}) = 1 = \beta_{0,(1,1,0)}$ $\quad x_1x_2$



usual corner we are made to sit in.



$K_b^\vee$



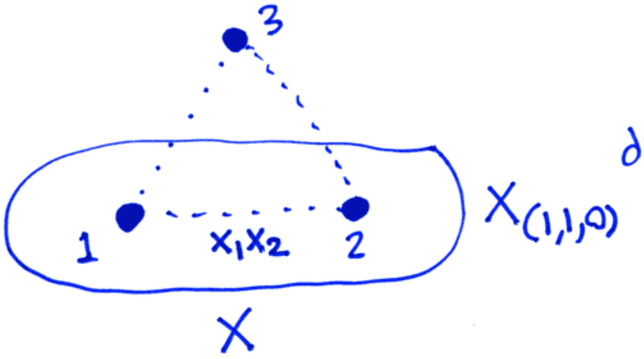
← we look down from this vertex

K_b

more interesting to dualize on <u>support</u> of $b = (1,1,0)$

K_b

$K_b^\vee = X_b$ full subcomplex of X on vertices in support of b



$\dim \widetilde{H}_0(X_{(1,1,0)}) = \beta_{0,(1,1,0)} = 1$

by Alexander duality

Theorem (Hochster) For $I_X \subseteq k[x_1,\dots,x_n]$ the Stanley-Reisner ideal of a simplicial complex X, we have $\beta_{i,b}=0$ unless $b\in\{0,1\}^n$, in which case

$$\beta_{i,b} = \dim \tilde{H}_{|b|-i-2}(X_b;k)$$

proof: Compute $\beta_{i,b}$ directly from K_b, and apply Alexander duality with respect to the support of b. //

In general, x^b is a minimal gen of I_X

$\iff$ $F = \operatorname{supp}(b)$ is a min nonface of X

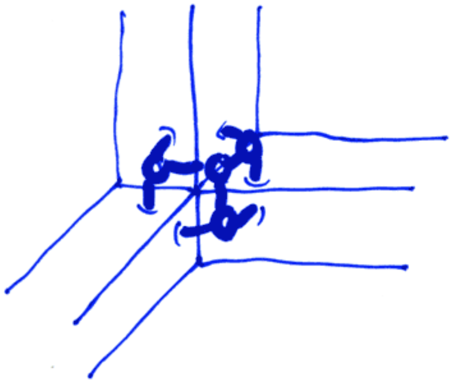
$\iff$ ~~so~~ $F \notin X$ but every facet of $F \in X$

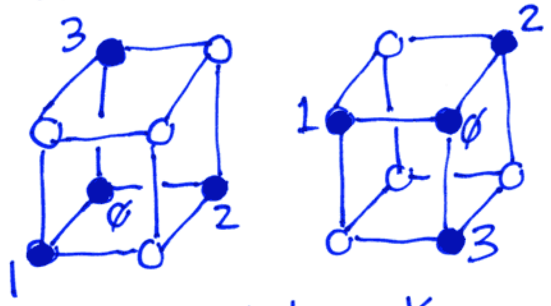
$\iff$ X_b is a $(|b|-2)$-sphere

$\iff$ $\tilde{H}_{|b|-0-2}(X_b) \cong k$

$\iff$ $\beta_{0,b} = 1$

For our example, $K_{(1,1,1)} = \{\emptyset, 1, 2, 3\}$ $\quad b=(1,1,1)$





X_b dual to K_b

$X_b = X =$


$\dim \tilde{H}_0(X_b) = 2 = \beta_{1,(1,1,1)}$

⑤ Bayer, Wed 7

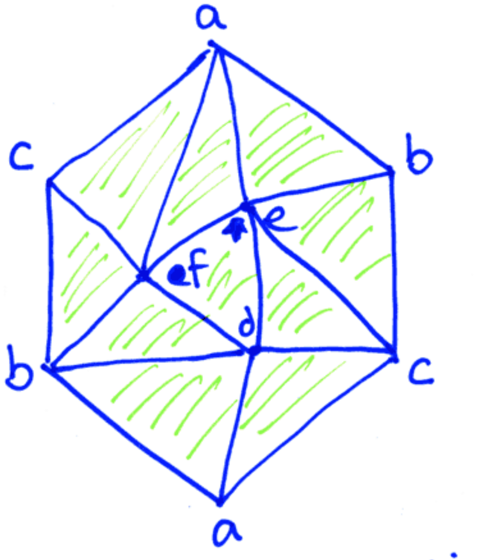
From Hochster's formula we get Macaulay Betti diagram for minimal free resolution of S/I_X:

$$\begin{array}{|lllll}
1 & - & - & - & \\
- & h_0(X_{..}) & h_0(X_{\therefore}) & h_0(X_{::}) & \cdots \\
- & h_1(X_{\therefore}) & h_1(X_{::}) & h_1(X_{:\cdot:}) & h_1(X_{:::}) \\
- & h_2(X_{::}) & h_2(X_{:\cdot:}) & h_2(X_{:::}) & \\
 & \equiv & & &
\end{array}$$

$h_i = \dim \tilde{H}_i$ $\qquad X_{:\cdot:}$ = full subcomplex on $\#(:\cdot:)$ points (we range over all possibilities)

In particular, we get an extremal syzygy from a top homology class of X itself.

Example ($\mathbb{RP}^2$)



(identify opposite vertices)

6 vertices a, b, c, d, e, f

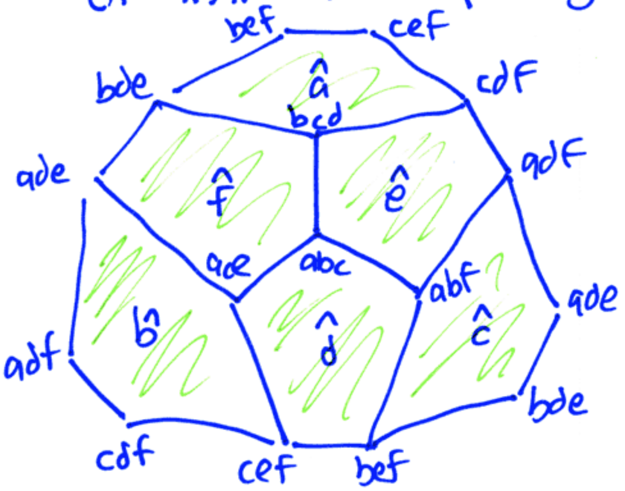
15 edges $\binom{6}{2} = 15$, so all edges

10 triangles $\binom{6}{3} = 20$, half of them

I = (the remaining 10 deg 3 products)

⑥ Bayer, wed7

The dual in $\mathbb{RP}^2$ of this triangulation is a cellular decomposition of $\mathbb{RP}^2$ as 6 pentagons



Does this support a cellular resolution of S/I_X ?

Obstruction is total space

$\mathbb{RP}^2$: $\tilde{H}_i(\mathbb{RP}^2;k) \cong k$ in char 2, $i=1,2$

$\cong 0$ otherwise

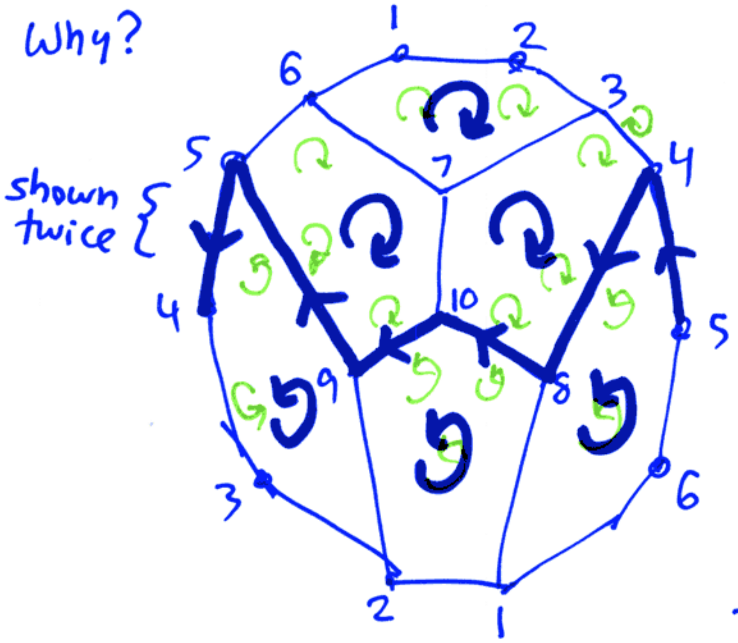
1	–	–	–
–	–	–	–
–	10	15	6

in char ≠ 2

1	–	–	–	–
–	–	–	–	–
–	10	15	6	1
–	–	–	1	–

in char 2

Why?



Orient 6 faces so boundary cancels to twice path shown.

In char 2

- these faces form a cycle so $\tilde{H}_2 \cong k$
- the path is not a boundary, so $\tilde{H}_1 \cong k$

In char ≠2 they cancel.

toric & lattice ideals

We define and study a class of ideals generalizing the defining ideals of projective toric varieties as studied in algebraic geometry. These ideals arise in diverse fields.

A *lattice* is a subgroup $L \subseteq \mathbb{Z}^n$, under $+$.

L is *saturated* $\Leftrightarrow$ L consists of all integer lattice points in its span in $\mathbb{R}^n$.

Otherwise, let $L^{sat} = \text{span}(L) \cap \mathbb{Z}^n$. Then L^{sat}/L is a group of finite order. We say L has finite *index* in its saturation.

Let L act on $\mathbb{Z}^n$ by translation. The orbits of this action are cosets of L in $\mathbb{Z}^n$.

Say two points $a, b \in \mathbb{N}^n$ are equivalent mod L

$$\begin{aligned} a \sim b &\Leftrightarrow a = b \text{ in } \mathbb{Z}^n/L \\ &\Leftrightarrow a, b \text{ are in same orbit} \\ &\Leftrightarrow a - b \in L \end{aligned}$$

The lattice ideal I_L expresses this equivalence relation in terms of monomials $x^a, x^b \in k[x_1, \dots, x_n]$:

$$x^a - x^b \in I_L \Leftrightarrow a - b \in L$$

② THUR 9, Bayer

We identify $\mathbb{N}^n$ with the monomials in $k[x_1,\dots,x_n]$ and think of S as the <u>semigroup algebra</u>

$$S = k[x_1,\dots,x_n] = k[\mathbb{N}^n]$$

Then

$$S/I_L = k[\underbrace{\mathbb{N}^n/L}]$$

this semigroup describes monomials in S/I_L, and gives mult rule (additively).

Conversely, every monomial subring

$$k[x^{a_1},\dots,x^{a_n}] \subseteq k[x_1,\dots,x_d]$$

can be modeled in this way:

Write

$$\begin{tikzcd} \mathbb{N}^n \arrow[r, "A"] & \mathbb{N}^d \end{tikzcd}$$

$$e_i = (0,..,1,..,0) \mapsto a_i$$

so the image of A is the semigroup generated by $a_1,\dots,a_n$ in $\mathbb{N}^d$. Then

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

is a $d \times n$ matrix giving a map $A: \mathbb{Z}^n \to \mathbb{Z}^d$, and $L = \ker(A)$ is a <u>saturated</u> <u>lattice</u> $L \subseteq \mathbb{Z}^n$.

We get

$$k[x^{a_1},\dots,x^{a_n}] \cong k[\mathbb{N}^n/L]$$

$$\cong k[y_1,\dots,y_n]/I_L$$

Example (twisted cubic)

The map

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, hook] & \mathbb{P}^3 \\ (s,t) \arrow[r, mapsto] & (s^3, s^2t, st^2, t^3) \end{tikzcd}$$

induces the ring map

$$\begin{tikzcd} k[s,t] & k[a,b,c,d] \arrow[l] \\ s^3 & a \arrow[l, mapsto] \\ s^2t & b \arrow[l, mapsto] \\ st^2 & c \arrow[l, mapsto] \\ t^3 & d \arrow[l, mapsto] \end{tikzcd}$$

with image the monomial subring $k[s^3, s^2t, st^2, t^3] \subseteq k[s,t]$.

Then $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ $\quad \mathbb{N}^2 \xleftarrow{A} \mathbb{N}^4$

defines the lattice $\quad L = \ker(A)$

$$L = \langle (-1,2,-1,0), (-1,1,1,-1) \rangle \subseteq \mathbb{Z}^4$$

and

$$I_L = \langle b^2 - ac,\ bc - ad,\ \underbrace{c^2 - bd}_{\substack{\text{minimal ideal generator,} \\ \text{redundant as lattice generator}}} \rangle \subseteq k[a,b,c,d]$$

is the ideal defining the twisted cubic curve in $\mathbb{P}^3$.

Caveats:

- Finding an integral basis for $\ker(A)$, considering A as a map $\mathbb{R}^n \to \mathbb{R}^d$, not the same as finding a lattice basis. Lattice generated by given basis might have finite index in desired answer. Use e.g. Smith normal form.
- Finding a lattice basis not the same as finding generators for I_L, as seen above.

④ Thurs 9, Bayer

a geometric view of binomial ideals:

Let $I = (x^{b_1} - x^{c_1}, \dots, x^{b_\ell} - x^{c_\ell}) \subseteq k[x_1, \dots, x_n]$

be an arbitrary ~~monomial~~ binomial ideal.

Could be strikingly poorly behaved: [Mayr, Meyer]

$$x^b - x^c \overset{?}{\in} I$$

can require double exponential degree to express if true, giving worst bounds known for general complexity of ideal membership, Gröbner bases, ... (not nec. binomial).

Noncommutative version is undecideable, famous word problem.

On the other hand, let

$$G \subseteq \mathbb{A}^n \setminus \{\text{coordinate hyperplanes}\}$$

be the set of points $g = (g_1, \dots, g_n)$ with all $g_i \neq 0$ on which I vanishes.

For any binomial $x^a - x^b \in I$ and any two points $g, h \in G$

if $gh := (g_1 h_1, \dots, g_n h_n)$ then

$$g^a = g^b \text{ and } h^a = h^b \implies (gh)^a = g^a h^a = g^b h^b = (gh)^b$$

if $g^{-1} := (g_1^{-1}, \dots, g_n^{-1})$ then

$$g^a = g^b \implies (g^{-1})^a = (g^a)^{-1} = (g^b)^{-1} = (g^{-1})^b$$

and for $\underline{1} = (1, \dots, 1)$,

$$\underline{1}^a = \underline{1}^b = 1.$$ Thus G is an abelian group.

What is the structure of G? [Sturmfels, Eisenbud]

Define a lattice $L \subseteq \mathbb{Z}^n$ from I

$$L := \langle a-b \mid x^a - x^b \in I \rangle$$

and also consider

$$L^{sat} = \operatorname{span}(L) \cap \mathbb{Z}^n$$

We can find a matrix $A : \mathbb{Z}^n \longrightarrow \mathbb{Z}^d$ so $L^{sat} = \ker(A)$, giving a rational parametrization of the component $G_{\underline{1}}$ through $\underline{1} = (1, \dots, 1)$ of G:

$$k[x_1, \dots, x_n] \longrightarrow k[y_1^{\pm 1}, \dots, y_d^{\pm 1}]$$

$$x^b \longmapsto y^{Ab}$$

$$\mathbb{A}^n \supseteq G \supseteq G_{\underline{1}} \longleftarrow \mathbb{A}^d \setminus \{\text{coord hyperplanes}\}$$

Also, $G/G_{\underline{1}} \cong L^{sat}/L$ describes the distinct components of G.

I differs from I_L precisely by the primary components supported on the coordinate hyperplanes of $\mathbb{A}^n$.

- To compute I_L from I, carry out an algebraic procedure corresponding to erasing these primary components supported on coordinate hyperplanes.
- To compute I_L from L, define

$$I = (x^{a_1} - x^{b_1}, \dots, x^{a_\ell} - x^{b_\ell})$$

for a basis $\{a_1 - b_1, \dots, a_\ell - b_\ell\}$ of L, and proceed as above.

Finally, an algorithm! How do we remove components?

Familiar effect:

Let $\{f_1, \ldots, f_\ell\}$, $\{g_1, \ldots, g_m\}$ be two generating sets for an inhomogeneous ideal $I \subseteq k[x_1, \ldots, x_n]$ defining an affine variety (zero locus) $X \subset \mathbb{A}^n$.

Homogenizing with a new variable x_0

$$\bar{f} = x_0^{\deg(f)} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

(mult each term by power of x_0, bringing all terms up to same deg)

then $I_1 = (\bar{f}_1, \ldots, \bar{f}_\ell)$,

$I_2 = (\bar{g}_1, \ldots, \bar{g}_m) \subseteq k[x_0, \ldots, x_n]$

can define distinct projective varieties $X_1, X_2 \subseteq \mathbb{P}^n$

but $X_1 \cap \mathbb{A}^n = X_2 \cap \mathbb{A}^n = X$.

Difference is (again) primary components supported on hyperplane at infinity $\{x_0 = 0\}$.

$\{f_1, \ldots, f_\ell\}$ is a homogenizing basis for I

$\Leftrightarrow$ $\bar{I} = (\bar{f}_1, \ldots, \bar{f}_\ell)$ defines $\bar{X}$ = closure of X in $\mathbb{P}^n$

Any reduced Gröbner basis for a degree-respecting term order on $k[x_1, \ldots, x_n]$ gives a homogenizing basis for I.

In other words, Gröbner bases can erase components (here, those supported at ∞)

⑦ ThuB 9, Bayer

Algebraic view: $I \subset S$, $f \in S$

$$(I : f^\infty) := \langle g \mid f^m g \in I \text{ for some } m \rangle$$

is the ideal formed from I by dividing by f whenever and as much as possible.

Suppose I is in fact primary ideal with associated prime P.

If $f \in P \Rightarrow f^m \in I$ for some m

$\Rightarrow 1 \in (I : f^\infty) \Rightarrow (1) = (I : f^\infty)$

$f \notin P \Rightarrow$ if $f^m g \in I$ then $g \in I$

$\Rightarrow I = (I : f^\infty)$

This operation commutes with primary decomposition, so in general it erases all primary components supported on $\{f = 0\}$, leaving rest intact.

Now take a homogeneous ideal $I \subseteq k[x_0, .., x_n]$ term order which avoids x_0 (sorts first by degree in $x_1, .., x_n$) $(I : x_0^\infty)$ can be computed by dividing whereever possible by x_0 in a Gröbner basis for I.

Given $f \in k[x_1, .., x_n]$ form $k[x_0, .., x_n]/(x_0 - f)$ and apply above to compute $(I : f^\infty)$

$$\boxed{I_L = (I : (x_1 \cdots x_n)^\infty)}$$

or erase one variable at a time

⑧ Thu 3 9, Bayer

Example (twisted cubic)

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \qquad \mathrm{Ker}(A) = L = \langle (-1,2,-1,0), (-1,1,1,-1) \rangle$$

$$I = (b^2 - ac,\ bc - ad)$$

Variety $X \subseteq \mathbb{P}^3$ defined by I consists of twisted cubic

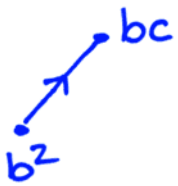
$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ (s,t) &\longmapsto (s^3, s^2t, st^2, t^3) \end{aligned}$$

and a line

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ (s,t) &\longrightarrow (0,0,s,t) \end{aligned}$$

(a or b appears in each term of gens of I)

We compute a Gröbner basis avoiding a to erase on $\{a=0\}$:



$$b(bc) - c(b^2) = 0$$

$$b(bc - ad) - c(b^2 - ac) = ac^2 - abd$$

dividing by a, get remaining generator

$c^2 - bd$ of I_L.

① Fri 10, Bayer

We want to study syzygies of lattice ideals, applying principles we learned from monomial case:

Stick to special case:

$R = k[y^{a_1}, \dots, y^{a_n}] \subseteq k[y_1, \dots, y_m]$ subring generated by monomials all of the same degree d.

$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$ m×n matrix, columns exponent vectors of generating monomials.

$L = \ker(A) \subseteq \mathbb{Z}^n$ lattice kernel of A

$I_L \subseteq k[x_1, \dots, x_n] =: S$ defined by

$I_L := \langle x^b - x^c \mid b - c \in L \rangle$

we have

$$S/I_L = k[x_1, \dots, x_n]/I_L \xrightarrow{\sim} k[y^{a_1}, \dots, y^{a_n}] = R$$

$$x_i \longmapsto y^{a_i}$$

$$x^b \longmapsto y^{Ab}$$

sanity check:

$$x^b \equiv x^c \mod I_L$$

$$\Leftrightarrow \quad y^{Ab} = y^{Ac}$$

$$\Leftrightarrow \quad Ab = Ac$$

$$\Leftrightarrow \quad b - c \in \ker(A) = L$$ ☑

② Fri 10, Bayer

Study syzygies. We recall for monomial ideals I that $\mathrm{Tor}_i(I,k)$ worked nicely. Here,

$$\mathrm{Tor}_i(I_L,k) \quad \text{or} \quad \mathrm{Tor}_i(S/I_L,k) \quad ??$$

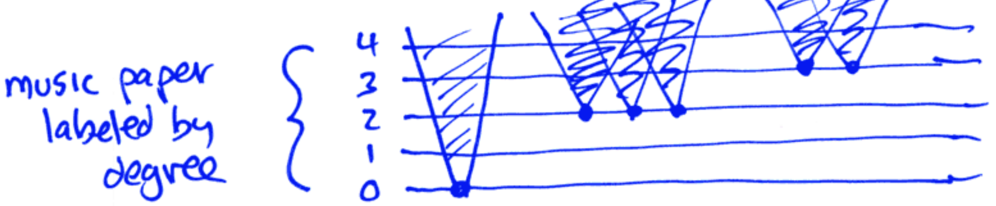
Hmm. We see two term expressions where? First syzygies of monomial ideals, or generators of binomial ideals. So lets choke up on the bat, and think of $1 \in S/I_L$ as its generating monomial.

How did $\mathrm{Tor}_i(S/I_L,k)$ go?

$k \cong S/(x_1,\dots,x_n)$ and $\otimes k$ lops off anything which is an x_j-multiple

e.g. twisted cubic, we compute $\mathrm{Tor}_*(S/I_L,k)$ by resolving S/I_L and $\otimes k$; then taking homology

$$0 \leftarrow S/I \leftarrow S \longleftarrow S^3 \longleftarrow S^2 \leftarrow 0$$



$$\Downarrow \quad \otimes_S k \qquad \text{(Kerchunk!)}$$

$$\begin{tikzcd}[row sep=small]
\text{deg } 3 & 0 & 0 \arrow[l] & 0 \arrow[l] & k^2 \arrow[l] & 0 \arrow[l] \\
2 & 0 & 0 \arrow[l] & k^3 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] \\
1 & 0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] \\
0 & 0 & k \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

there is a multigraded picture we can't draw

Single graded betti numbers: $\beta_{0,0}=1 \quad \beta_{1,2}=3 \quad \beta_{2,3}=2$

$$\beta_{i,d} = \dim \mathrm{Tor}_i(S/I_L,k)_d$$

③ Fri 10, Bayer

- We recall $\mathrm{Tor}_*(S/I_L, k)$ can also be computed by resolving $k \cong S/(x_1, \dots, x_n)$, and $\otimes S/I_L$, then taking homology

"Koszul complex" is just Taylor resolution of $(x_1, \dots, x_n)$ which we think of as simplicial resolution on the full simplex:

- Yoga here is: stare at a single multigraded strand, and recognize homology as simplicial homology of some simplicial complex. Figure out a way of saying it so we remember what that simplicial complex was!

Here goes:

$$0 \leftarrow k \leftarrow S \leftarrow S^n \leftarrow S^{\binom{n}{2}} \leftarrow \cdots \leftarrow S^{\binom{n}{n-1}} \leftarrow S \leftarrow 0$$

$$(\quad '' \quad '' \quad '' \qquad\qquad '' \quad '' \quad) \otimes_S S/I_L$$

(This generally is where one pauses, repeats "huh?" as a mantra, ...)

Ahh... What does $\otimes S/I_L$ really do?

$\otimes k$ lops off a bunch of stuff, but $(x_1, \dots, x_n)$ was a monomial ideal.

$\otimes S/I_L$ identifies monomials that map to the same monomial in $R = k[y^{a_1}, \dots, y^{a_n}]$

$$\boxed{\text{So} \quad S^\ell \otimes_S S/I_L \ \text{ turns } \ S^\ell \ \text{ into } \ R^\ell}$$

④ Fri 10, Bayer

we want homology of

$$0 \leftarrow R \leftarrow R^n \leftarrow R^{\binom{n}{2}} \leftarrow \dots \leftarrow R^n \leftarrow R \leftarrow 0$$

Since R is monomial subring of $k[y_1, \dots, y_m]$, everything is m-graded, and we can consider a single m-graded degree $\underline{b} \in \mathbb{N}^m$.

Look for simplicial homology. Two questions:

- what is "alive" in $R^{\binom{n}{i}}$ in degree $\underline{b}$?
- what do the maps look like?

$$R^{\binom{n}{i}} = \bigoplus R_F, \text{ summing over all subsets } F \subseteq \{x_1, \dots, x_n\}, \quad |F| = i$$

Think also of F as a 0-1 vector, picking out the subset.

In our m-grading, each x_i has degree a_i

the subset F has degree AF.

the summand R_F starts in degree AF

i.e. at the monomial y^{AF}

So $\dim (R_F)_{\underline{b}} = 1$ if y^{AF} divides y^b in R

$\iff y^{b-AF} \in R$

$= 0$ otherwise.

~~Looks familia~~

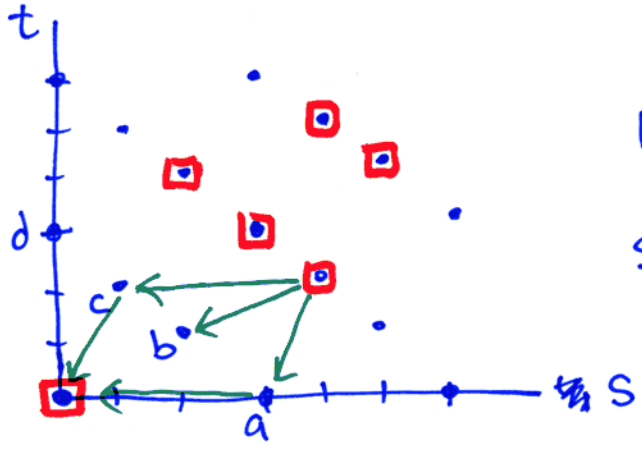
Looks familiar... for a monomial ideal,

$$K_b := \{ F \in \Delta \mid x^{b-F} \in I \}$$

So R replaces monomial ideal, A translates x-steps into y-steps

⑤ Fri 10, Bayer

Getting confusing w/o an example in mind, return to simplest example that could possibly help:



twisted cubic

$R = k[s^3, s^2t, st^2, t^3] \subseteq k[s,t]$

12

$S/I_L = k[a,b,c,d] \Big/ \begin{pmatrix} b^2-ac, \\ bc-ad, \\ c^2-bd \end{pmatrix}$

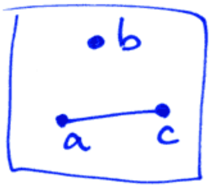
$\beta_{0,(0,0)} = 1$ counts generator of $S/I_L \cong R$

$K_{(0,0)} = \{\emptyset\}$ because $(0,0) \in$ semigroup but stepping down by a,b,c ord leaves.

$\tilde{H}_{-1}(\{\emptyset\}) \cong k$ picks up this Betti #.

$\beta_{1,(4,2)} = 1$ counts generator b^2-ac of I_L

$K_{(4,2)} = \{\emptyset, a, b, c, ac\} =$



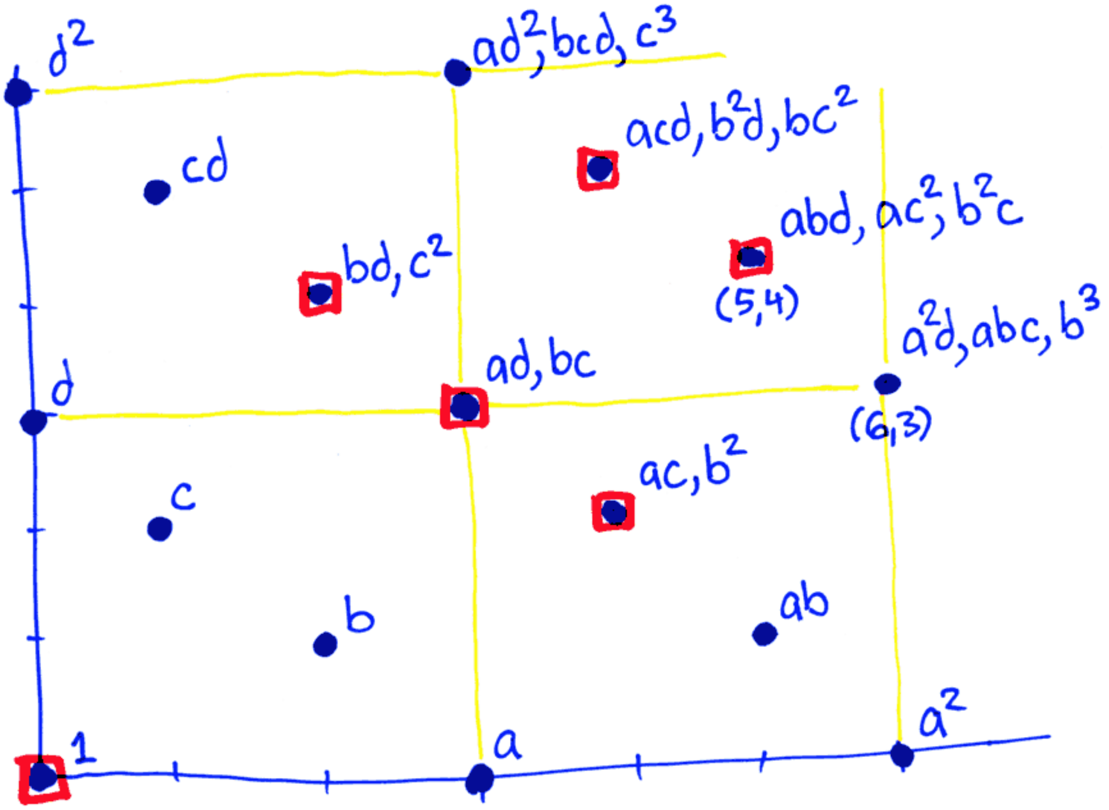
because these steps stay in semigroup.

$\tilde{H}_0(K_{(4,2)}) \cong k$ ☑

$\beta_{2,(5,4)} = 1$ counts syzygy $c(b^2-ac) - b(bc-ad) + a(c^2-bd) = 0$

$K_{(4,2)} = \{$ huh. There must be a better way to keep track...

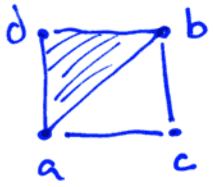
⑥ Fri 10, Bayer



List ways of getting around, by Pascal-triangle style recursion.

Square free supports of these monomials give maximal faces (facets) of K_b (and some nested faces)

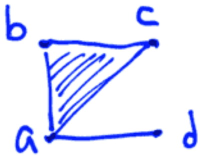
$K_{(5,4)}$ =



$\tilde{H}_1 \cong k$ ☑

from $\{abd, ac^2, b^2c\}$

$K_{(6,3)}$ =



$\tilde{H}_1 = 0$ ☑

from $\{a^2d, abc, b^3\}$

One checks that maps worked out right, so

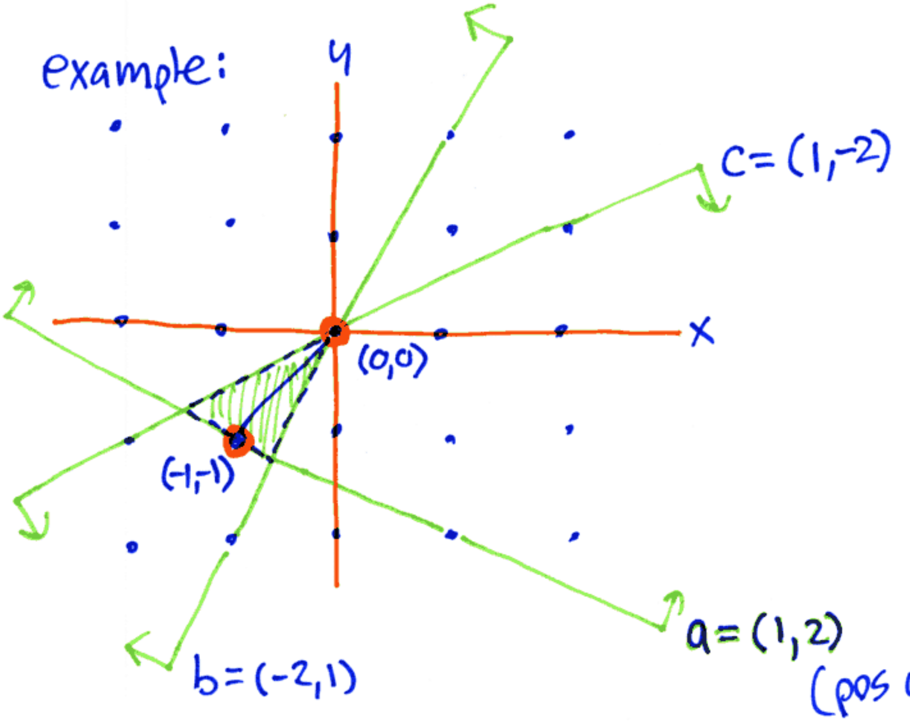
$$\boxed{\beta_{i,b} = {}_{\dim}\mathrm{Tor}_i(S/I_L, k) = \dim \tilde{H}_{i-1}(K_b)}$$

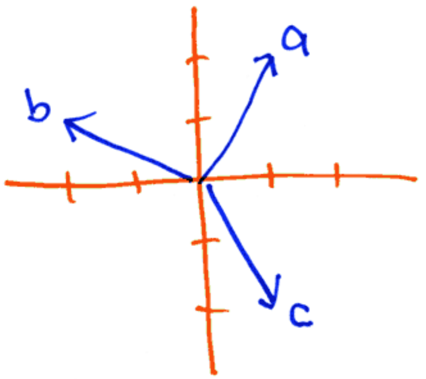
① Mon 13 July, Bayer

integer programming

Conti & Traverso
Thomas
Sturmfels

example:





$3a+4b+5c=0$

(pos coefs ⟺ bounded region)

Minimize $(0,-1)\cdot(x,y)$ given $(x,y)\in\mathbb{Z}^2$

constraints
$$\begin{bmatrix}1 & 2\\ -2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\geq\begin{bmatrix}-3\\ 0\\ 0\end{bmatrix}$$

and initial feasible point $(-1,-1)$

How to model using toric ideals?

slack variables

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}1 & 2\\ -2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}-\begin{bmatrix}-3\\ 0\\ 0\end{bmatrix}\geq\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

- Then $(a,b,c) \in \mathbb{N}^3$, can only move by changing

$$\begin{aligned} x &\leftrightarrow (1,-2,1) \\ y &\leftrightarrow (2,1,-2) \end{aligned}$$

- Initial point $(-1,-1)$ maps to $(a,b,c)=(0,1,1)$
- Find a weight vector $\omega=(0,1,2)$ which is multiple of cost $(0,-1)$ in (x,y) coords.

Define $L = \langle (1,-2,1),(2,1,-2) \rangle \subseteq \mathbb{Z}^3$

feasible points of IP 1:1 with orbit of $(0,1,1)$ in $\mathbb{N}^3$ under action of L. Want to find point minimizing $(0,1,2)\cdot(a,b,c)$

Reinterpret as Gröbner basis problem.

Define I_L as lattice ideal $\langle x^u - x^v \mid u-v \in L \rangle$ as usual.

We work in $k[a,b,c]$:

$$I = (b^2 - ac,\ c^2 - a^2 b) \text{ from lattice gens}$$

$$(I : abc^\infty) = (\text{"}, \text{"}, bc - a^3) = I_L$$

Now define term order:

$>$
- sort first by degree w.r.t. $\omega=(0,1,2)$
- break ties by reverse lex (avoid c)

$$\text{Gröbner basis} = \{ \underset{2}{\underline{\underline{b^2}}} - \underset{\substack{2\\ \uparrow \\ \text{avoid } c}}{ac},\ \underset{4}{\underline{\underline{c^2}}} - \underset{1}{a^2 b},\ \underset{3}{\underline{\underline{bc}}} - \underset{0}{a^3} \}$$

[Note I_L is homogeneous w.r.t. $(3,4,5)$-grading, from $3a+4b+5c=0$ before.]

Now IP translates to:

Reduce the monomial bc mod I_L w.r.t. term order $\succ$

Here, $bc \Longrightarrow a^3$ and no further reductions possible.

$bc - a^3$

3 0

$a^3 \Longleftrightarrow (0,0)$ in original picture: $\begin{bmatrix}3\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 2\\-2 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}-3\\0\\0\end{bmatrix}$

which is optimum vertex for IP.

Relates to <u>test</u> <u>set</u> methods:

Find a set of test vectors (directions to try moving) which suffice to minimize $Ax \geq b$ for any b and

- our specific cost vector

or

- <u>any</u> cost vector

Not obvious at first these sets are even finite?

<u>Graver</u> gives "Graver basis" for 2nd problem, <u>any</u> cost vector.

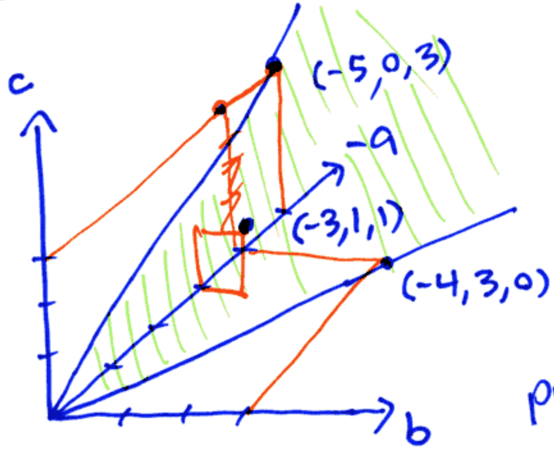
Take union of "Hilbert bases" in each orthant for lattice L.

In (x,y)-coordinates gives universal test set, for A.

④ Mon 13, Bayer

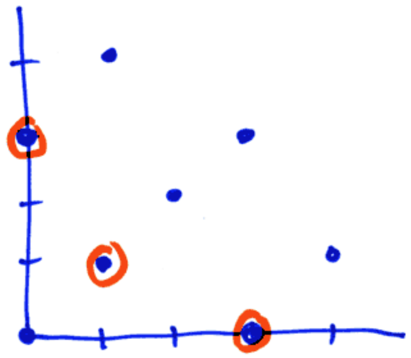
Here, answer is:

binomial	a	b	c	x	y	sign patterns −++	 +−+	 ++−
$b^2 - ac$	1	−2	1	1	0		×	
$c^2 - a^2b$	2	1	−2	0	1			×
$bc - a^3$	3	−1	−1	1	1	×		
$c^3 - ab^3$	1	3	−3	−1	1			×
$b^3 - a^4$	4	−3	0	2	1	×	×	
$c^3 - a^5$	5	0	−3	1	2	×		×
$c^4 - b^5$	0	5	−4	−2	1		×	×

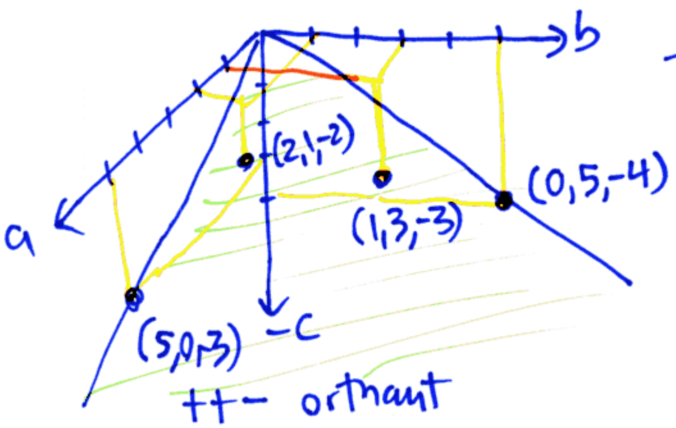


−++ orthant

⟹ project to bc-plane

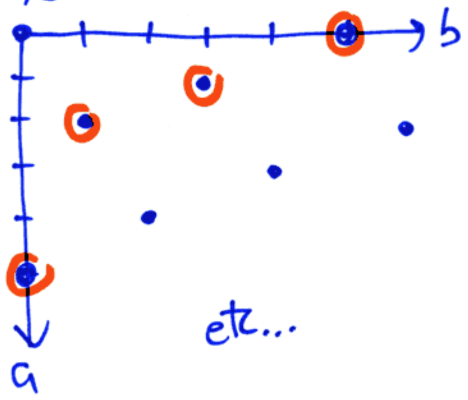


Hilbert basis is generating set for semigroup



++− orthant

⟹ to ab-plane

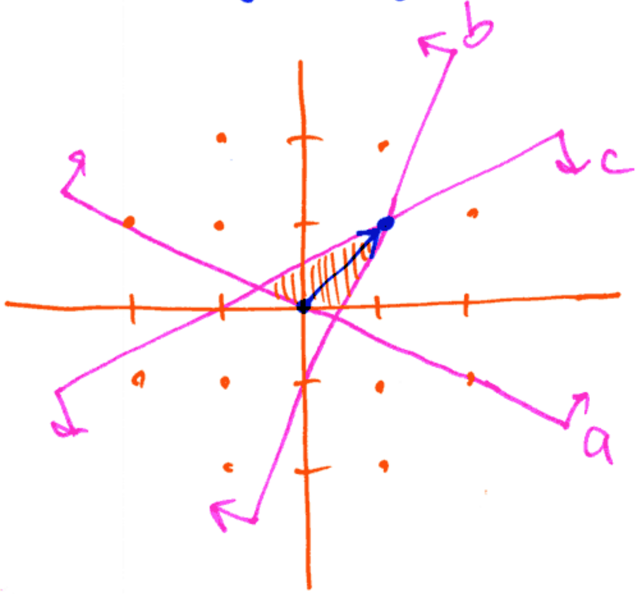


etc...

⑤ Mon 13, Bayer

Can we tell if we ever need each of these vectors?

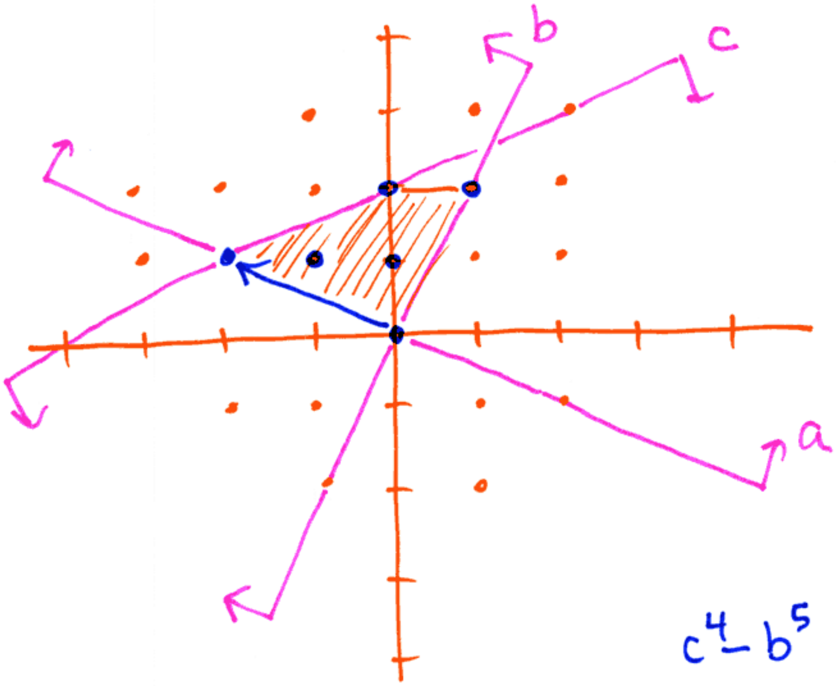
Plot them in original space, and shrink-wrap an integer program around endpoints.



..and this is translate of our original problem.

$bc - a^3$

$(1,1)$



$c^4 - b^5$

$(-2,1)$

Need this vector

cost $= (1,2) \pm (\varepsilon, 0)$

" ↑ "

edge is extremal in polytope, any other applicable vector says "stay put" on either end of this edge, but we want to move to minimal ~~edge~~. endpoint.

⑥ Mon 13, Bayer

Theorem (Sturmfels) [Gröbner Bases and Convex Polytopes]

- $\{$ lattice basis for $L\}$

$$\subseteq \{\text{ideal generators for } I_L\}$$
$$\subseteq \{\text{Gröbner basis w.r.t. some } \succ\}$$
$$\subseteq \{\text{Universal Gröbner basis}\}$$
$$\subseteq \{\text{Graver basis}\}$$

and each inclusion can be strict.

- A Graver basis element is in Universal Gröbner basis $\Leftrightarrow$ extremal edge in minimal IP in which it appears.

- Take complete homogenization in $2n$ vars

$$\mathbb{A}^n \subseteq \underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{n \text{ times}}$$

$$(a,A) \ \cdots \ (c,C)$$

Then $\{$ ideal gen set for $\bar{I}_L\} = \{$ Graver basis $\}$

Example: $\bar{I} = (b^2AC - B^2ac,\ c^2A^2B - C^2a^2b)$

$$(\bar{I} : abcABC^{\infty}) = (\,'',\ '',\ bcA^3 - a^3BC,\ c^3AB^3 - C^3ab^3,$$
$$b^3A^4 - B^3a^4,\ c^3A^5 - C^3a^5,\ c^4B^5 - C^4b^5)$$

gets Graver basis in entirety.

This relates to Lawrence lifting $\begin{bmatrix} A & 0 \\ I & I \end{bmatrix}$

in polytope theory. See Ziegler's book...

① Tues 14, Bayer

term orders, state polytopes, and regular triangulations

Theorem Let $\succ$ be a term order on $S=k[x_1,\dots,x_n]$, and let $E \subseteq \mathbb{N}^n$ represent a finite set of monomials in S (x^b for each $b \in E$). Then there exists a weight vector $w \in \mathbb{Z}^n$ so for $a,b \in E$, $x^a \succ x^b \Leftrightarrow w\cdot a > w\cdot b$.

---

proof. Let $V = \{ a-b \mid a,b \in E \text{ and } x^a \succ x^b \}$.

We can find $w \in \mathbb{Z}^n$ so $w\cdot v > 0$ for all $v \in V$, unless the convex hull of V contains $\vec{0}$. But if it does, $\sum a_i v_i = 0$ for $v_i \in V$, $a_i \in \mathbb{N}$. Rewrite as $\sum_{j=1}^{m} v_{i_j} = 0$, allowing repeats. For sufficiently large (in all coords) $a \in \mathbb{N}^n$ we have $a_\ell = \sum_{j=1}^{\ell} v_{i_j} + a \in \mathbb{N}^n$ for $\ell = 1..m$

Then $x^a \prec x^{a_1} < x^{a_2} < \dots \prec x^{a_m} = x^a$, a contradiction. //

For a given ideal, universal bound (regularity) on degrees of generators of any ideal with same Hilbert polynomial, hence on degree of any Gröbner basis element for any order. Let

$E \subseteq \mathbb{N}^n$ be the set of monomials up to this degree.

$\succ_1 \sim \succ_2 \Leftrightarrow$ have same effect on E.

w.r.t. ideal I

Thus only finitely many equivalence classes of $\succ$ for I

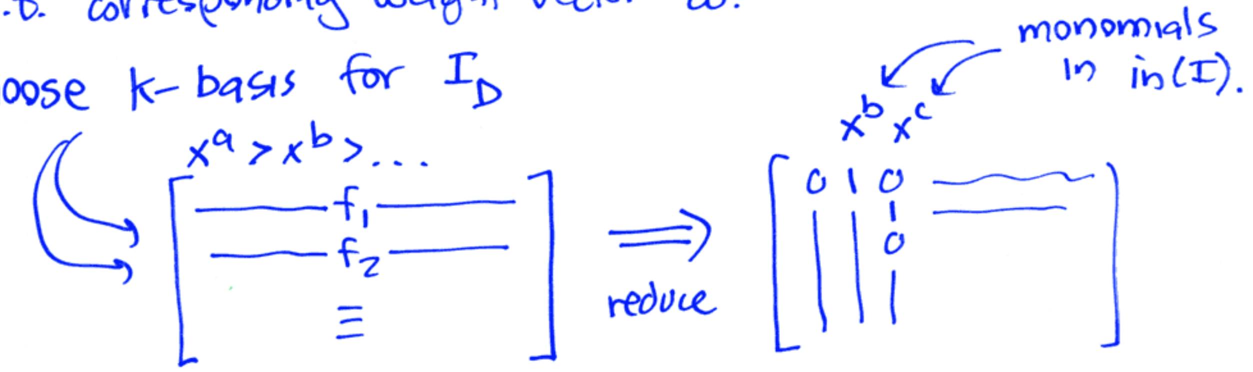
$\Rightarrow$ only finitely many $\text{in}(I)$.

② Tues 14, Bayer

How to classify? Choose large degree D ($>$ regularity bound).

Term order $\succ$ orders monomials in S_D by decreasing weight w.r.t. corresponding weight vector ω.

choose k-basis for I_D



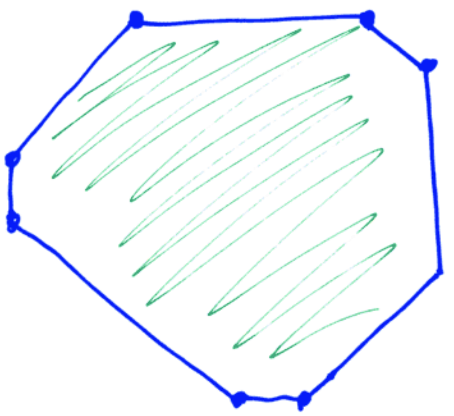
Among all nonzero minors of this matrix, $\text{in}(I)_D$ is minor of ~~least~~ greatest weight w.r.t. ω.

So plot all minors by their column exponent sums $b+c+\ldots \in \mathbb{N}^n$

$\{$vertices of convex hull$\} \underset{1:1}{\Longleftrightarrow} \{\text{in}(I)$ for different $\succ\}$

This is State polytope (Bayer, Morrison) of I.

Dual fan is Gröbner fan (Mora, Robbiano) whose interiors give weight vectors inducing strict term orders.

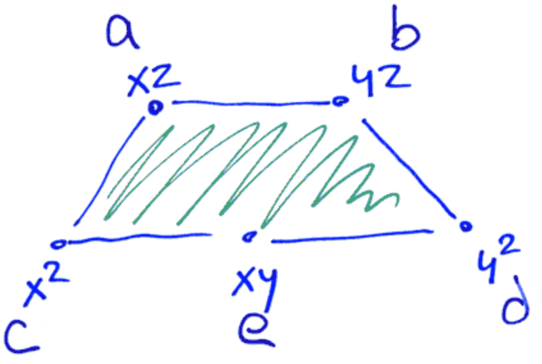


Interpretation as toric variety inside Hilbert scheme

③ Tues 14, Bayer

Example Monomial subring $k[xz, yz, x^2, y^2, xy] \subseteq k[x,y,z]$.

$= k[a,b,c,d,e]/I_L$.



$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccccc} a & b & c & d & e \\ \end{array}$$

$$A = \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$L = \langle (1,-1,0,1,-1), (-1,1,1,0,-1) \rangle$

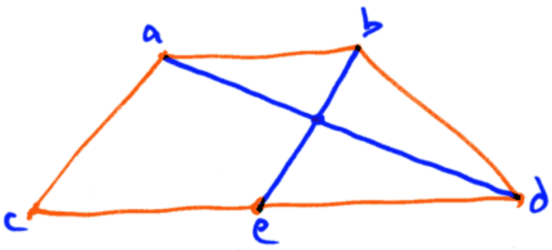
$I = (ad-be, \; bc-ae)$

$(I : (abcde)^\infty) = I_L = \langle \;'' ,\; '' ,\; e^2 - cd \rangle$

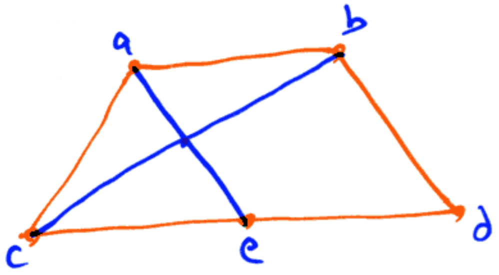
Lawrence lifting to compute Graver basis (Universal Gröbner basis):

$J = (adBE - beAD, \; bcAE - BCae)$

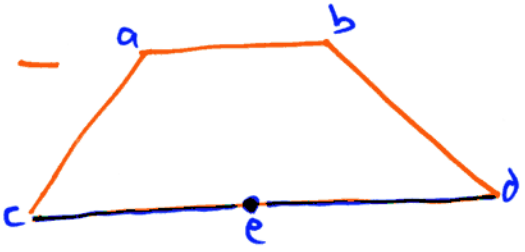
$(J : (abcde\,ABCDE)^\infty) = (\;'' ,\; '' ,\; e^2CD - cdE^2, \; a^2dB^2C - b^2cA^2D)$



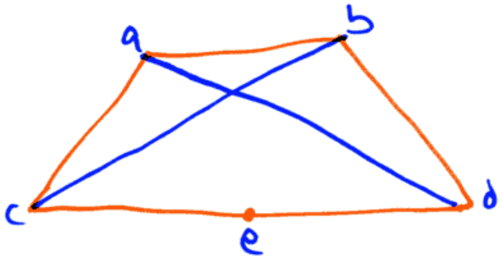
ad−be



bc−ae



$e^2 - cd$



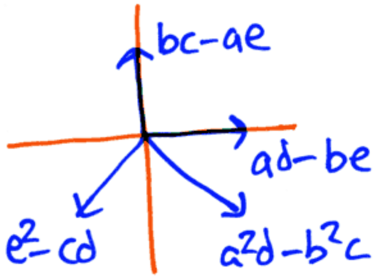
$a^2d - b^2c$

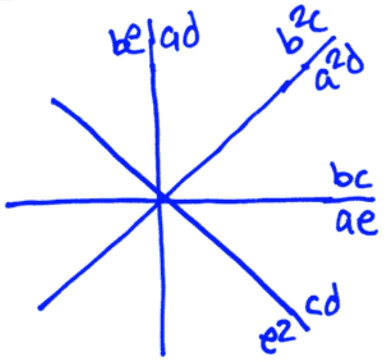
(incompatible choices in a triangulation)

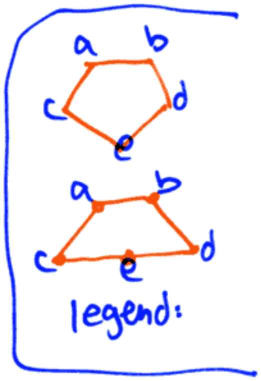
④ Tues 14, Bayer

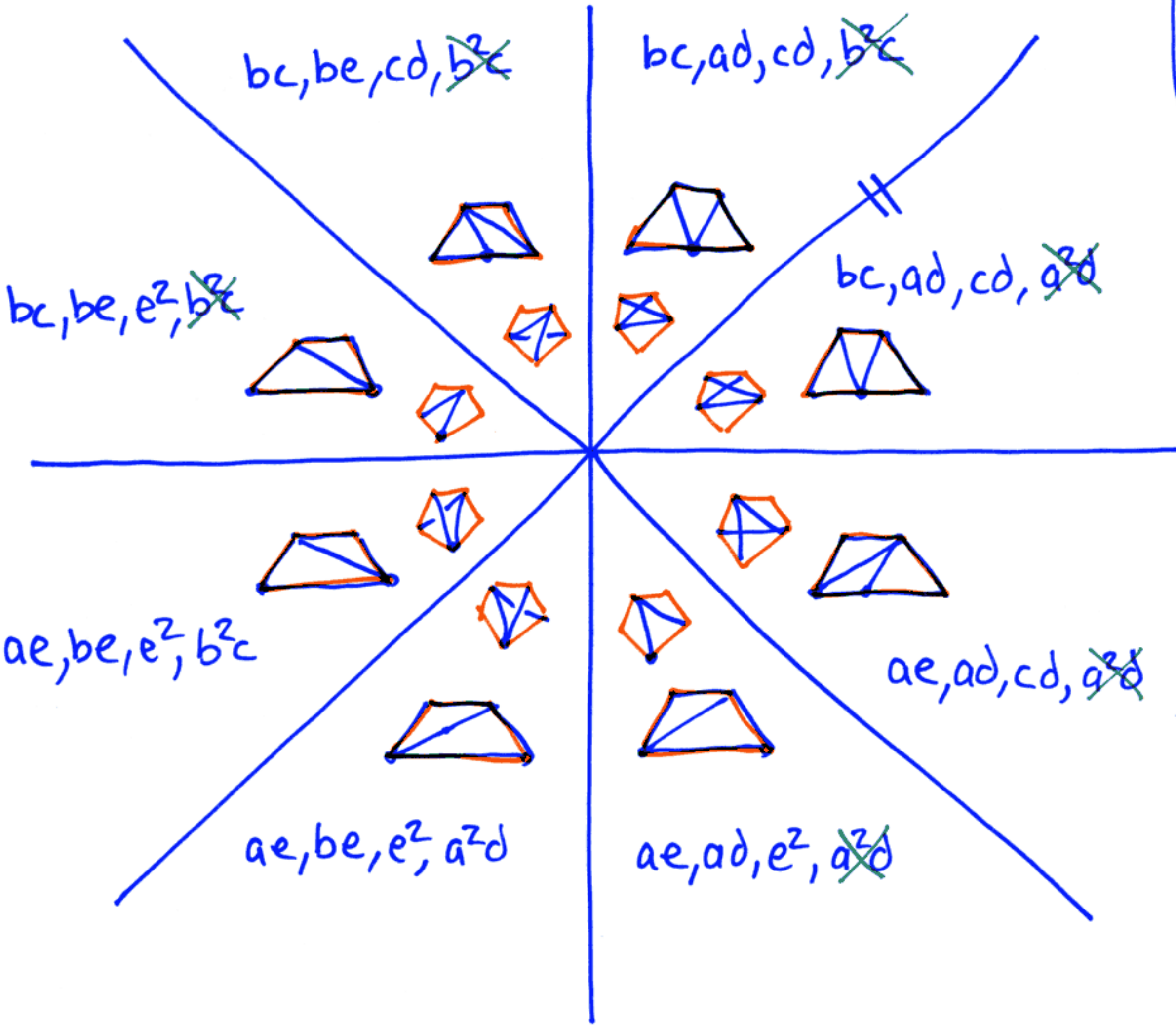
Plot all this in lattice coords $L \cong \mathbb{Z}^2$:

	a	b	c	d	e		
$ad-be$	1	-1	0	1	-1	1	0
$bc-ae$	-1	1	1	0	-1	0	1
e^2-cd	0	0	-1	-1	2	-1	-1
a^2d-b^2c	2	-2	-1	1	0	1	-1









⑤ Tues 14, Bayer

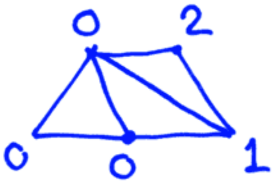
So we see $\sqrt{in(I)}$ is always a square free monomial ideal, the Stanley-Reisner ideal of the simplicial complex of a triangulation of the point set.

Which triangulations arise? (Sturmfels)

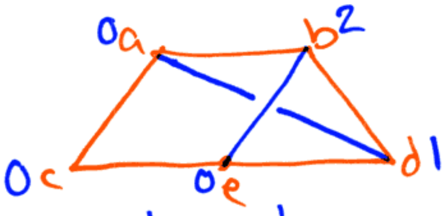
<u>Regular triangulations</u>

Assign a height function to vertices of point configuration. Lift to this height in new coord, take lower boundary of convex hull.

<u>Example</u>:

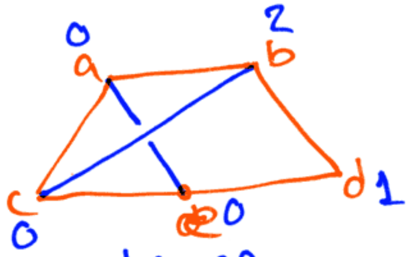


Translates, to term order. Given any binomial $x^a - x^b \in I_L$, interpret as two ways of writing same point as positive combination of vertices. Lead term is more expensive (higher) way of writing. (Optimization problem)



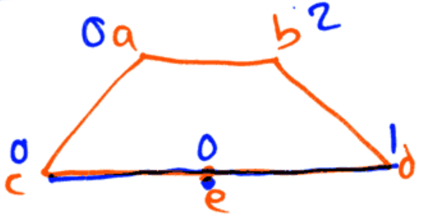
$\underline{be} - ad$

1   1/2



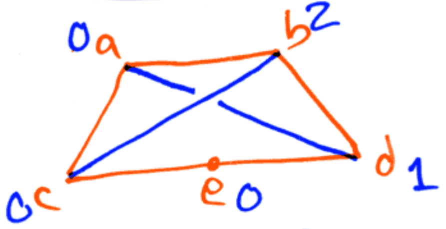
$\underline{bc} - ae$

1   0



$\underline{cd} - e^2$

1/2   0

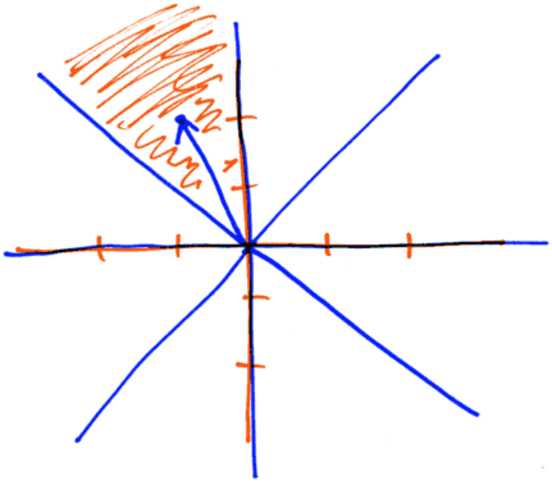


$\underline{b^2c} - a^2d$

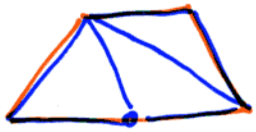
4/3   1/3

We can confirm by writing height function $w = (0,2,0,1,0)$ in L-coords:

$$(0,2,0,1,0) \cdot (1,-1,0,1,-1) = -1$$
$$(-1,1,1,0,-1) = 2$$



bc, be, cd, ~~b^2c~~



which agrees with our previous calculation.

① Thurs 16, Bayer

Lattice -vs- monomial ideals

Acyclic cocomplex of recent papers following work of Scarf

Why are constructions in toric ideals so similar to constructions in monomial ideals?

pick one →

- They're both combinatorial.
- I don't like coefficients.
- They're really the same. ← **best answer**

Intriguing similarity: Where do you see two term expressions in commutative algebra?

hmm... {
- ⇨ binomial generators of lattice ideals
- ⇨ first syzygies of monomial ideals

Generator $1 \in S$ for a binomial quotient S/I_L is somehow same as generator $x^a \in I$ for a monomial ideal J

While we're at it,

$L \subseteq \mathbb{Z}^n$ is a group acting on $\mathbb{Z}^n$ by translation.

We've been studying restriction to its action on $\mathbb{N}^n$.

Complicated effects e.g. where are generators of I_L come from banging into walls of $\mathbb{N}^n$ as first orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$.

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What's wrong with this picture?

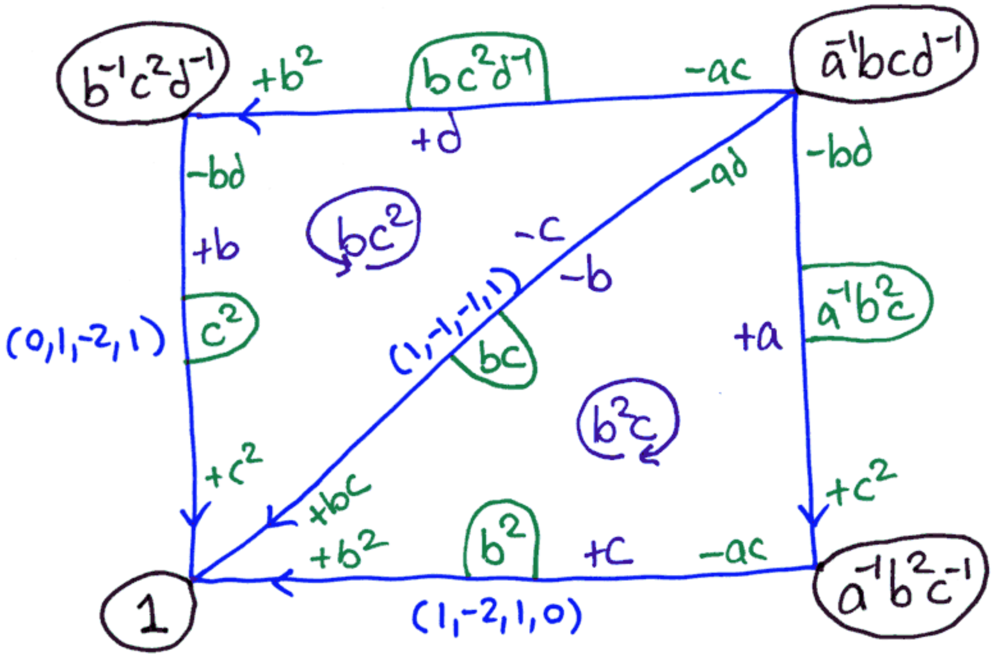
⇨ Better to have L act freely for us.

⇨ $\mathrm{Span}(L)/L$ is a torus. Where's the torus?

Example (twisted cubic)

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \qquad L = \langle \underset{\updownarrow}{(-1,2,-1,0)}, \underset{\updownarrow}{(0,-1,2,-1)} \rangle$$

$$I_L = (\underset{\substack{\updownarrow \\ (-1,1,1,-1)}}{bc-ad},\quad b^2-ac,\quad c^2-bd\ ) \subseteq \underset{\substack{\| \\ K[a,b,c,d]}}{S}$$



$$0 \leftarrow S/I_L \leftarrow S \xleftarrow{[b^2-ac,\ bc-ad,\ c^2-bd]} S^3 \xleftarrow{\begin{bmatrix} d & c \\ -c & -b \\ b & a \end{bmatrix}} S^2 \leftarrow 0$$

Take cellular resolution, identify faces mod L,
get minimal resolution of S/I_L.

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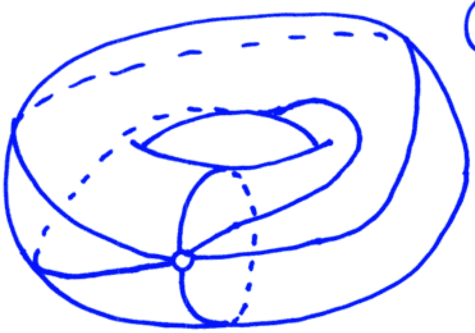


Cellular resolution is supported on a torus!

What did we do here?

Given $L \subseteq \mathbb{Z}^n$ so $L \cap \mathbb{N}^n = \{\vec{0}\}$

$S = k[x_1, \ldots, x_n]$ poly ring

$T = k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ <u>Laurent</u> poly ring

define

$I_L = \langle x^a - x^b \mid a-b \in L \rangle \subseteq S$ as usual

$M_L = \langle x^a \mid a \in L \rangle \subseteq T$ same idea

"lattice module"

M_L is generated by its minimal monomials

$\Leftrightarrow$ no decreasing infinite chains under divisibility

M_L is an $S[L]$-module, $\mathbb{Z}^n$-graded

⇨ S acts as usual

⇨ L acts by translation

S/I_L is $\mathbb{Z}^n/L$-graded S module,

⇨ resolution is $\mathbb{Z}^n/L$-homogeneous of degree 0.

Theorem (—, Sturmfels)

Let $\mathcal{A} = \{\mathbb{Z}^n\text{-graded } S[L]\text{-modules}\}$

maps are $\mathbb{Z}^n$-graded $S[L]$-module homomorphisms of deg 0

$\mathcal{B} = \{\mathbb{Z}^n/L\text{-graded } S\text{-modules}\}$

maps are $\mathbb{Z}^n/L$-graded S-module homomorphisms of deg 0

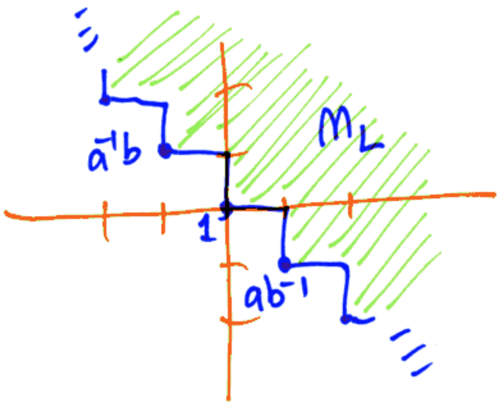
The categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

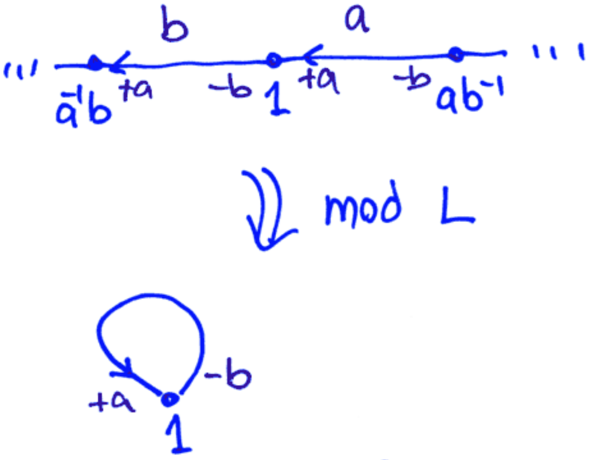
Equivalence maps resolutions to resolutions.

We mapped a cellular resolution

Simpler example: $L = \langle (1,-1) \rangle \subseteq \mathbb{Z}^2$

$I_L = (a-b) \subseteq k[a,b] = S$





$$0 \leftarrow k[a,b]/(a-b) \leftarrow S \xleftarrow{[a-b]} S \leftarrow 0$$

⑤ Thur 16, Bayer

Applications:

A corner of S/I_L is an equivalence class of (identical) corners of M_L, under action of L.

Recall we compute Betti numbers of $\left\{ \begin{matrix} \text{monomial } J \\ \text{toric } I_L \end{matrix} \right\}$

by suspiciously similar complexes $K_{\underline{b}}$:

For J monomial: "walking down by F stays in ideal"

$K_b := \{ F \subseteq \{1,..,n\} \mid \underline{b} - F$ is an exponent in $J \}$

gives simplicial complex describing corner at $\underline{b}$.

For I_L lattice ideal:

$K_b := \{ F \subseteq \{1,..,n\} \mid$ walking down from degree $\underline{b}$ in S/I_L by steps given by directions in F stay in ring $\}$

"walking down by F stays in ring"

Take any $\underline{c} \in \mathbb{Z}^n$ whose image $\in \mathbb{Z}^n/L$ is $\underline{b}$

K_b describes the corner at $\underline{c}$ of M_L.

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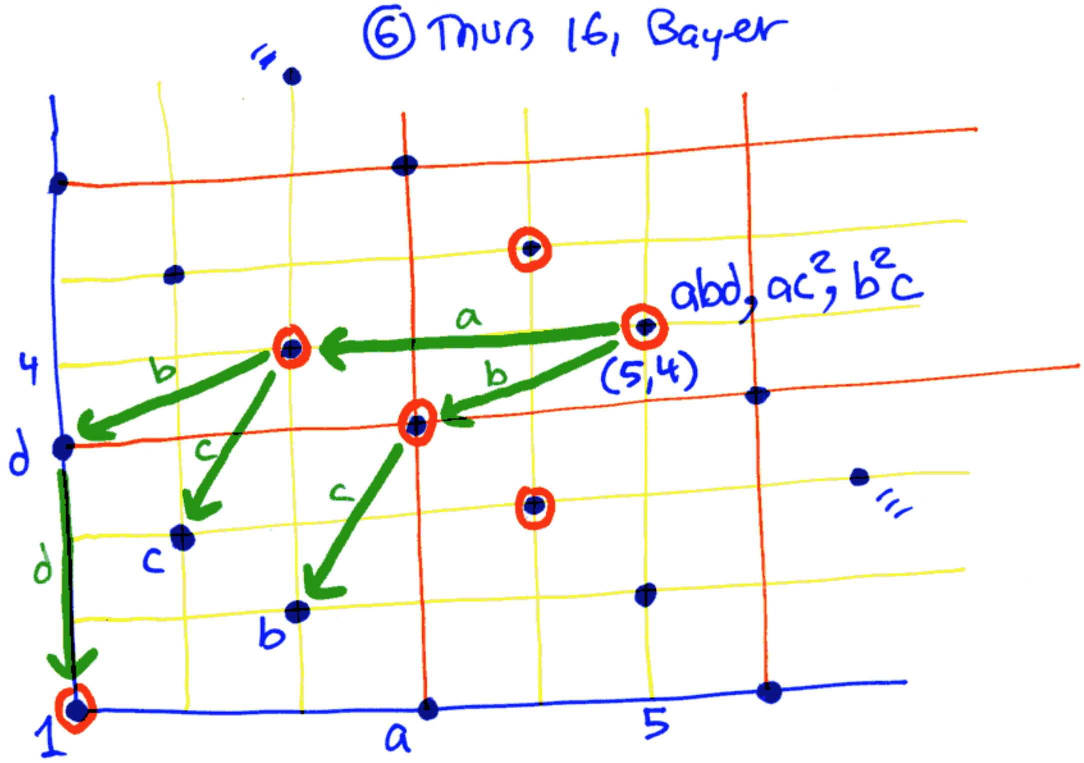
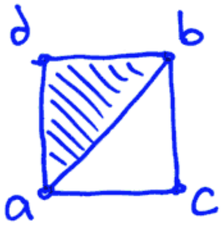


We computed

$K_{(5,4)} =$



from supports of abd, ac^2, b^2c

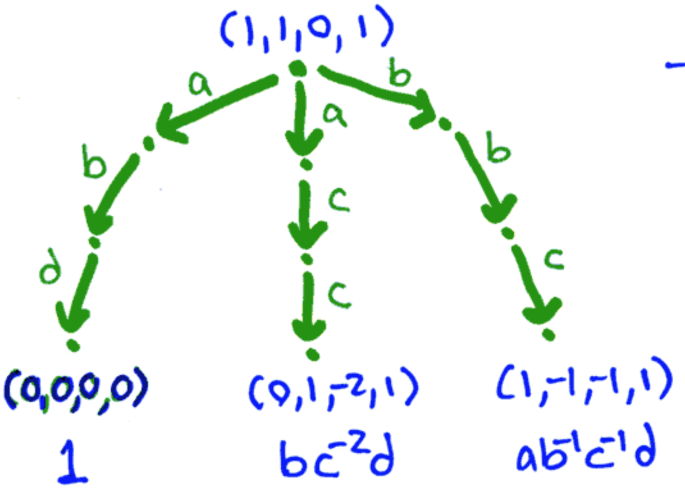
as paths within subring

abd $\cdot \xrightarrow{a} \cdot \xrightarrow{b} \cdot \xrightarrow{d} \cdot$

ac $\cdot \xrightarrow{a} \cdot \xrightarrow{c} \cdot$

bc $\cdot \xrightarrow{b} \cdot \xrightarrow{c} \cdot$

Take $\underline{c} = (1,1,0,1) \in \mathbb{Z}^4$ maps to $\underline{b} = (5,4) \in \mathbb{Z}^4/L$



corner formed by generators of M_L

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The hull resolution

(—, Sturmfels)
again algebraic interp/generalization
of work of Scarf.

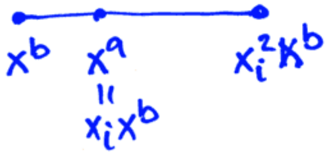
We work with monomial modules $M \subseteq T = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ (generated by their minimal monomials).
Think of monomial ideal $I \subseteq S = k[x_1, \ldots, x_n]$ as basic example; we use modules for toric case. Fix $t > (n+1)!$.

Define:
$$\boxed{P_t = \operatorname{conv}\{ t^a = (t^{a_1}, \ldots, t^{a_n}) \in \mathbb{R}^n \mid x^a \in M \}}$$

lemma P_t closed polyhedron $\Leftarrow$ M gen by min monomials

Prop Vertices of $P_t \Leftrightarrow$ min generators of M

proof. If x^a not min gen, then $x^a = x_i x^b$ for some $x^b \in M$.
Look at line segment $[t^b, t^{b+2e_i}] \subseteq P_t$



If x^a is min gen, let $v = t^{-a}$, so $t^a \cdot v = n$.
For any other $x^b \in M$, $b_i > a_i$ for some i, so
$$t^b \cdot v = \sum_{j=1}^{n} t^{b_j - a_j} \geq t^{b_i - a_i} \geq t > (n+1)! > n.$$
//

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Theorem The combinatorial structure (face poset) of P_t indep of t for $t > (n+1)!$ Same for bounded faces.

idea of proof: Look at $(n+1)\times(n+1)$ matrices of form

$$\text{various } e_i \left\{ \left[\begin{array}{ccc|c} 0 & 0 & 0 & \\ 1 & \vdots & \vdots & \cdots\ t^{a_j}\ \cdots \\ & 0 & & \\ & 1 & & \\ & 0 & 0 & \\ 0 & & 1 & \\ \hline 0 & \cdots & 0 & 1 \cdots 1 \end{array}\right] \right\} \text{various } t^{a_j} \text{ for min gens } x^{a_j} \in M$$

The sign of the determinant is indep of $t > (n+1)!$, and this pattern of signs determines combinatorics of P_t and its bounded faces.

Theorem The cell complex hull(M) supports a free resolution of M, where hull(M) = complex of bounded faces of P_t.

proof. We need to show that $X = (\text{hull}(M))_{\le b}$ is acyclic for each degree $b \in \mathbb{Z}^n$.

Immediate unless X has ≥ 2 vertices. So assume it does.

Again let $v = t^{-b}$. Same argument as before: x^a min gen of M:

- If t^a is a vertex of X, $a < b$ so
$$t^a \cdot v = t^{-b} \cdot t^a < t^{-b} \cdot t^b = n$$
- If t^a not a vertex of X, $a_i > b_i$ for some i, so
$$t^a \cdot v = t^{-b} \cdot t^a > t^{a_i - b_i} \ge t > n$$

Thus $x \cdot v = n$ is a separating hyperplane

vertices of X / other vertices of P_t

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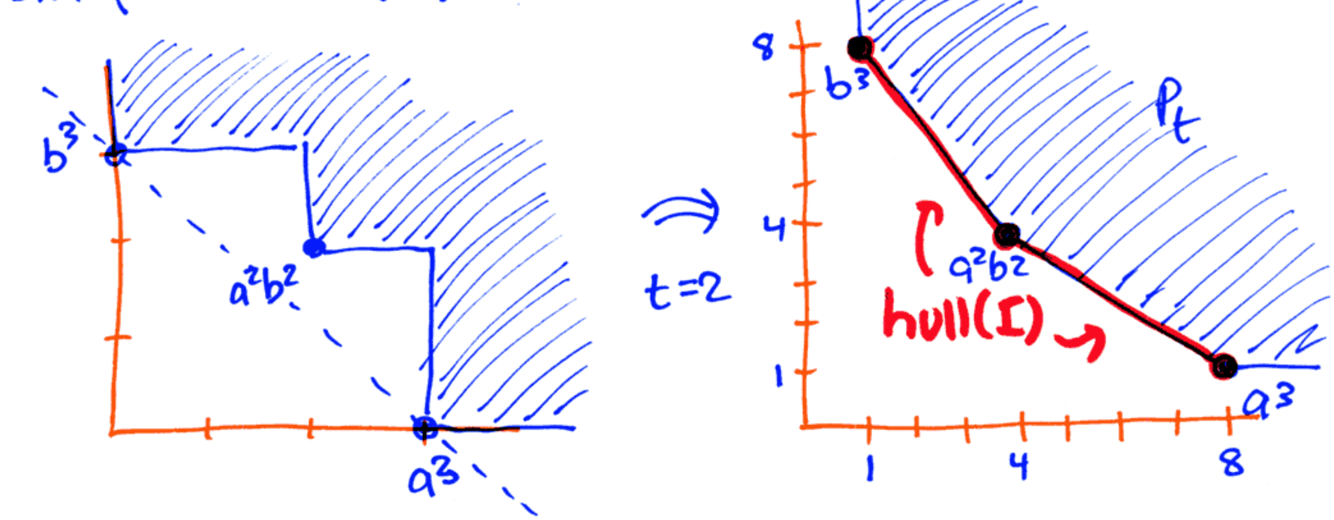
Move this hyperplane to ∞ by projective transformation. X is now bounded faces of convex polyhedron, known to be contractible. //

Example: $I = (a^3, a^2b^2, b^3) \subseteq k[a,b]$



a^2b^2 inside convex hull of (a^3, b^3), but after Scarf lifting ↑

$$\begin{tikzcd} b^3 \arrow[r, "a^2b^3"] & a^2b^2 \arrow[r, "a^3b^2"] & a^3 \end{tikzcd}$$

$$0 \leftarrow I \xleftarrow{[a^3, a^2b^2, b^3]} \underbrace{S^3 \xleftarrow{\begin{bmatrix} b^2 & 0 \\ -a & b \\ 0 & -a^2 \end{bmatrix}} S^2}_{\text{here, minimal resolution.}} \leftarrow 0$$

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This generalizes Scarf complex of [–, Peeva, Sturmfels]:

<u>Def</u> A monomial ideal $I \subseteq S = k[x_1, \ldots, x_n]$ is <u>generic</u>

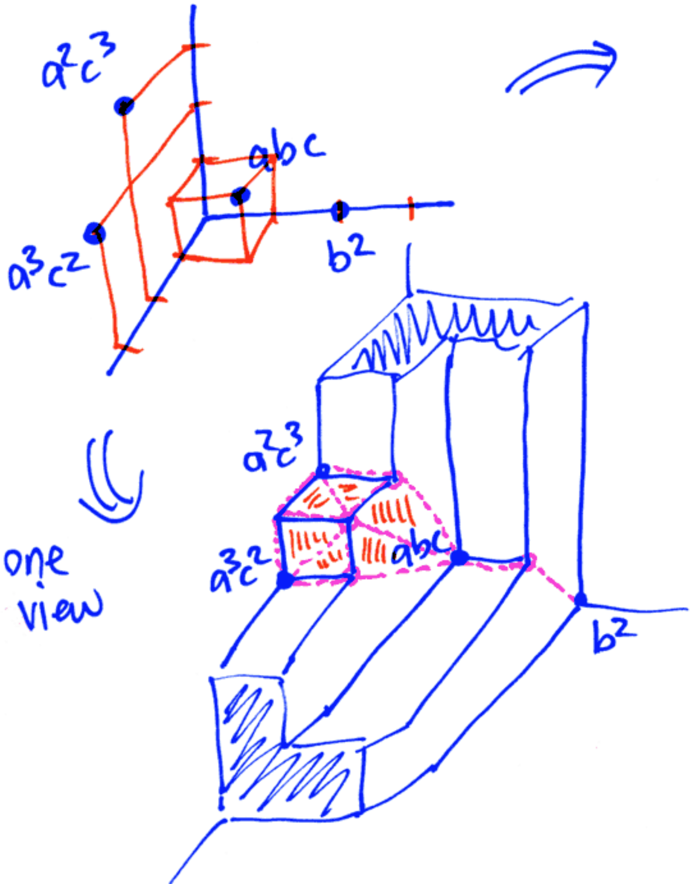
$\Leftrightarrow$ distinct min generators have distinct nonzero exponents.

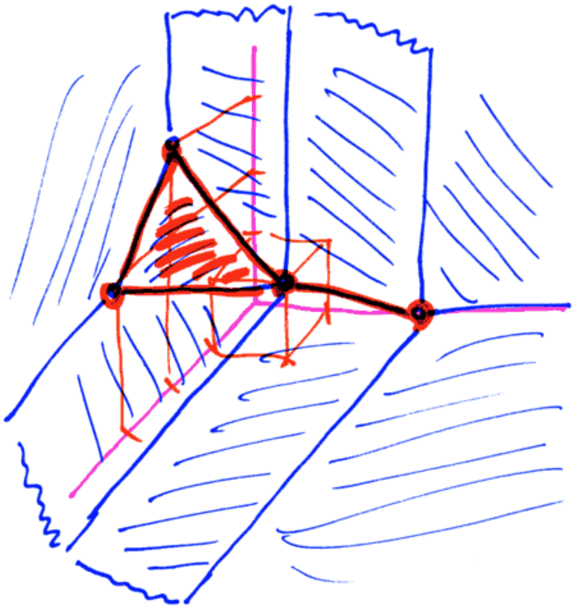
[Motivation: move generating exponents around in $\mathbb{R}^n$. (!)
Better behavior in "general position", just like convex hulls become simplicial. Some integer exponent sets already in "general position", and above defn captures this idea.]

example: $I = (a^3c^2, a^2c^3, abc, b^2) \subseteq k[a,b,c]$.

Look at exponents as columns of a matrix:

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 3 & 1 & 0 \end{bmatrix} \leftarrow \text{nonzero entries in each row are distinct}$$





<u>Thm</u> hull(M) supports a minimal resolution of a generic monomial ideal.

Application: toric ideals.

Theorem Let $L \subseteq \mathbb{Z}^n$ be a lattice so $L \cap \mathbb{N}^n = \{\vec{0}\}$.

Let $I_L = \langle x^a - x^b \mid a-b \in L \rangle \subseteq S = k[x_1, \dots, x_n]$

$M_L = \langle x^a \mid a \in L \rangle \subseteq T = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

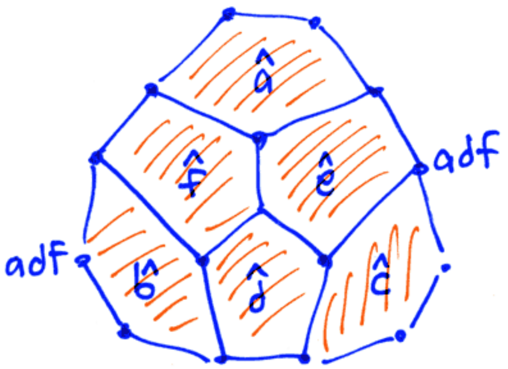
The hull resolution of M_L is L-invariant, and gives a resolution of S/I_L. //

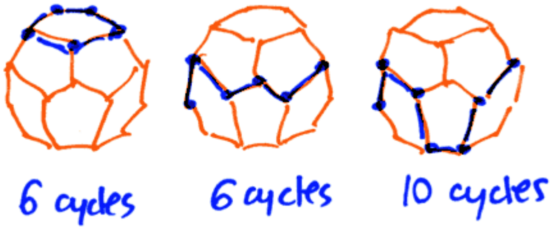
This *was* our resolution yesterday of twisted cubic.

Application: squarefree monomial ideals.

Enough to take original exponents, same hull

$a = 0$ or 1 $\quad t^a \underbrace{= 1 \text{ or } 2}_{\text{translate back to 0 or 1}}$ for $t=2$, which suffices.





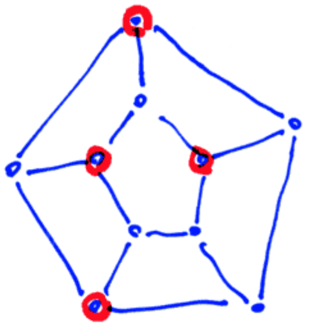
$I = ($10 cubic squarefree monoms$)$

$\subseteq k[a, \dots, f]$

Min res supported on cell complex $\approx \mathbb{RP}^2$

if $\mathrm{char}(k) \neq 2$.

Hull(I) is 5-dim polytope w/ 22 facets, cor. to cycles above.



Application (graph theory)

Let G = graph on n vertices

Let $I = (x_i x_j \mid (i,j) \text{ is an edge of } G)$
$\subseteq S = k[x_1, \dots, x_n]$.

(●●●● is facet of X)

Gens of I $\Longleftrightarrow$ min nonfaces of a simplicial complex
$X \subseteq \Delta = 2^{\{1,\dots,n\}}$

A face $F \in X$ is therefore a set of vertices of G
not connected by any edge.

Known in graph theory as stable sets or independent sets

hull(I) related (by Alexander duality) to
stable set polytope $S(G) = \text{conv}\left\{ \begin{matrix}\text{incidence vectors of} \\ \text{faces of } X\end{matrix} \right\} \subseteq \mathbb{R}^n$.

Grötschel et. al. study; see survey B. Toft in Handbook Comb. I

Thm If G is planar, G has a stable set on $\geq n/4$ vertices.

proof. 4-color theorem; use "biggest" coloring.

(!?) No other proof known. Hmm...

Any resolution of I determines dim X

$\Rightarrow$ must be very hard to determine structure of hull(I)??

$\Rightarrow$ must be very hard to understand K_b, X_b here ??