Cellular Resolutions of Monomial Modules

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Abstract: *We construct a canonical free resolution for arbitrary monomial modules and lattice ideals. This includes monomial ideals and defining ideals of toric varieties, and it generalizes our joint results with Irena Peeva for generic ideals [BPS],[PS].*

Introduction

Given a field k, we consider the Laurent polynomial ring $T = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ as a module over the polynomial ring $S = k[x_1, \ldots, x_n]$. The module structure comes from the natural inclusion of semigroup algebras $S = k[\mathbb{N}^n] \subset k[\mathbb{Z}^n] = T$. A *monomial module* is an S-submodule of T which is generated by monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, $\mathbf{a} \in \mathbb{Z}^n$. Of special interest are the two cases when M has a minimal monomial generating set which is either finite or forms a group under multiplication. In the first case M is isomorphic to a *monomial ideal* in S. In the second case M coincides with the *lattice module*

$$M_L \quad := \quad S\,\{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in L\} \quad = \quad k\,\{\mathbf{x}^{\mathbf{b}} \mid \mathbf{b} \in \mathbb{N}^n + L\} \quad \subset \quad T.$$

for some sublattice $L \subset \mathbb{Z}^n$ whose intersection with $\mathbb{N}^n$ is the origin $\mathbf{0} = (0, \ldots, 0)$.

We shall derive free resolutions of M from regular cell complexes whose vertices are the generators of M and whose faces are labeled by the least common multiples of their vertices. The basic theory of such cellular resolutions is developed in Section 1.

Our main result is the construction of the *hull resolution* in Section 2. We rescale the exponents of the monomials in M, so that their convex hull in $\mathbb{R}^n$ is a polyhedron P_t whose bounded faces support a free resolution of M. This resolution is new and interesting even for monomial ideals. It need not be minimal, but, unlike minimal resolutions, it respects symmetry and is free from arbitrary choices.

In Section 3 we relate the lattice module M_L to the $\mathbb{Z}^n/L$-graded *lattice ideal*

$$I_L \quad = \quad \left\langle\ \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \ \mid\ \mathbf{a} - \mathbf{b} \in L\ \right\rangle \quad \subset \quad S.$$

This class of ideals includes ideals defining *toric varieties*. We express the cyclic S-module S/I_L as the quotient of the infinitely generated S-module M_L by the action of L. In fact, we like to think of M_L as the "universal cover" of I_L. Many questions about I_L can thus be reduced to questions about M_L. In particular, we obtain the hull resolution of a lattice ideal I_L by taking the hull resolution of M_L modulo L.

This paper is inspired by the work of Barany, Howe and Scarf [BHS] who introduced the polyhedron P_t in the context of integer programming. The hull resolution generalizes results in [BPS] for generic monomial ideals and in [PS] for generic lattice ideals. In these generic cases the hull resolution is minimal.

1 Cellular resolutions

Let $M \subset T$ be a monomial module. We write $\min(M)$ for the set of *minimal monomials* in M, that is, $\min(M) := \{\, \mathbf{x}^{\mathbf{a}} \in M \,:\, \mathbf{x}^{\mathbf{a}}/x_i \notin M \text{ for } i = 1, \dots, n \,\}$.

Remark 1.1 For a monomial module M the following are equivalent:
(1) M is generated by its minimal monomials, that is, $M = S \cdot \min(M)$.
(2) There is no infinite decreasing (under divisibility) sequence of monomials in M.
(3) For all $\mathbf{b} \in \mathbb{Z}^n$, the set of monomials in M of degree $\preceq \mathbf{b}$ is finite.

We call M *co-Artinian* if these three equivalent conditions hold. For instance, T itself is a monomial module which is not co-Artinian, since $\min(T) = \emptyset$.

In this section we consider an arbitrary co-Artinian monomial module M. The generating set $\min(M) = \{\, m_j = \mathbf{x}^{\mathbf{a}_j} \mid j \in I \,\}$ is identified with the subset $\{\, \mathbf{a}_j \mid j \in I \,\} \subset \mathbb{Z}^n$, and I is a totally ordered index set which need not be finite.

Let X be a *regular cell complex* having I as its set of vertices, and equipped with a choice of an *incidence function* $\varepsilon(F, F')$ on pairs of faces. We recall from [BH, Section 6.2] that ε takes values in $\{0, 1, -1\}$, that $\varepsilon(F, F') = 0$ unless F' is a facet of F, that $\varepsilon(\{j\}, \emptyset) = 1$ for all vertices $j \in I$, and that for any codimension 2 face F' of F,

$$\varepsilon(F, F_1)\varepsilon(F_1, F') \;+\; \varepsilon(F, F_2)\varepsilon(F_2, F') \quad = \quad 0$$

where F_1, F_2 are the two facets of F containing F'. The prototype of a regular cell complex is the set of faces of a convex polytope. The incidence function ε defines a differential ∂ which can be used to compute the homology of X. Define the *augmented oriented chain complex* $\widetilde{C}(X; k) = \bigoplus_{F \in X} kF$, with differential

$$\partial F \quad = \quad \sum_{F' \in X} \varepsilon(F, F')\, F'.$$

The *reduced cellular homology group* $\widetilde{H}_i(X; k)$ is the ith homology of $\widetilde{C}(X; k)$, where faces of X are indexed by their dimension. The *oriented chain complex* $C(X; k) = \bigoplus_{F \in X,\, F \neq \emptyset} kF$ is obtained from $\widetilde{C}(X; k)$ by dropping the contribution of the empty face. It computes the ordinary homology groups $H_i(X; k)$ of X.

The cell complex X inherits a $\mathbb{Z}^n$-grading from the generators of M as follows. Let F be a nonempty face of X. We identify F with its set of vertices, a finite subset of I. Set $m_F := \operatorname{lcm}\{\, m_j \mid j \in F \,\}$. The exponent vector of the monomial m_F is the *join* $\mathbf{a}_F := \bigvee \{\, \mathbf{a}_j \mid j \in F \,\}$ in $\mathbb{Z}^n$. We call $\mathbf{a}_F$ the *degree* of the face F.

Homogenizing the differential ∂ of $C(X; k)$ yields a $\mathbb{Z}^n$-graded chain complex of S-modules. Let SF be the free S-module with one generator F in degree $\mathbf{a}_F$. The *cellular complex* $\mathbf{F}_X$ is the $\mathbb{Z}^n$-graded S-module $\bigoplus_{F \in X,\, F \neq \emptyset} SF$ with differential

$$\partial\, F \quad = \quad \sum_{F' \in X,\, F' \neq \emptyset} \varepsilon(F, F')\, \frac{m_F}{m_{F'}}\, F'.$$

The homological degree of each face F of X is its dimension.

For each degree $\mathbf{b} \in \mathbb{Z}^n$, let $X_{\preceq \mathbf{b}}$ be the subcomplex of X on the vertices of degree $\preceq \mathbf{b}$, and let $X_{\prec \mathbf{b}}$ be the subcomplex of $X_{\preceq \mathbf{b}}$ obtained by deleting the faces of degree $\mathbf{b}$. For example, if there is a unique vertex j of degree $\preceq \mathbf{b}$, and $\mathbf{a}_j = \mathbf{b}$, then $X_{\preceq \mathbf{b}} = \{\{j\}, \emptyset\}$ and $X_{\prec \mathbf{b}} = \{\emptyset\}$. A full subcomplex on no vertices is the acyclic complex $\{\}$, so if there are no vertices of degree $\preceq \mathbf{b}$, then $X_{\preceq \mathbf{b}} = X_{\prec \mathbf{b}} = \{\}$.

The following proposition generalizes [BPS, Lemma 2.1] to cell complexes:

Proposition 1.2 *The complex $\mathbf{F}_X$ is a free resolution of M if and only if $X_{\preceq \mathbf{b}}$ is acyclic over k for all degrees $\mathbf{b}$. In this case we call $\mathbf{F}_X$ a cellular resolution of M.*

Proof. The complex $\mathbf{F}_X$ is $\mathbb{Z}^n$-graded. The degree $\mathbf{b}$ part of $\mathbf{F}_X$ is precisely the oriented chain complex $C(X_{\preceq \mathbf{b}}; k)$. Hence $\mathbf{F}_X$ is a free resolution of M if and only if $H_0(X_{\preceq \mathbf{b}}; k) \cong k$ for $\mathbf{x}^{\mathbf{b}} \in M$, and otherwise $H_i(X_{\preceq \mathbf{b}}; k) = 0$ for all i and all $\mathbf{b}$. This condition is equivalent to $\widetilde{H}_i(X_{\preceq \mathbf{b}}; k) = 0$ for all i (since $\mathbf{x}^{\mathbf{b}} \in M$ if and only if $\emptyset \in X_{\preceq \mathbf{b}}$) and thus to $X_{\preceq \mathbf{b}}$ being acyclic. ∎

For $\mathbf{b} \in \mathbb{Z}^n$ we let $M_{\preceq \mathbf{b}}$ denote the monomial module generated by all monomials in M of degree $\preceq \mathbf{b}$. Since M is co-Artinian, by part (3) of Remark 1.1, $M_{\preceq \mathbf{b}}$ is isomorphic (up to a degree shift) to a monomial ideal. The minimal generators of $M_{\preceq \mathbf{b}}$ are the monomials in $\min(M)$ which have degree $\preceq \mathbf{b}$.

Corollary 1.3 *The cellular complex $\mathbf{F}_X$ is a resolution of M if and only if the cellular complex $\mathbf{F}_{X_{\preceq \mathbf{b}}}$ is a resolution of the monomial ideal $M_{\preceq \mathbf{b}}$ for all $\mathbf{b} \in \mathbb{Z}^n$.*

Proof. This follows from Proposition 1.2 and the identity $(X_{\preceq \mathbf{b}})_{\preceq \mathbf{c}} = X_{\preceq \mathbf{b} \wedge \mathbf{c}}$. ∎

Remark 1.4 A cellular resolution $\mathbf{F}_X$ is a minimal resolution if and only if any two comparable faces $F' \subset F$ of the complex X have distinct degrees $\mathbf{a}_F \neq \mathbf{a}_{F'}$.

The simplest example of a cellular resolution is the *Taylor resolution* for monomial ideals [Tay]. The Taylor resolution is easily generalized to arbitrary monomial modules M as follows. Let $\{ m_j \mid j \in I \}$ be the minimal generating set of M. Define Δ to be the simplicial complex consisting of all finite subsets of I, equipped with the standard incidence function $\varepsilon(F, F') = (-1)^j$ if $F \setminus F'$ consists of the jth element of F. The Taylor complex of M is the cellular complex $\mathbf{F}_\Delta$.

Proposition 1.5 *The Taylor complex $\mathbf{F}_\Delta$ is a resolution of M.*

Proof. By Proposition 1.2 we need to show that each subcomplex $\Delta_{\preceq \mathbf{b}}$ of Δ is acyclic. $\Delta_{\preceq \mathbf{b}}$ is the full simplex on the set of vertices $\{ j \in I \mid \mathbf{a}_j \preceq \mathbf{b} \}$. This set is finite by Remark 1.1 (3). Hence $\Delta_{\preceq \mathbf{b}}$ is a finite simplex, which is acyclic. ∎

The Taylor resolution $\mathbf{F}_\Delta$ is typically far from minimal. If M is infinitely generated, then Δ has faces of arbitrary dimension and $\mathbf{F}_\Delta$ has infinite length. Following [BPS, §2] we note that every simplicial complex $X \subset \Delta$ defines a submodule $\mathbf{F}_X \subset \mathbf{F}_\Delta$ which is closed under the differential ∂. We call $\mathbf{F}_X$ the *restricted Taylor complex* supported on X. $\mathbf{F}_X$ is a resolution of M if and only if the hypothesis of Proposition 1.2 holds, with cellular homology specializing to simplicial homology.

Example 1.6 Consider the monomial ideal $M = \langle\, a^2b,\ ac,\ b^2,\ bc^2 \,\rangle$ in $S = k[a,b,c]$. Figure 1 shows a truncated "staircase diagram" of unit cubes representing the monomials in $S\backslash M$, and shows two simplicial complexes X and Y on the generators of M. Both are two triangles sharing an edge. Each vertex, edge or triangle is labeled by its degree. The notation **210**, for example, represents the degree $(2,1,0)$ of a^2b.

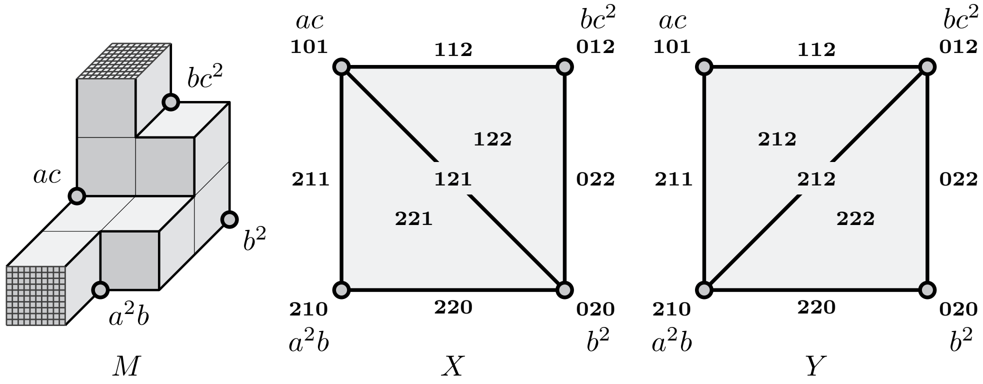


Figure 1

By Proposition 1.2, the complex X supports the minimal free resolution $\mathbf{F}_X =$

$$0 \to S^2 \xrightarrow{\begin{bmatrix} -b & 0 \\ c & 0 \\ 0 & -b \\ -a & c \\ 0 & -a \end{bmatrix}} S^5 \xrightarrow{\begin{bmatrix} c & b & 0 & 0 & 0 \\ -ab & 0 & bc & b^2 & 0 \\ 0 & 0 & -a & 0 & b \\ 0 & -a^2 & 0 & -ac & -c^2 \end{bmatrix}} S^4 \xrightarrow{[\ a^2b \ \ ac \ \ bc^2 \ \ b^2\ ]} M \to 0.$$

The complex Y fails the criterion of Proposition 1.2, and hence $\mathbf{F}_Y$ is not exact: if $\mathbf{b} = (1,2,1)$ then $Y_{\preceq \mathbf{b}}$ consists of the two vertices ac and b^2, and is not acyclic. ∎

We next present four examples which are not restricted Taylor complexes.

Example 1.7 Let M be a *Gorenstein ideal of height* 3 generated by m monomials. It is shown in [BH1, §6] that the minimal free resolution of M is the cellular resolution $\mathbf{F}_X : 0 \to S \to S^m \to S^m \to S \to 0$ supported on a *convex m-gon.*

Example 1.8 A monomial ideal M is *co-generic* if its no variable occurs to the same power in two distinct irreducible components $\langle x_{i_1}^{r_1}, x_{i_2}^{r_2}, \ldots, x_{i_s}^{r_s} \rangle$ of M. It is shown in [Stu2] that the minimal resolution of a co-generic monomial ideal is a cellular resolution $\mathbf{F}_X$ where X is the complex of bounded faces of a *simple polyhedron.*

Example 1.9 Let $u_1, \ldots, u_n$ be distinct integers and M the module generated by the $n!$ monomials $x_{\pi(1)}^{u_1} x_{\pi(2)}^{u_2} \cdots x_{\pi(n)}^{u_n}$ where π runs over all permutations of $\{1, 2, \ldots, n\}$. Let X be the complex of all faces of the *permutohedron* [Zie, Example

0.10], which is the convex hull of the $n!$ vectors $\big(\pi(1),,\dots,\pi(n)\big)$ in $\mathbb{R}^n$. It is known [BLSWZ, Exercise 2.9] that the i-faces F of X are indexed by chains

$$\emptyset \;=\; A_0 \subset A_1 \subset \;\; \dots \;\; \subset A_{n-i-1} \subset A_{n-i} \;=\; \{1,2,\dots,n\}.$$

We assign the following monomial degree to the i-face F indexed by this chain:

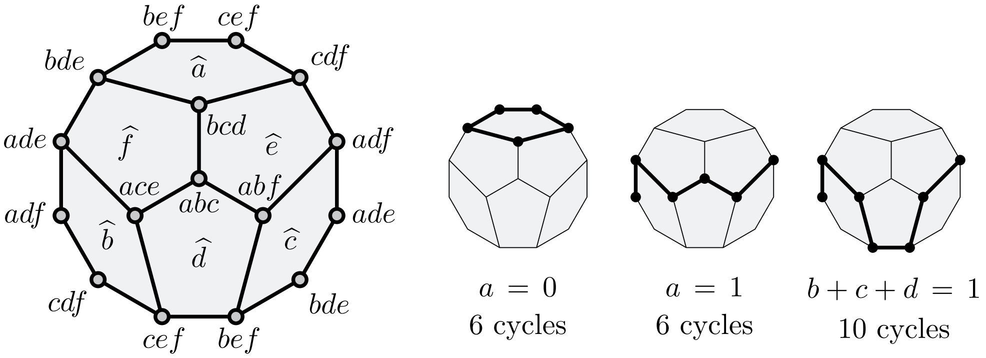
$$\mathbf{x}^{\mathbf{a}_F} \quad = \quad \prod_{j=1}^{n-i} \; \prod_{r\in A_j\setminus A_{j-1}} x_r^{\max\{\, u_\ell \;\mid\; |A_{j-1}|<\ell\le|A_j|\,\}}.$$

It can be checked (using our results in §2) that the conditions in Proposition 1.2 and Remark 1.4 are satisfied. Hence $\mathbf{F}_X$ is the minimal free resolution of M. ∎

Example 1.10 Let $S = k[a,b,c,d,e,f]$. Following [BH, page 228] we consider the Stanley-Reisner ideal of the minimal triangulation of the *real projective plane* $\mathbb{RP}^2$,

$$M \;=\; \langle\, abc,\ abf,\ ace,\ ade,\ adf,\ bcd,\ bde,\ bef,\ cdf,\ cef \,\rangle.$$

The dual in $\mathbb{RP}^2$ of this triangulation is a cell complex X consisting of six pentagons. The ten vertices of X are labeled by the generators of M. We illustrate $X \simeq \mathbb{RP}^2$ as the disk shown on the left in Figure 2; antipodal points on the boundary are to be identified. The small pictures on the right will be discussed in Example 2.14.



$a \;=\; 0$
6 cycles

$a \;=\; 1$
6 cycles

$b+c+d \;=\; 1$
10 cycles

Figure 2

If $\operatorname{char} k \neq 2$ then X is acyclic over k and the cellular complex $\mathbf{F}_X$ coincides with the minimal free resolution $0 \to S^6 \to S^{15} \to S^{10} \to M$. If $\operatorname{char} k = 2$ then X is not acyclic over k, and the cellular complex $\mathbf{F}_X$ is not a resolution of M. ∎

Returning to the general theory, we next present a formula for the *Betti number* $\beta_{i,\mathbf{b}} = \dim \operatorname{Tor}_i(M,k)_{\mathbf{b}}$ which is the number of minimal ith syzygies in degree $\mathbf{b}$. The degree $\mathbf{b} \in \mathbb{Z}^n$ is called a *Betti degree* of M if $\beta_{i,\mathbf{b}} \neq 0$ for some i.

Theorem 1.11 *If $\mathbf{F}_X$ is a cellular resolution of a monomial module M then*

$$\beta_{i,\mathbf{b}} \quad = \quad \dim H_i(X_{\preceq\mathbf{b}}, X_{\prec\mathbf{b}}; k) \quad = \quad \dim \widetilde{H}_{i-1}(X_{\prec\mathbf{b}}; k),$$

where H_* denotes relative homology and $\widetilde{H}_*$ denotes reduced homology.

Proof. We compute $\mathrm{Tor}_i(M,k)_{\mathbf{b}}$ as the ith homology of the complex of vector spaces $(\mathbf{F}_X \otimes_S k)_{\mathbf{b}}$. This complex equals the chain complex $\widetilde{C}(X_{\preceq \mathbf{b}}, X_{\prec \mathbf{b}}; k)$ which computes the relative homology with coefficients in k of the pair $(X_{\preceq \mathbf{b}}, X_{\prec \mathbf{b}})$. Thus

$$\mathrm{Tor}_i(M,k)_{\mathbf{b}} \quad = \quad H_i(X_{\preceq \mathbf{b}}, X_{\prec \mathbf{b}}; k).$$

Since $X_{\preceq \mathbf{b}}$ is acyclic, the long exact sequence of homology groups looks like

$$0 \,=\, \widetilde{H}_i(X_{\preceq \mathbf{b}}; k) \,\to\, H_i(X_{\preceq \mathbf{b}}, X_{\prec \mathbf{b}}; k) \,\to\, \widetilde{H}_{i-1}(X_{\prec \mathbf{b}}; k) \,\to\, \widetilde{H}_{i-1}(X_{\preceq \mathbf{b}}; k) \,=\, 0.$$

We conclude that the two vector spaces in the middle are isomorphic. ∎

A subset $Q \subset \mathbb{Z}^n$ is an *order ideal* if $\mathbf{b} \in Q$ and $\mathbf{c} \in \mathbb{N}^n$ implies $\mathbf{b} - \mathbf{c} \in Q$. For a $\mathbb{Z}^n$-graded cell complex X and an order ideal Q we define the *order ideal complex* $X_Q = \{\, F \in X \;\mid\; \mathbf{a}_F \in Q \,\}$. Note that $X_{\prec \mathbf{b}}$ and $X_{\preceq \mathbf{b}}$ are special cases of this.

Corollary 1.12 *If $\mathbf{F}_X$ is a cellular resolution of M and $Q \subset \mathbb{Z}^n$ an order ideal which contains the Betti degrees of M, then $\mathbf{F}_{X_Q}$ is also a cellular resolution of M.*

Proof. By Corollary 1.3 and the identity $(X_Q)_{\preceq \mathbf{b}} = (X_{\preceq \mathbf{b}})_Q$, it suffices to prove this for the case where M is a monomial ideal and X is finite. We proceed by induction on the number of faces in $X \backslash X_Q$. If $X_Q = X$ there is nothing to prove. Otherwise pick $\mathbf{c} \in \mathbb{Z}^n \backslash Q$ such that $X_{\preceq \mathbf{c}} = X$ and $X_{\prec \mathbf{c}} \neq X$. Since $\mathbf{c}$ is not a Betti degree, Theorem 1.11 implies that the complex $X_{\prec \mathbf{c}}$ is acyclic. For any $\mathbf{b} \in \mathbb{Z}^n$, the complex $(X_{\prec \mathbf{c}})_{\preceq \mathbf{b}}$ equals either $X_{\prec \mathbf{c}}$ or $X_{\preceq\, \mathbf{b} \wedge \mathbf{c}}$ and is hence acyclic. At this point we replace X by the proper subcomplex $X_{\prec \mathbf{c}}$, and we are done by induction. ∎

By Proposition 1.5 and Theorem 1.11, the Betti numbers $\beta_{i,\mathbf{b}}$ of M are given by the reduced homology of $\Delta_{\prec \mathbf{b}}$. Let us compare that formula for $\beta_{i,\mathbf{b}}$ with the following formula which is due independently to Hochster [Ho] and Rosenknop [Ros].

Corollary 1.13 *The Betti numbers of M satisfy $\beta_{i,\mathbf{b}} \,=\, \dim \widetilde{H}_{i-1}(K_{\mathbf{b}}; k)$ where $K_{\mathbf{b}}$ is the simplicial complex $\{\, \sigma \,\subseteq\, \{1, \ldots, n\} \,|\, M$ has a generator of degree $\preceq \mathbf{b} - \sigma\}$. Here each face σ of $K_{\mathbf{b}}$ is identified with its characteristic vector in $\{0,1\}^n$.*

Proof. For $i \in \{1, \ldots, n\}$ consider the subcomplex of $\Delta_{\prec \mathbf{b}}$ consisting of all faces F with degree $\mathbf{a}_F \preceq \mathbf{b} - \{i\}$. This subcomplex is a full simplex. Clearly, these n simplices cover $\Delta_{\prec \mathbf{b}}$. The nerve of this cover by contractible subsets is the simplicial complex $K_{\mathbf{b}}$. Therefore, $K_{\mathbf{b}}$ has the same reduced homology as $\Delta_{\prec \mathbf{b}}$. ∎

2 The hull resolution

Let M be a monomial module in $T = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. For $\mathbf{a} \in \mathbb{Z}^n$ and $t \in \mathbb{R}$ we abbreviate $t^{\mathbf{a}} = (t^{a_1}, \ldots, t^{a_n})$. Fix any real number t larger than $(n+1)! = 2 \cdot 3 \cdot \cdots \cdot (n+1)$. We define P_t to be the convex hull of the point set

$$\{\, t^{\mathbf{a}} \mid \mathbf{a} \text{ is the exponent of a monomial } \mathbf{x}^{\mathbf{a}} \in M \,\} \quad \subset \quad \mathbb{R}^n.$$

The set P_t is an unbounded n-dimensional convex polyhedron.

Lemma 2.1 *The polyhedron P_t is closed if and only if M is co-Artinian.*

Proof. We first prove the "only-if" direction. Suppose that M is not co-Artinian. Then there exists a monomial $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{x}^{\mathbf{a}} \cdot x_i^j \in M$ for all $j \in \mathbb{Z}$. Without loss of generality, we may assume $i = 1$ and $\mathbf{x}^{\mathbf{a}} = x_2^{a_2} \cdots x_n^{a_n}$. Note that $t^{\mathbf{a}} = (1, t^{a_2}, \ldots, t^{a_n})$. The condition $\mathbf{x}^{\mathbf{a}} \cdot x_1^j \in M$ for all j implies that the set $\{(t^j, t^{a_2}, \ldots, t^{a_n}) : j = 0, -1, -2, \ldots\}$ is contained in P_t. The point $(0, t^{a_2}, \ldots, t^{a_n})$ lies in the closure of this set, but it does not lie in P_t because all points in P_t have positive coordinates. Hence P_t is not a closed subset of $\mathbb{R}^n$.

For the converse assume that M is co-Artinian, that is, $M = S \cdot \min(M)$. We shall prove the following identity, which shows that P_t is a closed subset of $\mathbb{R}^n$:

$$P_t \;=\; \mathbb{R}^n_+ \;+\; \text{conv}\,\{\, t^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}} \in \min(M) \,\} \tag{2.1}$$

Here $\mathbb{R}^n_+$ denotes the closed orthant consisting of all non-negative real vectors.

We first prove the inclusion $\subseteq$ in (2.1). Let $\mathbf{x}^{\mathbf{b}}$ be any monomial in M. Then there exists a minimal generator $\mathbf{x}^{\mathbf{a}} \in \min(M)$ which divides $\mathbf{x}^{\mathbf{b}}$. This implies $t^{a_i} \leq t^{b_i}$ for all i, and hence $t^{\mathbf{b}} - t^{\mathbf{a}} \in \mathbb{R}^n_+$. Thus $t^{\mathbf{b}}$ lies in the right hand side of (2.1). Since the right hand side of (2.1) is a convex set, it therefore contains P_t.

For the converse it suffices to prove that $t^{\mathbf{a}} + \mathbb{R}^n_+ \subseteq P_t$ for all $\mathbf{x}^{\mathbf{a}} \in \min(M)$. Fix $t^{\mathbf{a}} + \mathbf{u} \in t^{\mathbf{a}} + \mathbb{R}^n_+$ where $\mathbf{u} = (u_1, \ldots, u_n)$ is a non-negative real vector. Choose a positive integer r such that $0 \leq u_j \leq t^{a_j + r} - t^{a_j}$ for $j = 1, \ldots, n$. Let C be the convex hull of the 2^n points $t^{\mathbf{a}} + \sum_{j \in J} (t^{a_j + r} - t^{a_j}) \cdot \mathbf{e}_j$ where J runs over all subsets of $\{1, \ldots, n\}$. These points represent the monomials $\mathbf{x}^{\mathbf{a}} \cdot \prod_{j \in J} x_j^r$. The cube C is contained in P_t and it contains $t^{\mathbf{a}} + \mathbf{u}$, so we are done. ∎

From now on let M be a co-Artinian monomial module. In this section we apply convexity methods to construct a canonical cellular resolution of M.

Proposition 2.2 *The vertices of the polyhedron P_t are precisely the points $t^{\mathbf{a}} = (t^{a_1}, \ldots, t^{a_n})$ for which the monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ is a minimal generator of M.*

Proof. Suppose $\mathbf{x}^{\mathbf{a}} \in M$ is not a minimal generator of M. Then $t^{\mathbf{a}}$ is not a vertex of P_t, by formula (2.1). Conversely, suppose $\mathbf{x}^{\mathbf{a}} \in M$ is a minimal generator of M. Let $\mathbf{v} = t^{-\mathbf{a}}$, so $t^{\mathbf{a}} \cdot \mathbf{v} = n$. For any other exponent $\mathbf{b}$ of a monomial in M, we have $b_i \geq a_i + 1$ for some i, so

$$t^{\mathbf{b}} \cdot \mathbf{v} \quad = \quad \sum_{j=1}^{n} t^{b_j - a_j} \quad \geq \quad t^{b_i - a_i} \quad \geq \quad t \quad > \quad (n+1)! \quad > \quad n.$$

Thus, the inner normal vector $\mathbf{v}$ supports $t^{\mathbf{a}}$ as a vertex of P_t. ∎

Our first goal is to establish the following combinatorial result.

Theorem 2.3 *The face poset of the polyhedron* P_t *is independent of* t *for* $t > (n+1)\,!$. *The same holds for the subposet of all bounded faces of* P_t.

Proof. The face poset of P_t can be computed as follows. Let $C_t \subset \mathbb{R}^{n+1}$ be the cone spanned by the vectors $(t^{\mathbf{a}}, 1)$ for all minimal generators $\mathbf{x}^{\mathbf{a}}$ of M together with the unit vectors $(\mathbf{e}_i, 0)$ for $i = 1, \ldots, n$. The faces of P_t are in order-preserving bijection with the faces of C_t which do not lie in the hyperplane "at infinity" $x_{n+1} = 0$. A face of P_t is bounded if and only if the corresponding face of C_t contains none of the vectors $(\mathbf{e}_i, 0)$. It suffices to prove that the face poset of C_t is independent of t.

For any $(n+1)$-tuple of generators of C_t consider the sign of their determinant

$$\text{sign}\ \det \begin{bmatrix} \mathbf{e}_{i_0} & \cdots & \mathbf{e}_{i_r} & t^{\mathbf{a}_{j_1}} & \cdots & t^{\mathbf{a}_{j_{n-r}}} \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix} \quad \in \quad \{-1, 0, +1\}. \tag{2.2}$$

The list of these signs forms the (possibly infinite) *oriented matroid* associated with the cone C_t. It is known (see e.g. [BLWSZ]) that the face poset of C_t is determined by its oriented matroid. It therefore suffices to show that the sign of the determinant in (2.2) is independent of t for $t > (n+1)\,!$. This follows from the next lemma. ∎

Lemma 2.4 *Let* a_{ij} *be integers for* $1 \le i, j \le r$. *Then the Laurent polynomial* $f(t) = \det\big((t^{a_{ij}})_{1 \le i,j \le r}\big)$ *either vanishes identically or has no real roots for* $t > r\,!$.

Proof. Suppose that f is not zero and write $f(t) = c_\alpha t^\alpha + \sum_\beta c_\beta t^\beta$, where the first term has the highest degree in t. For $t > r!$ we have the chain of inequalities

$$|\sum_\beta c_\beta \cdot t^\beta| \le \sum_\beta |c_\beta| \cdot t^\beta \le (\sum_\beta |c_\beta|) \cdot t^{\alpha-1} < r\,! \cdot t^{\alpha-1} < t^\alpha \le |c_\alpha \cdot t^\alpha|.$$

Therefore $f(t)$ is nonzero, and $\text{sign}\big(f(t)\big) = \text{sign}(c_\alpha)$. ∎

In the proof of Theorem 2.3 we are using Lemma 2.4 for $r = n+1$. Lev Borisov and Sorin Popescu constructed examples of matrices which show that the exponential lower bound for t is necessary in Lemma 2.4, and also in Theorem 2.3.

We are now ready to define the hull resolution and state our main result. The *hull complex* of a monomial module M, denoted hull(M), is the complex of bounded faces of the polyhedron P_t for large t. Theorem 2.3 ensures that hull(M) is well-defined and depends only on M. The vertices of hull(M) are labeled by the generators of M, by Proposition 2.2, and hence the complex hull(M) is $\mathbb{Z}^n$-graded. Let $\mathbf{F}_{\text{hull}(M)}$ be the complex of free S-modules derived from hull(M) as in Section 1.

Theorem 2.5 *The cellular complex* $\mathbf{F}_{\text{hull}(M)}$ *is a free resolution of* M.

Proof. Let $X = (\text{hull}(M))_{\preceq\mathbf{b}}$ for some degree $\mathbf{b}$; by Proposition 1.2 we need to show that X is acyclic. This is immediate if X is empty or a single vertex. Otherwise choose $t > (n+1)\,!$ and let $\mathbf{v} = t^{-\mathbf{b}}$. If $t^{\mathbf{a}}$ is a vertex of X then $\mathbf{a} \prec \mathbf{b}$, so

$$t^{\mathbf{a}} \cdot \mathbf{v} \quad = \quad t^{-\mathbf{b}} \cdot t^{\mathbf{a}} \quad < \quad t^{-\mathbf{b}} \cdot t^{\mathbf{b}} \quad = \quad n,$$

while for any other minimal generator $\mathbf{x}^{\mathbf{c}} \in M$ we have $c_i \geq b_i + 1$ for some i, so

$$t^{\mathbf{c}} \cdot \mathbf{v} \quad = \quad t^{-\mathbf{b}} \cdot t^{\mathbf{c}} \quad > \quad t^{c_i - b_i} \quad \geq \quad t \quad > \quad n.$$

Thus, the hyperplane H defined by $\mathbf{x} \cdot \mathbf{v} = n$ separates the vertices of X from the remaining vertices of P_t. Make a projective transformation which moves H to infinity. This expresses X as the complex of bounded faces of a convex polyhedron, a complex which is known to be contractible, e.g. [BLSWZ, Exercise 4.27 (a)]. ∎

We call $\mathbf{F}_{\text{hull}(M)}$ the *hull resolution* of M. Let us see that the hull resolution generalizes the *Scarf complex* introduced in [BPS]. This is the simplical complex

$$\Delta_M \quad = \quad \{\; F \subseteq I \;\mid\; m_F \neq m_G \text{ for all } G \subseteq I \text{ other than } F \;\}.$$

The Scarf complex Δ_M defines a subcomplex $\mathbf{F}_{\Delta_M}$ of the Taylor resolution $\mathbf{F}_{\Delta}$.

Proposition 2.6 *For any monomial module M, the Scarf complex Δ_M is a subcomplex of the hull complex* hull(M).

Proof. Let $F = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_p}\}$ be a face of Δ_M with $m_F = \text{lcm}(F) = \mathbf{x}^{\mathbf{u}}$. Consider any injective map $\sigma : \{1, \ldots, p\} \to \{1, \ldots, n\}$ such that $a_{i,\sigma(i)} = u_i$ for all i. Compute the inverse over $\mathbb{Q}(t)$ of the $p \times p$-matrix $(t^{a_{i,\sigma(j)}})$, and let $\mathbf{v}^{\sigma}(t)'$ be the sum of the column vectors of that inverse matrix. By augmenting the p-vector $\mathbf{v}^{\sigma}(t)'$ with additional zero coordinates, we obtain an n-vector $\mathbf{v}^{\sigma}(t)$ with the following properties:

(i) $t^{\mathbf{a}_1} \cdot \mathbf{v}^{\sigma}(t) = t^{\mathbf{a}_2} \cdot \mathbf{v}^{\sigma}(t) = \cdots = t^{\mathbf{a}_p} \cdot \mathbf{v}^{\sigma}(t) \;=\; 1$;
(ii) $v^{\sigma}_j(t) \;=\; 0$, for all $j \notin \text{image}(\sigma)$;
(iii) $v^{\sigma}_j(t) \;=\; t^{-u_j} +$ *lower order terms in* t, for all $j \in \text{image}(\sigma)$.

By taking a convex combination of the vectors $\mathbf{v}^{\sigma}(t)$ for all possible injective maps σ as above, we obtain a vector $\mathbf{v}(t)$ with the following properties:

(iv) $t^{\mathbf{a}_1} \cdot \mathbf{v}(t) = t^{\mathbf{a}_2} \cdot \mathbf{v}(t) = \cdots = t^{\mathbf{a}_p} \cdot \mathbf{v}(t) \;=\; 1$;
(v) $v_j(t) \;=\; c_j \cdot t^{-u_j} +$ *lower order terms in* t with $c_j > 0$, for all $j \in \{1, \ldots, n\}$.

For any $\mathbf{x}^{\mathbf{b}} \in M$ which is not in F there exists an index ℓ such that $b_\ell \geq u_\ell + 1$. This implies $\mathbf{v}(t) \cdot t^{\mathbf{b}} \geq c_\ell \cdot t^{b_\ell - u_\ell} +$ *lower order terms in* t, and therefore $\mathbf{v}(t) \cdot t^{\mathbf{b}} > 1$ for $t \gg 0$. We conclude that F defines a face of P_t with inner normal vector $\mathbf{v}(t)$. ∎

A binomial first syzygy of M is called *generic* if it has full support, i.e., if no variable x_i appears with the same exponent in the corresponding pair of monomial generators. We call M *generic* if it has a basis of generic binomial first syzygies. This is a translation-invariant generalization of the definition of genericity in [BPS].

Lemma 2.7 *If M is generic, then for any pair of generators m_i, m_j either the corresponding binomial first syzygy is generic, or there exists a third generator m which strictly divides the least common multiple of m_i and m_j in all coordinates.*

Proof. Suppose that the syzygy formed by m_i and m_j is not generic, and induct on the length of a chain of generic syzygies needed to express it. If the chain has length two, then the middle monomial m divides $\operatorname{lcm}(m_i, m_j)$. Moreover, because the two syzygies involving m are generic, this division is strict in each variable. If the chain is longer, then divide it into two steps. Either each step represents a generic syzygy, and we use the above argument, or by induction there exists an m strictly dividing the degree of one of these syzygies in all coordinates, and we are again done. ∎

Lemma 2.8 *Let M be a monomial module and F a face of* $\operatorname{hull}(M)$*. For every monomial $m \in M$ there exists a variable x_j such that* $\deg_{x_j}(m) \geq \deg_{x_j}(m_F)$.

Proof. Suppose that $m = \mathbf{x}^{\mathbf{u}}$ strictly divides m_F in each coordinate. Let $t^{\mathbf{a}_1}, \ldots, t^{\mathbf{a}_p}$ be the vertices of F and consider their barycenter $\mathbf{v}(t) = \frac{1}{p} \cdot (t^{\mathbf{a}_1} + \cdots + t^{\mathbf{a}_p}) \in F$. The jth coordinate of $\mathbf{v}(t)$ is a polynomial in t of degree equal to $\deg_{x_j}(m_F)$. The jth coordinate of $t^{\mathbf{u}}$ is a monomial of strictly lower degree. Hence $t^{\mathbf{u}} < \mathbf{v}(t)$ coordinatewise for $t \gg 0$. Let $\mathbf{w}$ be a nonzero linear functional which is nonnegative on $\mathbb{R}^n_+$ and whose minimum over P_t is attained at the face F. Then $\mathbf{v}(t) \cdot \mathbf{w} = \mathbf{a}_1 \cdot \mathbf{w} = \cdots = \mathbf{a}_p \cdot \mathbf{w}$, but our discussion implies $t^{\mathbf{u}} \cdot \mathbf{w} < \mathbf{v}(t) \cdot \mathbf{w}$, a contradiction. ∎

Theorem 2.9 *If M is a generic monomial module then* $\operatorname{hull}(M)$ *coincides with the Scarf complex Δ_M of M, and the hull resolution $\mathbf{F}_{\operatorname{hull}(M)} = \mathbf{F}_{\Delta_M}$ is minimal.*

Proof. Let F be any face of $\operatorname{hull}(M)$ and $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_p}$ the generators of M corresponding to the vertices of F. Suppose that F is not a face of Δ_M. Then either

(i) $\operatorname{lcm}(\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{i-1}}, \mathbf{x}^{\mathbf{a}_{i+1}}, \ldots, \mathbf{x}^{\mathbf{a}_p}) = m_F$ for some $i \in \{1, \ldots, p\}$, or

(ii) there exists another generator $\mathbf{x}^{\mathbf{u}}$ of M which divides m_F and such that $t^{\mathbf{u}} \notin F$.

Consider first case (i). By Lemma 2.8 applied to $m = \mathbf{x}^{\mathbf{a}_i}$ there exists x_j such that $\deg_{x_j}(\mathbf{x}^{\mathbf{a}_i}) = \deg_{x_j}(m_F)$, and hence $\deg_{x_j}(\mathbf{x}^{\mathbf{a}_i}) = \deg_{x_j}(\mathbf{x}^{\mathbf{a}_k})$ for some $k \neq i$. The first syzygy between $\mathbf{x}^{\mathbf{a}_i}$ and $\mathbf{x}^{\mathbf{a}_k}$ is not generic, and, by Lemma 2.7, there exists a generator m of M which strictly divides $\operatorname{lcm}(\mathbf{x}^{\mathbf{a}_i}, \mathbf{x}^{\mathbf{a}_k})$ in all coordinates. Since $\operatorname{lcm}(\mathbf{x}^{\mathbf{a}_i}, \mathbf{x}^{\mathbf{a}_k})$ divides m_F, we get a contradiction to Lemma 2.8.

Consider now case (ii). For any variable x_j there exists $i \in \{1, \ldots, p\}$ such that $\deg_{x_j}(\mathbf{x}^{\mathbf{a}_i}) = \deg_{x_j}(m_F) \geq \deg_{x_j}(\mathbf{x}^{\mathbf{u}})$. If the inequality $\geq$ is an equality $=$, then the first syzygy between $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{a}_i}$ is not generic, and Lemma 2.7 gives a new monomial generator m which strictly divides m_F in all coordinates, a contradiction to Lemma 2.8. Therefore $\geq$ is a strict inequality $>$ for all variables x_j. This means that $\mathbf{x}^{\mathbf{u}}$ strictly divides m_F in all coordinates, again a contradiction to Lemma 2.8.

Hence both (i) and (ii) lead to a contradiction, and we conclude that every face of $\operatorname{hull}(M)$ is a face of Δ_M. This implies $\operatorname{hull}(M) = \Delta_M$ by Proposition 2.6. The resolution $\mathbf{F}_{\Delta_M}$ is minimal because no two faces in Δ_M have the same degree. ∎

In this paper we are mainly interested in nongeneric monomial modules for which the hull complex is typically not simplicial. Nevertheless the possible combinatorial types of facets seem to be rather limited. Experimental evidence suggests:

Conjecture 2.10 *Every face of* hull(M) *is affinely isomorphic to a subpolytope of the* $(n-1)$*-dimensional permutohedron and hence has at most* $n\,!$ *vertices.*

By Example 1.9 it is easy to see that any subpolytope of the $(n-1)$-dimensional permutohedron can be realized as the hull complex of suitable monomial ideal.

The following example, found in discussions with Lev Borisov, shows that the hull complex of a monomial module need not be locally finite:

Example 2.11 Let $n = 3$ and M the monomial module generated by $x_1^{-1}x_2$ and $\{\, x_2^i x_3^{-i} \mid i \in \mathbb{Z} \,\}$. Then every triangle of the form $\{x_1^{-1}x_2,\, x_2^i x_3^{-i},\, x_2^{i+1}x_3^{-i-1}\}$ is a facet of hull(M). In particular, the vertex $x_1^{-1}x_2$ of hull(M) has infinite valence. ∎

For a generic monomial module M we have the following important identity

$$\mathrm{hull}(M_{\preceq \mathbf{b}}) \quad = \quad \mathrm{hull}(M)_{\preceq \mathbf{b}}.$$

See equation (5.1) in [BPS]. This identity can fail if M is not generic:

Example 2.12 Consider the monomial ideal $M = \langle\, a^2b,\, ac,\, b^2,\, bc^2 \,\rangle$ studied in Example 1.6 and let $\mathbf{b} = (2,1,2)$. Then hull$(M_{\preceq \mathbf{b}})$ is a triangle, while hull$(M)_{\preceq \mathbf{b}}$ consists of two edges. The vertex b^2 of hull(M) "eclipses" the facet of hull$(M_{\preceq \mathbf{b}})$ ∎

The hull complex hull(M) is particularly easy to compute if M is a squarefree monomial ideal. In view of the identity $(t-1)\cdot \mathbf{a} + (1,1,\ldots,1) = t^{\mathbf{a}}$ for 0-1-vectors $\mathbf{a}$, we can identify P_t for $t > 1$ with the convex hull of all exponent vectors appearing in M. Moreover, if all square-free generators of M have the same total degree, then the faces of the convex hull of their exponent vectors are precisely the bounded faces of P_t. Theorem 2.5 implies the following corrollary.

Corollary 2.13 *Let* $\mathbf{a}_1, \ldots, \mathbf{a}_p$ *be* 0-1*-vectors having the same coordinate sum. Then their boundary complex, consisting of all faces of the convex polytope* $P =$ conv$\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$, *defines a cellular resolution of the ideal* $M = \langle \mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_p} \rangle$.

Example 2.14 Corollary 2.13 applies to the Stanley-Reisner ideal of the real projective plane in Example 1.10. Here P is a 5-dimensional polytope with 22 facets, corresponding to the 22 cycles on the 2-complex X of length ≤ 6. Representatives of these three cycle types, and supporting hyperplanes of the corresponding facets of P, are shown on the right in Figure 2. This example illustrates how the hull resolution encodes combinatorial information without making arbitrary choices. ∎

3 Lattice ideals

Let $L \subset \mathbb{Z}^n$ be a lattice such that $L \cap \mathbb{N}^n = \{0\}$. In this section we study (cellular) resolutions of the lattice module M_L and of the lattice ideal I_L. Our hypothesis $L \cap \mathbb{N}^n = \{0\}$ guarantees that M_L is co-Artinian, so all the results in Sections 1 and 2 are applicable to M_L.

Let $S[L]$ be the group algebra of L over S. We realize $S[L]$ as the subalgebra of $k[x_1, \ldots, x_n, z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ spanned by all monomials $\mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}}$ where $\mathbf{a} \in \mathbb{N}^n$ and $\mathbf{b} \in L$. Note that $S = S[L]/\langle\, \mathbf{z}^{\mathbf{a}} - 1 \mid \mathbf{a} \in L\,\rangle$.

Lemma 3.1 *The lattice module M_L is an $S[L]$-module, and $M_L \otimes_{S[L]} S = S/I_L$.*

Proof. The k-linear map $\phi : S[L] \to M_L$, $\mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}} \mapsto \mathbf{x}^{\mathbf{a}+\mathbf{b}}$ defines the structure of an $S[L]$-module on M_L. Its kernel $\ker(\phi)$ is the ideal in $S[L]$ generated by all binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}\mathbf{z}^{\mathbf{u}-\mathbf{v}}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ and $\mathbf{u} - \mathbf{v} \in L$. Clearly, we obtain I_L from $\ker(\phi)$ by setting all $\mathbf{z}$-variables to 1, and hence $(S[L]/\ker(\phi)) \otimes_{S[L]} S = S/I_L$. ∎

We define a $\mathbb{Z}^n$-grading on $S[L]$ via $\deg(\mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}}) = \mathbf{a} + \mathbf{b}$. Let $\mathcal{A}$ be the category of $\mathbb{Z}^n$-graded $S[L]$-modules, where the morphisms are $\mathbb{Z}^n$-graded $S[L]$-module homomorphisms of degree $\mathbf{0}$. The polynomial ring $S = k[x_1, \ldots, x_n]$ is graded by the quotient group $\mathbb{Z}^n/L$ via $\deg(\mathbf{x}^{\mathbf{a}}) = \mathbf{a} + L$. Let $\mathcal{B}$ be the category of $\mathbb{Z}^n/L$-graded S-modules, where the morphisms are $\mathbb{Z}^n/L$-graded S-module homomorphisms of degree $\mathbf{0}$. Clearly, M_L is an object in $\mathcal{A}$, and $M_L \otimes_{S[L]} S = S/I_L$ is an object in $\mathcal{B}$.

Theorem 3.2 *The categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent.*

Proof. Define a functor $\pi : \mathcal{A} \to \mathcal{B}$ by the rule $\pi(M) := M \otimes_{S[L]} S$. This functor weakens the $\mathbb{Z}^n$-grading of objects in $\mathcal{A}$ to a $\mathbb{Z}^n/L$-grading. The properties of π cannot be deduced from the tensor product alone, which is poorly behaved when applied to arbitrary $S[L]$-modules; e.g., S is not a flat $S[L]$-module. Further, the categories $\mathcal{A}$ and $\mathcal{B}$ are *not isomorphic*; we are only claiming that they are *equivalent*. For instance, $S[L]$ and $S[L](\mathbf{a})$ are different objects in $\mathcal{A}$ even for $\mathbf{a} \in L$, but they are isomorphic under multiplication by the unit $\mathbf{z}^{\mathbf{a}}$.

We apply condition iii) of [Mac, §IV.4, Theorem 1]: It is enough to prove that π is full and faithful, and that each object $N \in \mathcal{B}$ is isomorphic to $\pi(M)$ for some object $M \in \mathcal{A}$. To prove that π is full and faithful, we show that for any two modules M, $M' \in \mathcal{A}$ it induces an identification $\mathrm{Hom}_{\mathcal{A}}(M, M') = \mathrm{Hom}_{\mathcal{B}}(\pi(M), \pi(M'))$.

Because each module $M \in \mathcal{A}$ is $\mathbb{Z}^n$-graded, the lattice $L \subset S[L]$ acts on M as a group of automorphisms, i.e. the multiplication maps $\mathbf{z}^{\mathbf{b}} : M_{\mathbf{a}} \to M_{\mathbf{a}+\mathbf{b}}$ are isomorphisms of k-vector spaces for each $\mathbf{b} \in L$, compatible with multiplication by each x_i. For each $\alpha \in \mathbb{Z}^n/L$, the functor π identifies the spaces $M_{\mathbf{a}}$ for $\mathbf{a} \in \alpha$ as the single space $\pi(M)_\alpha$. A morphism $f : M \to M'$ in $\mathcal{A}$ is a collection of k-linear maps $f_{\mathbf{a}} : M_{\mathbf{a}} \to M'_{\mathbf{a}}$, compatible with the action by L and with multiplication by each x_i. A morphism $g : \pi(M) \to \pi(M')$ in $\mathcal{B}$ is a collection of k-linear maps $g_\alpha : \pi(M)_\alpha \to \pi(M')_\alpha$, compatible with multiplication by each x_i. For each $\alpha \in \mathbb{Z}^n/L$, the functor π identifies the maps $f_{\mathbf{a}}$ for $\mathbf{a} \in \alpha$ as the single map $\pi(f)_\alpha$.

It is clear from this discussion that π takes distinct morphisms to distinct morphisms. Given a morphism $g \in \mathrm{Hom}_{\mathcal{B}}(\pi(M), \pi(M'))$, define a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(M, M')$ by the rule $f_{\mathbf{a}} = g_\alpha$ for $\mathbf{a} \in \alpha$. We have $\pi(f) = g$, establishing the desired identification of Hom-sets. Hence π is full and faithful.

Finally, let $N = \bigoplus_{\alpha \in \mathbb{Z}^n/L} N_\alpha$ be any object in $\mathcal{B}$. We define an object $M = \oplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ in $\mathcal{A}$ by setting $M_{\mathbf{a}} := N_\alpha$ for each $\mathbf{a} \in \alpha$, by lifting each multiplication map $x_i : N_\alpha \to N_{\alpha + \mathbf{e}_i}$ to maps $x_i : M_{\mathbf{a}} \to M_{\mathbf{a}+\mathbf{e}_i}$ for $\mathbf{a} \in \alpha$, and by letting $\mathbf{z}^{\mathbf{b}}$ act on M as the identity map from $M_{\mathbf{a}}$ to $M_{\mathbf{a}+\mathbf{b}}$ for $\mathbf{b} \in L$. The module M satisfies $\pi(M) = N$, showing that π is an equivalence of categories. ∎

Theorem 3.2 allows us to resolve the lattice module $M_L \in \mathcal{A}$ in order to resolve the quotient ring $\pi(M_L) = S/I_L \in \mathcal{B}$, and conversely.

Corollary 3.3 *A $\mathbb{Z}^n$-graded complex of free $S[L]$-modules,*

$$C: \qquad \cdots \xrightarrow{f_2} S[L]^{\beta_1} \xrightarrow{f_1} S[L]^{\beta_0} \xrightarrow{f_0} S[L] \to M_L \to 0,$$

is a (minimal) free resolution of M_L if and only if its image

$$\pi(C) : \cdots \xrightarrow{\pi(f_2)} S^{\beta_1} \xrightarrow{\pi(f_1)} S^{\beta_0} \xrightarrow{\pi(f_0)} S \to S/I_L \to 0,$$

is a (minimal) $\mathbb{Z}^n/L$-graded resolution of S/I_L by free S-modules.

Proof. This follows immediately from Lemma 3.1 and Theorem 3.2. ∎

Since $S[L]$ is a free S-module, every resolution C as in the previous corollary gives rise to a resolution of M_L as a $\mathbb{Z}^n$-graded S-module. We demonstrate in an example how resolutions of M_L over S are derived from resolutions of S/I_L over S.

Example 3.4 Let $S = k[x_1, x_2, x_3]$ and $L = \ker [\,1\ 1\ 1\,] \subset \mathbb{Z}^3$. Then $\mathbb{Z}^3/L \simeq \mathbb{Z}$, $I_L = \langle\, x_1 - x_2, x_2 - x_3 \,\rangle$, and M_L is the module generated by all monomials of the form $x_1^i x_2^j x_3^{-i-j}$. The ring S/I_L is resolved by the Koszul complex

$$0 \longrightarrow S(-2) \xrightarrow{\begin{bmatrix} x_2 - x_3 \\ x_2 - x_1 \end{bmatrix}} S(-1)^2 \xrightarrow{[\ x_1 - x_2 \quad x_2 - x_3\ ]} S \longrightarrow S/I_L.$$

This is a $\mathbb{Z}^3/L$-graded complex of free S-modules. An inverse image under π equals

$$\begin{aligned} 0 \longrightarrow S[L]\big(-(1,1,0)\big) &\xrightarrow{\begin{bmatrix} x_2 - x_3 z_2 z_3^{-1} \\ x_2 - x_1 z_2 z_1^{-1} \end{bmatrix}} S[L]\big(-(1,0,0)\big) \oplus S[L]\big(-(0,1,0)\big) \\ &\xrightarrow{[\ x_1 - x_2 z_1 z_2^{-1} \quad x_2 - x_3 z_2 z_3^{-1}\ ]} S[L] \longrightarrow M_L. \end{aligned}$$

Writing each term as a direct sum of free S-modules, for instance, $S[L]\bigl(-(1,1,0)\bigr) = \oplus_{i+j+k=2} S\bigl(-(i,j,k)\bigr)$, we get a $\mathbb{Z}^3$-graded minimal free resolution of M_L over S:

$$0 \to \bigoplus_{i+j+k=2} S\bigl(-(i,j,k)\bigr) \to \bigoplus_{i+j+k=1} S\bigl(-(i,j,k)\bigr)^2 \to \bigoplus_{i+j+k=0} S\bigl(-(i,j,k)\bigr) \to M_L.$$

∎

Our goal is to define and study cellular resolutions of the lattice ideal I_L. Let X be a $\mathbb{Z}^n$-graded cell complex whose vertices are the generators of M_L. Each cell $F \in X$ is identified with its set of vertices, regarded as a subset of L. The cell complex X is called *equivariant* if $F \in X$ and $\mathbf{b} \in L$ implies that $F + \mathbf{b} \in X$, and if the incidence function satisfies $\varepsilon(F, F') = \varepsilon(F + \mathbf{b}, F' + \mathbf{b})$ for all $\mathbf{b} \in L$.

Lemma 3.5 *If X is an equivariant $\mathbb{Z}^n$-graded cell complex on M_L then the cellular complex $\mathbf{F}_X$ has the structure of a $\mathbb{Z}^n$-graded complex of free $S[L]$-modules.*

Proof. The group L acts on the faces of X. Let X/L denote the set of orbits. For each orbit $\mathcal{F} \in X/L$ we select a distinguished representative $F \in \mathcal{F}$, and we write $\mathrm{Rep}(X/L)$ for the set of representatives. The following map is an isomorphism of $\mathbb{Z}^n$-graded S-modules, which defines the structure of a free $S[L]$-module on $\mathbf{F}_X$:

$$\bigoplus_{F \in \mathrm{Rep}(X/L)} S[L] \cdot e_F \quad \simeq \quad \bigoplus_{F \in X} S \cdot e_F \quad = \quad \mathbf{F}_X, \quad \mathbf{z}^{\mathbf{b}} \cdot e_F \;\mapsto\; e_{F+\mathbf{b}} \; .$$

The differential ∂ on $\mathbf{F}_X$ is compatible with the $S[L]$-action on $\mathbf{F}_X$ because the incidence function is L-invariant. For each $F \in \mathrm{Rep}(X/L)$ and $\mathbf{b} \in L$ we have

$$\begin{aligned} \partial(\mathbf{z}^{\mathbf{b}} \cdot e_F) \quad &= \quad \partial(e_{F+\mathbf{b}}) \quad = \quad \sum_{F' \in X,\, F' \neq \emptyset} \varepsilon(F{+}\mathbf{b}, F'{+}\mathbf{b}) \, \frac{m_{F+\mathbf{b}}}{m_{F'+\mathbf{b}}} \, e_{F'+\mathbf{b}} \\ &= \quad \sum_{F' \in X,\, F' \neq \emptyset} \varepsilon(F, F') \, \frac{m_F}{m_{F'}} \, \mathbf{z}^{\mathbf{b}} \cdot e_{F'} \quad = \quad \mathbf{z}^{\mathbf{b}} \cdot \partial(e_F). \end{aligned}$$

Clearly, the differential ∂ is homogeneous of degree 0, which proves the claim. ∎

Corollary 3.6 *If X is an equivariant $\mathbb{Z}^n$-graded cell complex on M_L then the cellular complex $\mathbf{F}_X$ is exact over S if and only if it is exact over $S[L]$.*

Proof. The $\mathbb{Z}^n$-graded components of $\mathbf{F}_X$ are complexes of k-vector spaces which are independent of our interpretation of $\mathbf{F}_X$ as an S-module or $S[L]$-module. ∎

If X is an equivariant $\mathbb{Z}^n$-graded cell complex on M_L such that $\mathbf{F}_X$ is exact, then we call $\mathbf{F}_X$ an *equivariant cellular resolution* of M_L.

Corollary 3.7 *If $\mathbf{F}_X$ is an equivariant cellular (minimal) resolution of M_L then $\pi(\mathbf{F}_X)$ is a (minimal) resolution of S/I_L by $\mathbb{Z}^n/L$-graded free S-modules.* ∎

We call $\pi(\mathbf{F}_X)$ a *cellular resolution* of the lattice ideal I_L. Let Q be an order ideal in the quotient poset $\mathbb{N}^n/L$. Then $Q + L$ is an order ideal in $\mathbb{N}^n + L$, and the restriction $\mathbf{F}_{X_{Q+L}}$ is a complex of $\mathbb{Z}^n$-graded free $S[L]$-modules. We set $\pi(\mathbf{F}_X)_Q := \pi(\mathbf{F}_{X_{Q+L}})$. This is a complex of $\mathbb{Z}^n/L$-graded free S-modules. Corollary 1.12 implies

Proposition 3.8 *If $\pi(\mathbf{F}_X)$ is a cellular resolution of I_L and Q is an order ideal in $\mathbb{N}^n/L$ which contains all Betti degrees then $\pi(\mathbf{F}_X)_Q$ is a cellular resolution of I_L.*

In what follows we shall study two particular cellular resolutions of I_L.

Theorem 3.9 *The Taylor complex Δ on M_L and the hull complex* $\mathrm{hull}(M_L)$ *are equivariant. They define cellular resolutions $\pi(\mathbf{F}_\Delta)$ and $\pi(\mathbf{F}_{\mathrm{hull}(M_L)})$ of I_L.*

Proof. The Taylor complex Δ consists of all finite subsets of generators of M_L. It has an obvious L-action. The hull complex also has an L-action: if $F = \mathrm{conv}\bigl(\{t^{\mathbf{a}_1}, \ldots, t^{\mathbf{a}_s}\}\bigr)$ is a face of $\mathrm{hull}(M_L)$ then $\mathbf{z}^b \cdot F = \mathrm{conv}\bigl(\{t^{\mathbf{a}_1+\mathbf{b}}, \ldots, t^{\mathbf{a}_s+\mathbf{b}}\}\bigr)$ is also a face of $\mathrm{hull}(M_L)$ for all $\mathbf{b} \in L$. In both cases the incidence function ε is defined uniquely by the ordering of the elements in L. To ensure that ε is L-invariant, we fix an ordering which is L-invariant; for instance, order the elements of L by the value of an $\mathbb{R}$-linear functional whose coordinates are $\mathbb{Q}$-linearly independent.

Both $\pi(\mathbf{F}_\Delta)$ and $\pi(\mathbf{F}_{\mathrm{hull}(M_L)})$ are cellular resolutions of I_L by Corollary 3.7. ∎

The *Taylor resolution* $\pi(\mathbf{F}_\Delta)$ of I_L has the following explicit description. For $\alpha \in \mathbb{N}^n/L$ let $\mathrm{fiber}(\alpha)$ denote the (finite) set of all monomials $\mathbf{x}^\mathbf{b}$ with $\mathbf{b} \in \alpha$. Thus $S_\alpha = k \cdot \mathrm{fiber}(\alpha)$. Let $E_i(\alpha)$ be the collection of all i-element subsets I of $\mathrm{fiber}(\alpha)$ whose greatest common divisor $\gcd(I)$ equals 1. For $I \in E_i(\alpha)$ set $\deg(I) := \alpha$.

Proposition 3.10 *The Taylor resolution $\pi(\mathbf{F}_\Delta)$ of a lattice ideal I_L is isomorphic to the $\mathbb{Z}^n/L$-graded free S-module $\bigoplus_{\alpha\in\mathbb{N}^n/L} S \cdot E_i(\alpha)$ with the differential*

$$\partial(I) \quad = \quad \sum_{m\in I} \mathrm{sign}(m, I) \cdot \gcd(I\backslash\{m\}) \cdot [I\backslash\{m\}]. \tag{3.1}$$

In this formula, $[I\backslash\{m\}]$ denotes the element of $E_{i-1}\bigl(\alpha - \deg(\gcd(I\backslash\{m\}))\bigr)$ which is obtained from $I\backslash\{m\}$ by removing the common factor $\gcd(I\backslash\{m\})$.

Proof. For $\mathbf{b} \in \mathbb{Z}^n$ let $F_i(\mathbf{b})$ denote the collection of i-element subsets of generators of M_L whose least common multiple equals $\mathbf{b}$. For $J \in F_i(\mathbf{b})$ we have $\mathrm{lcm}(J) = \mathbf{x}^\mathbf{b}$. The Taylor resolution $\mathbf{F}_\Delta$ of M_L equals $\bigoplus_{\mathbf{b}\in\mathbb{N}^n+L} S \cdot F_i(\mathbf{b})$ with differential

$$\partial(J) \quad = \quad \sum_{m\in J} \mathrm{sign}(m, J) \cdot \frac{\mathrm{lcm}(J)}{\mathrm{lcm}(J\backslash\{m\})} \cdot J\backslash\{m\}. \tag{3.2}$$

There is a natural bijection between $F_i(\mathbf{b})$ and $E_i(\mathbf{b} + L)$, namely, $J \mapsto \{\mathbf{x}^\mathbf{b}/\mathbf{x}^\mathbf{c} \mid \mathbf{x}^\mathbf{c} \in J\} = I$. Under this bijection we have $\frac{\mathbf{x}^\mathbf{b}}{\mathrm{lcm}(J\backslash\{m\})} = \gcd(I\backslash\{m\})$. The functor π identifies each $F_i(\mathbf{b})$ with $E_i(\mathbf{b} + L)$ and it takes (3.2) to (3.1). ∎

Corollary 3.11 *Let Q be an order ideal in $\mathbb{N}^n/L$ which contains all Betti degrees. Then $\pi(\mathbf{F}_\Delta)_Q = \bigoplus_{\alpha\in Q} SE_i(\alpha)$ with differential (3.1) is a cellular resolution of I_L.*

Proof. This follows from Proposition 3.8, Theorem 3.9 and Proposition 3.10. ∎

Example 3.12 *(Generic lattice ideals)* The lattice module M_L is generic (in the sense of §2) if and only if the ideal I_L is generated by binomials with full support. Suppose that this holds. It was shown in [PS] that the Betti degrees of I_L form an order ideal Q in $\mathbb{N}^n/L$. Theorem 2.9 and Proposition 3.8 imply that the resolution $\pi(\mathbf{F}_\Delta)_Q$ is minimal and coincides with the hull resolution $\pi(\mathbf{F}_{\mathrm{hull}(M_L)})$. ∎

The remainder of this section is devoted to the hull resolution of I_L. We next show that the hull complex $\mathrm{hull}(M_L)$ is locally finite. This fact is nontrivial, in view of Example 2.11. It will imply that the hull resolution has finite rank over S.

Write each vector $\mathbf{a} \in L \subset \mathbb{Z}^n$ as difference $\mathbf{a} = \mathbf{a}^+ - \mathbf{a}^-$ of two nonnegative vectors with disjoint support. A nonzero vector $\mathbf{a} \in L$ is called *primitive* if there is no vector $\mathbf{b} \in L \backslash \{\mathbf{a}, \mathbf{0}\}$ such that $\mathbf{b}^+ \leq \mathbf{a}^+$ and $\mathbf{b}^- \leq \mathbf{a}^-$. The set of primitive vectors is known to be finite [St, Theorem 4.7]. The set of binomials $\mathbf{x}^{\mathbf{a}^+} - \mathbf{x}^{\mathbf{a}^-}$ were $\mathbf{a}$ runs over all primitive vectors in L is called the *Graver basis* of the ideal I_L. The Graver basis contains the universal Gröbner basis of I_L [St, Lemma 4.6].

Lemma 3.13 *If* $\{\mathbf{0}, \mathbf{a}\}$ *is an edge of* $\mathrm{hull}(M_L)$ *then* $\mathbf{a}$ *is a primitive vector in* L.

Proof. Suppose that $\mathbf{a} = (a_1, \ldots, a_n)$ is a vector in L which is not primitive, and choose $\mathbf{b} = (b_1, \ldots, b_n) \in L \backslash \{\mathbf{a}, \mathbf{0}\}$ such that $\mathbf{b}^+ \leq \mathbf{a}^+$ and $\mathbf{b}^- \leq \mathbf{a}^-$. This implies $t^{b_i} + t^{a_i - b_i} \leq 1 + t^{a_i}$ for $t \gg 0$ and $i \in \{1, \ldots, n\}$. In other words, the vector $t^{\mathbf{b}} + t^{\mathbf{a}-\mathbf{b}}$ is componentwise smaller or equal to the vector $t^{\mathbf{0}} + t^{\mathbf{a}}$. We conclude that the midpoint of the segment $\mathrm{conv}\,\{t^{\mathbf{0}}, t^{\mathbf{a}}\}$ lies in $\mathrm{conv}\,\{t^{\mathbf{b}}, t^{\mathbf{a}-\mathbf{b}}\} + \mathbb{R}^n_+$, and hence $\mathrm{conv}\,\{t^{\mathbf{0}}, t^{\mathbf{a}}\}$ is not an edge of the polyhedron $P_t \,=\, \mathrm{conv}\,\{\, t^{\mathbf{c}} \,:\, \mathbf{c} \in L\} \,+\, \mathbb{R}^n_+$. ∎

Theorem 3.14 *The hull resolution* $\pi(\mathbf{F}_{\mathrm{hull}(M_L)})$ *is finite as an* S*-module.*

Proof. By Lemma 3.13 the vertex $\mathbf{0}$ of $\mathrm{hull}(M_L)$ lies in only finitely many edges. It follows that $\mathbf{0}$ lies in only finitely many faces of $\mathrm{hull}(M_L)$. The lattice L acts transitively on the vertices of $\mathrm{hull}(M_L)$, and hence every face of $\mathrm{hull}(M_L)$ is L-equivalent to a face containing $\mathbf{0}$. The faces containing $\mathbf{0}$ generate $\mathbf{F}_{\mathrm{hull}(M_L)}$ as an $S[L]$-module, and hence they generate $\pi(\mathbf{F}_{\mathrm{hull}(M_L)})$ as an S-module. ∎

A minimal free resolution of a lattice ideal I_L generally does not respect symmetries, but the hull resolution does. The following example illustrates this point.

Example 3.15 *(The hypersimplicial complex as a hull resolution)*
The lattice $L = \ker_{\mathbb{Z}} (\, 1 \;\; 1 \;\; \cdots \;\; 1 \,)$ in $\mathbb{Z}^n$ defines the toric ideal

$$I_L \quad = \quad \langle\, x_i - x_j \;:\; 1 \leq i < j \leq n \,\rangle.$$

The minimal free resolution of I_L is the Koszul complex on $n-1$ of the generators $x_i - x_j$. Such a minimal resolution does not respect the action of the symmetric group S_n on I_L. The hull resolution is the Eagon-Northcott complex of the matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}$. This resolution is not minimal but it retains the S_n-symmetry of I_L.

It coincides with the *hypersimplicial complex* studied by Gel'fand and MacPherson in [GM, §2.1.3]. The basis vectors of the hypersimplicial complex are denoted Δ_ℓ^I where I is a subset of $\{1, 2, \ldots, n\}$ with $|I| \geq 2$ and ℓ is an integer with $1 \leq \ell \leq |I|-1$. We have $\Delta_1^{\{i,j\}} \mapsto x_i - x_j$ and the higher differentials act as

$$\Delta_\ell^I \quad \mapsto \quad \sum_{i \in I} \operatorname{sign}(i, I) \cdot x_i \cdot \Delta_{\ell-1}^{I \backslash \{i\}} \; - \; \sum_{i \in I} \operatorname{sign}(i, I) \cdot \Delta_\ell^{I \backslash \{i\}},$$

where the first sum is zero if $\ell = 1$ and the second sum is zero if $\ell = |I| - 1$. ∎

Remark 3.16 Our study suggests a *curious duality* of toric varieties, under which the coordinate ring of the primal variety is resolved by a discrete subgroup of the dual variety. More precisely, the hull resolution of I_L is gotten by taking the convex hull in $\mathbb{R}^n$ of the points $t^{\mathbf{a}}$ for $\mathbf{a} \in L$. The Zariski closure of these points (as t varies) is itself an affine toric variety, namely, it is the variety defined by the lattice ideal $I_{L^\perp}$ where $L^\perp$ is the lattice dual to L under the standard inner product on $\mathbb{Z}^n$.

For instance, in Example 3.15 the primal toric variety is the line $(t, t, \ldots, t)$ and the dual toric variety is the hypersurface $x_1 x_2 \cdots x_n = 1$. That hypersurface forms a group under coordinatewise multiplication, and we are taking the convex hull of a discrete subgroup to resolve the coordinate ring of the line $(t, t, \ldots, t)$. ∎

Example 3.17 *(The rational normal quartic curve in P^4)*
Let $L = \ker_{\mathbb{Z}} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$. The minimal free resolution of the lattice ideal I_L looks like $0 \rightarrow S^3 \rightarrow S^8 \rightarrow S^6 \rightarrow I_L$. The primal toric variety in the sense of Remark 3.16 is a curve in P^4 and the dual toric variety is the embedding of the 3-torus into affine 5-space given by the equations $x_2 x_3^2 x_4^3 x_5^4 = x_1^4 x_2^3 x_3^2 x_4 = 1$. Here the hull complex hull(M_L) is simplicial, and the hull resolution of I_L has the format $0 \rightarrow S^4 \rightarrow S^{16} \rightarrow S^{20} \rightarrow S^9 \rightarrow I_L$. The nine classes of edges in hull(M_L) are the seven quadratic binomials in I_L and the two cubic binomials $x_3 x_4^2 - x_1 x_5^2$, $x_2^2 x_3 - x_1^2 x_5$.

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