Combinatorics
January 18, 2022

Class #1

binomial coefficients and inclusion-exclusion

$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ $\qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

integer

$0! = 1$

---

permutations $\quad n=3 \quad \{1,2,3\}$

123
132
213
231
312
321

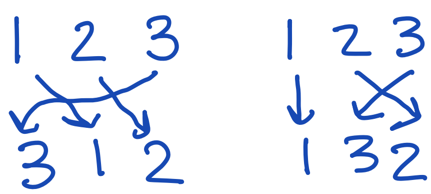
$n!$

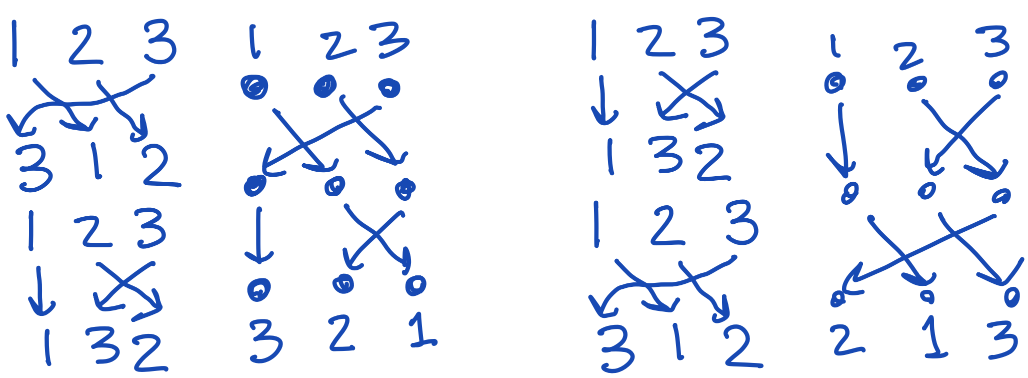
$3! = 3\cdot 2\cdot 1 = 6$

linear algebra

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} 3 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

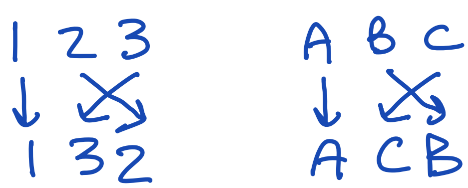




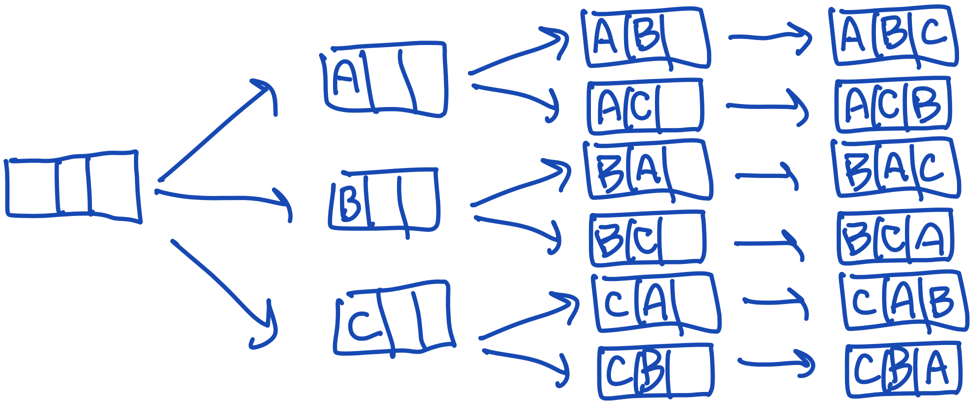
$n=3$

A	B	C
A	C	B
B	A	C
B	C	A
C	A	B
C	B	A

(23)



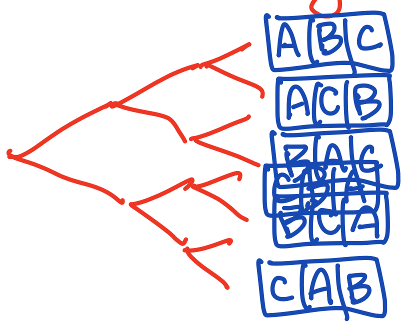
$n!$



$\times 3$ $\times 2$ $\times 1$

$3\cdot2\cdot1$

n log n steps to sort n elements



n choose k $\binom{n}{k}$

= # subsets size k from n things

$\binom{4}{2}$

A	B	C	D	
X	X			AB
X		X		AC
X			X	AD
	X	X		BC
	X		X	BD
		X	X	CD

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n_k}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

1	3	5
2	6	
4	7	

$$\binom{4}{2} = \frac{4!}{2!\,2!} \quad \frac{4\cdot3\cdot\cancel{2\cdot1}}{2\cdot1\cdot\cancel{2\cdot1}} = \frac{4\cdot3}{2\cdot1} = \frac{12}{2} = 6$$

counting technique: overcount, divide by overcount

ABCD $\binom{4}{2}$

write perm of ABCD (4) $4! = 24$

choose first two entries as a set

ignore order

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

ABCD

$\longleftrightarrow$ swap

swap $\updownarrow$

ABCD BACD
ABDC BADC
AB

ADBC DABC
ADCB DACB
AD

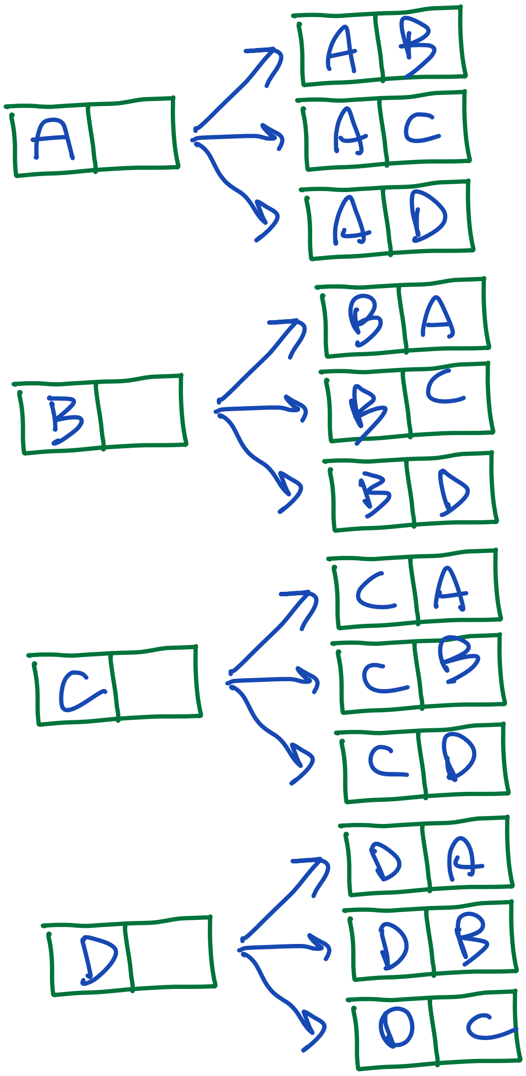
ACBD CABD
ACDB CADB
AC

BDAC DBAC
BDCA DBCA
BD

BCAD CBAD
BCDA CBDA
BC

CDAB DCAB
CDBA DCBA
CD

fill in two from ABCD



$$\binom{n}{k} = \frac{n_k}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

A B	B A	AB
A C	C A	AC
A D	D A	AD
B C	C B	BC
B D	D B	BD
C D	D C	CD

1

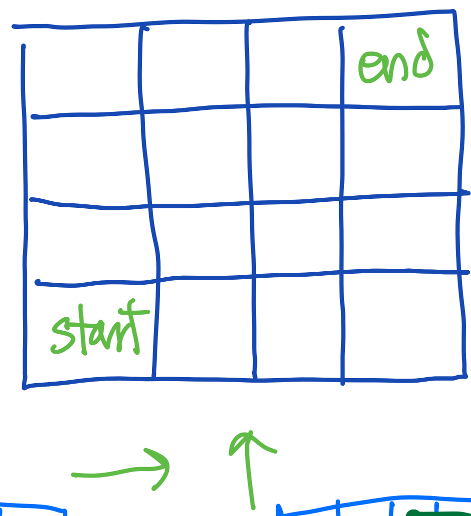
1 1

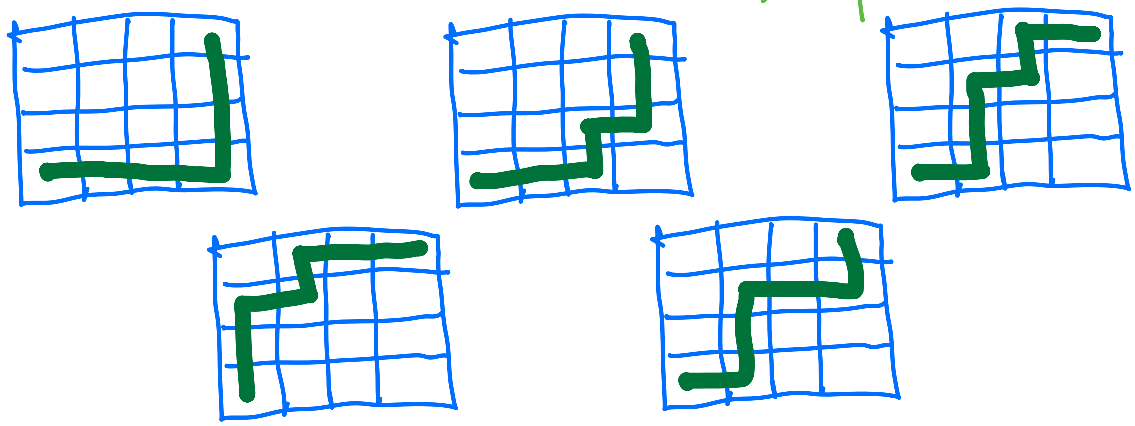
1 2 1

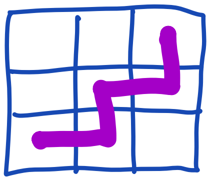
1 3 3 1

1 4 6 4 1

1 5 10 10 5 1







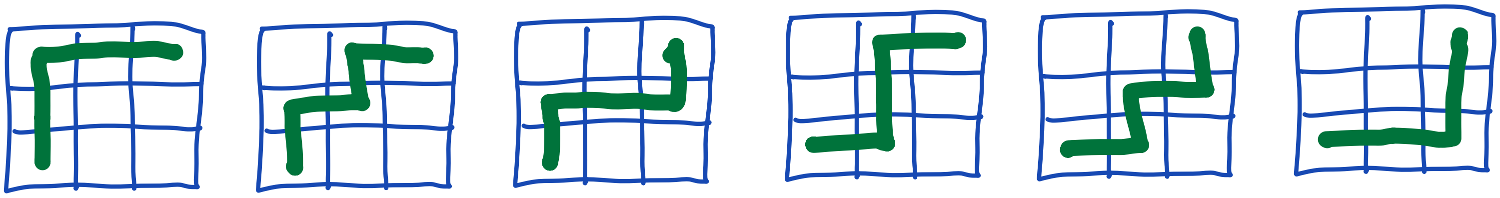
24

$\rightarrow\ \rightarrow\ \uparrow\ \uparrow$

$\rightarrow\ \uparrow\ \rightarrow\ \uparrow$

1 2 3 4

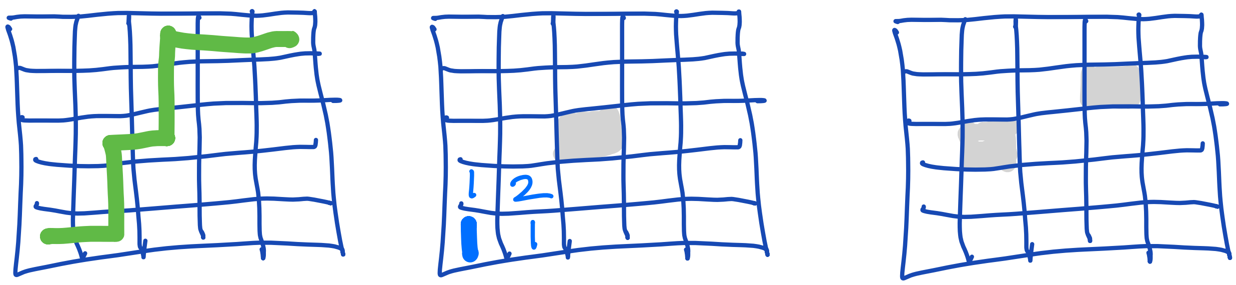
which steps are ↑ (not →)



12 13 14 23 24 34

inclusion-exclusion

How many paths if certain squares are forbidden?



$\rightarrow 4 \quad \uparrow 4$

$$\binom{8}{4} = \frac{\cancel{8}^{2}\cdot 7\cdot \cancel{6}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 70$$

1	5	15	35	70
1	4	10	20	35
1	3	6	10	15
1	2	3	4	5
1	1	1	1	1

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

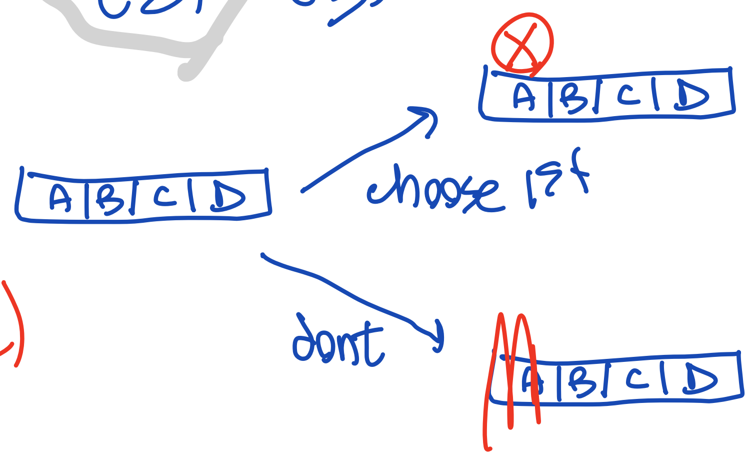
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{3}{2} = \binom{2}{1} + \binom{2}{2}$$

$$\binom{n}{k} = \underbrace{\binom{n-1}{k-1}}_{\text{choose 1}} + \underbrace{\binom{n-1}{k}}_{\text{don't}}$$

$\binom{4}{2}$



Combinatorics Jan 20, 2022 #2

Binomial Coefficients

$\binom{n}{k}$ = # ways to choose k elements from n possibilities

$$= \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1}$$

Pascal's triangle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
```

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$3 + 3$
$= 6$

$\binom{3}{1} + \binom{3}{2}$
$= \binom{4}{2}$

$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

use first element? $\{1, \ldots\}$ don't $\{\ldots\}$

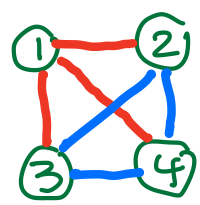
$\{1, \ldots n\}$

1	2	3	4	
x	x			12
x		x		13
x			x	14
	x	x		23
	x		x	24
		x	x	34

$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$

6 = 3 + 3

12, 13, 14 | 23, 24, 34



```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1     16
  1   5  10  10   5   1
```

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

```
     1
   1   1
  1   2   1
 1   3   3   1
```

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}y^n$$

$$= (x+y)(x+y)^{n-1}$$

$$(x+y)^{n-1} = \binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \binom{n-1}{2}x^{n-3}y^2 + \dots$$

$$x\left[\binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \binom{n-1}{2}x^{n-3}y^2 + \dots\right]$$

$$+y\left[\binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \binom{n-1}{2}x^{n-3}y^2 + \dots\right.$$

$$1\ x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$x=1\ y=1$

$(1+1)^4 = 2^4 = 16$

$$(x+y)(x+y)(x+y)(x+y)$$

x x x x

x x x y

x x y x

x x y y

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$1 = 1$

$1+2 = 3$

$1+2+3 = 6$

$1+2+3+4 = 10$

1 2 3 4 5 6 $\quad 3 \cdot 7 = 21$

even $\quad 1\ 2\ \dots\ n-1\ n \qquad \frac{n}{2}(n+1) \quad \frac{(n+1)n}{2\cdot 1} = \binom{n+1}{2}$

1 2 3 4 5 6 7

$$\frac{n-1}{2}(n+1)+\frac{1}{2}(n+1) = \frac{n}{2}(n+1)$$

picture for $1+2+\dots+n$

$n=4$

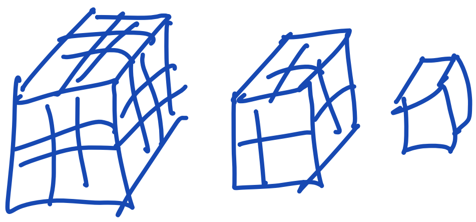
$$\frac{n(n+1)}{2} = \binom{n+1}{2}$$

$f(1) = 1+2+\dots+n = \binom{n+1}{2}$

$f(2) = 1^2+2^2+\dots+n^2 =$

$f(3) = 1^3+2^3+\dots+n^3 =$

$f(3) = f(1)^2$



$$\frac{d}{dx}f(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$$

discrete $\Delta f(n) \underset{(\varepsilon=1)}{=} f(n+1)-f(n)$

e^x x^n x^{n-1} ...

2^n

$2^{n+1}-2^n = 2^n$

$$\Delta\binom{n}{k} = \binom{n+1}{k}-\binom{n}{k} = \binom{n}{k-1}$$

$$\binom{n}{k} = \binom{n-1}{k-1}+\binom{n-1}{k}$$

$$\binom{n+1}{k} = \binom{n}{k-1}+\binom{n}{k}$$

$$\frac{\partial}{\partial x}x^n = nx^{n-1}$$

$$\Delta\binom{n}{k} = \binom{n}{k-1}$$

what is $f(n) = 1 + 2 + \_ + n$?

$$\Delta f(n) = f(n+1) - f(n) = n+1$$

$$= \binom{n}{1} + \binom{n}{0}$$

$$\Longrightarrow f(n) = \binom{n}{2} + \binom{n}{1} + C$$

$f(2) = 1+2=3$

$$F(2) = \underset{1}{\binom{2}{2}} + \underset{2}{\binom{2}{1}} + 0$$

$$\boxed{\binom{n}{2} + \binom{n}{1} = \binom{n+1}{2}}$$

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

n-dim symmetric matrices

$$\frac{n(n+1)}{2} \qquad \frac{3 \cdot 4}{2} = \frac{12}{2} = 6$$

$n=3$

6 abcdef

$$3 \begin{bmatrix} a & a & d & e \\ d & b & b & f \\ e & f & c & c \end{bmatrix} \div 2$$

4

How many monomials of deg d in n vars?

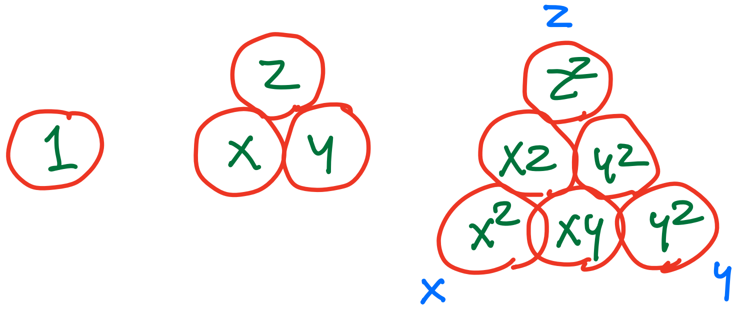
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$n=1$: $1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5$
1 1 1 1 1 1

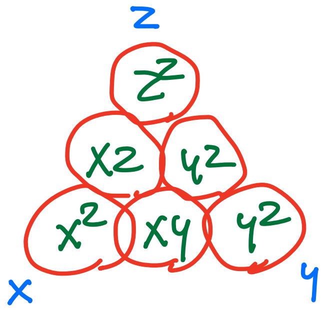
$n=2$: $1 \quad x, y \quad x^2, xy, y^2 \quad x^3, x^2y, xy^2, y^3$
1 2 3 4

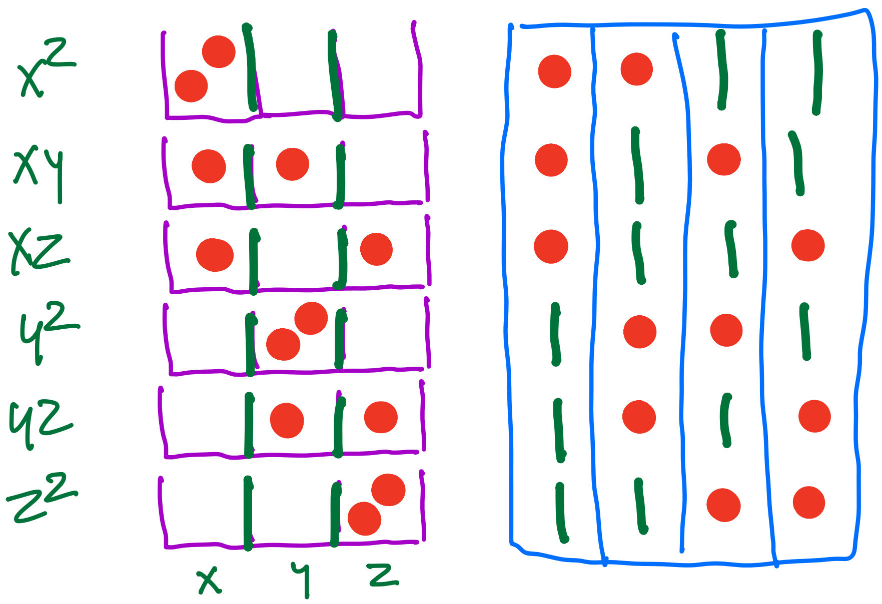
$n=3$: $1 \quad x, y, z \quad x^2, y^2, z^2, xy, xz, yz$
1 3 6

$1 = 1$
$1+2 = 3$
$1+2+3 = 6$
$1+2+3+4 = 10$



$\boxed{\text{Bars \& Stars argument}}$



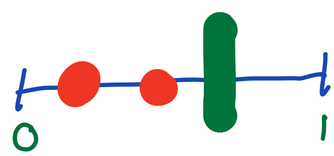


n vars deg d $\qquad \binom{n-1+d}{n-1} = \binom{n-1+d}{d}$

n=3, d varies $\qquad \binom{d+2}{2} \qquad \binom{n+1}{2}$

1, 3, 6, 10

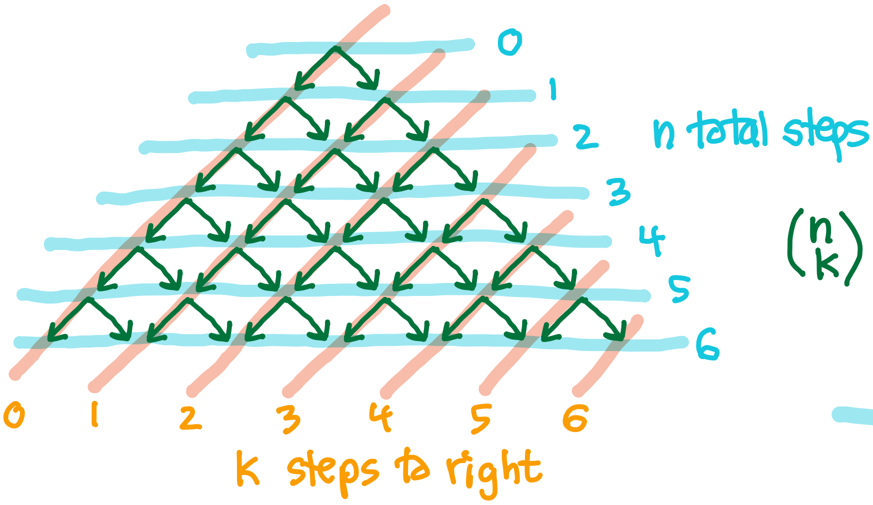
Bose Einstein statistics



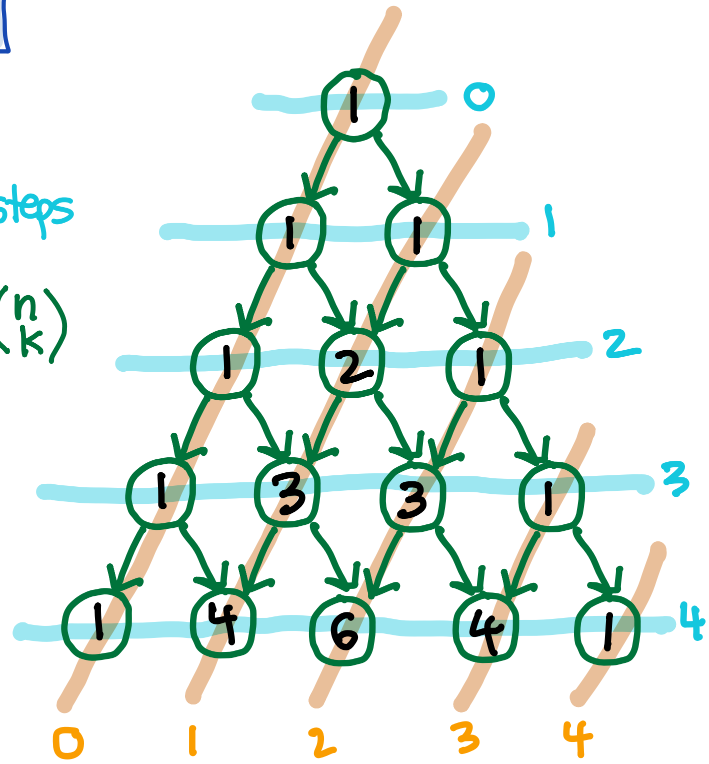
A, B |

AB|	1/4	1/3	••|
A|B B|A	1/2	1/3	•|•
|AB	1/4	1/3	••|

Combinatorics #3, January 25, 2022



$\binom{n}{k}$



$\binom{n-1}{k-1} \quad \binom{n-1}{k} \rightarrow \binom{n}{k}$

$1 \quad 3 \rightarrow 4$

each path comes from left or right

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1	4	10	20	end
1	3	6	10	
1	2	3	4	
1	1	1	1	

start

$$\binom{6}{3} = \frac{\cancel{6}\cdot 5\cdot 4}{\cancel{3\cdot 2}\cdot 1} = 20$$

(all over)

→→→↑↑↑

inclusion-exclusion

avoid

1	5	9	17	34
1	4	4	8	17
1	3	0	4	9
1	2	3	4	5
1	1	1	1	1

=

free-for-all

1	5	15	35	70
1	4	10	20	35
1	3	6	10	15
1	2	3	4	5
1	1	1	1	1

−

must use

		6	18	36
		6	12	18
1	3	6	6	6
1	2	3		
1	1	1		

$$\binom{8}{4} - \binom{4}{2}\binom{4}{2}$$

34

$$\frac{\overset{2}{\cancel{8}}\cdot 7\cdot \cancel{6}\cdot 5}{\cancel{4\cdot 3\cdot 2}\cdot 1} = 70 \quad - \quad 6\cdot 6$$

1	1	2	4	7
1		1	2	3
1	1	1	1	1

avoid

1	3	6	10	15
1	2	3	4	5
1	1	1	1	1

all

	2	4	6	8
1	2	2	2	2
1	1			

use

1	2
1	1

1	2	3	4
1	1	1	1

$$\binom{6}{2} - \binom{2}{1}\binom{4}{1}$$

$$\frac{6\cdot5}{2\cdot1} - 2\cdot4$$

$$7 = 15 - 8$$

1	2	4	4	7
1	1	2	B	3
1	A	1	2	3
1	1	1	1	1

$\emptyset$ = no properties or more
A = at least property A
B = at least property B
AB = at least properties A and B

$$\text{avoid} = \emptyset - A - B + AB$$

$$7 \qquad 35 - 20 - 20 + 12$$

1	4	10	20	35
1	3	6	10	15
1	2	3	4	5
1	1	1	1	1

$\emptyset$

	2	6	12	20
	2	4	6	8
1	2	2	2	2
1	1			

A

			10	20
1	3	6	10	10
1	2	3	4	
1	1	1	1	

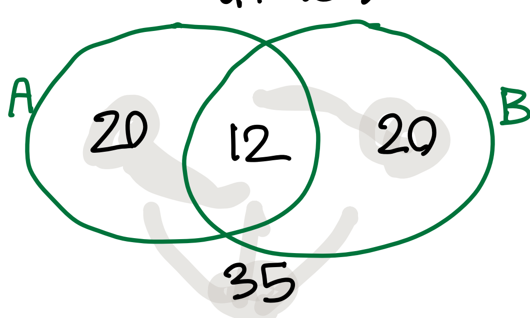
B

			6	12
	2	4	6	6
1	2	2	2	
1	1			

AB

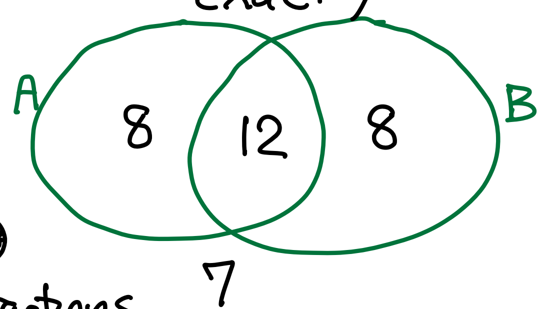
Venn diagram

at least



subtractions

exactly



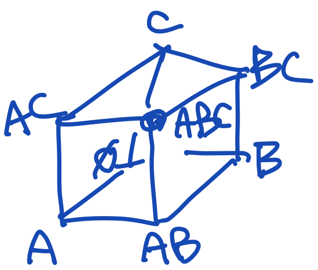
$$\begin{array}{c} \emptyset \\ A \\ B \\ AB \end{array} \begin{bmatrix} 35 \\ 20 \\ 20 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 8 \\ 12 \end{bmatrix}$$

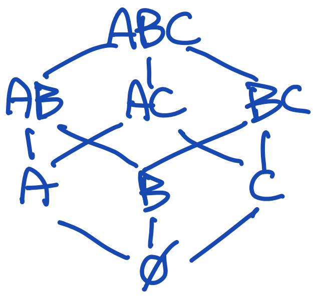
we know at least

we want exactly

inclusion-exclusion is solving matrix

inclusion-exclusion
Möbius inversion





x^2 xy y^2
A B C

AC = ABC

How many integers 1,.., 30 not divisible by 2, 3, or 5

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

properties A = divisible by 2
B 3
C 5

	Ø	− A	− B	− C	+ AB	+ AC	+ BC	− ABC	
Ø	+								1
A	+	−							0
AB	+	−	−		+				0
ABC	+	−	−	−	+	+	+	−	0

1..30

$$\emptyset - A - B - C + AB + AC + BC - ABC$$

$$\frac{30}{1} \quad \frac{30}{2} \quad \frac{30}{3} \quad \frac{30}{5} \quad \frac{30}{6} \quad \frac{30}{10} \quad \frac{30}{15} \quad \frac{30}{30}$$

all multiples of

$$8 = 30 \underbrace{-15 - 10 - 6}_{-31} + 5 + 3 + 2 - 1$$

Hat check problem

How many permutations have no fixed points?

$1 \to 2$, $2 \to 1$, $3 \to 3$

$1 \to 3$, $2 \to 1$, $3 \to 2$

n=2

~~1~~ ~~2~~

2 1

n=3

~~1~~ ~~2~~ ~~3~~

~~1~~ 3 2

2 1 ~~3~~

2 3 1

3 1 2

3 ~~2~~ 1

n=4

~~1~~ ~~2~~ ~~3~~ ~~4~~	2 1 ~~3~~ ~~4~~	3 1 2 ~~4~~	$\boxed{4\ 1\ 2\ 3}$
~~1~~ 2 4 3	$\boxed{2\ 1\ 4\ 3}$	$\boxed{3\ 1\ 4\ 2}$	4 1 ~~3~~ 2
~~1~~ 3 2 ~~4~~	2 3 1 ~~4~~	3 ~~2~~ 1 ~~4~~	4 ~~2~~ 1 3
~~1~~ 3 4 2	$\boxed{2\ 3\ 4\ 1}$	3 ~~2~~ 4 1	4 2 ~~3~~ 1
~~1~~ 4 2 3	$\boxed{2\ 4\ 1\ 3}$	$\boxed{3\ 4\ 1\ 2}$	$\boxed{4\ 3\ 1\ 2}$
~~1~~ 4 ~~3~~ 2	2 4 ~~3~~ 1	$\boxed{3\ 4\ 2\ 1}$	$\boxed{4\ 3\ 2\ 1}$

Find formula for n agrees with

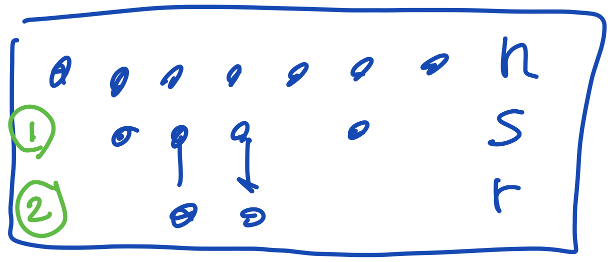
n	2	3	4
#	1	2	9

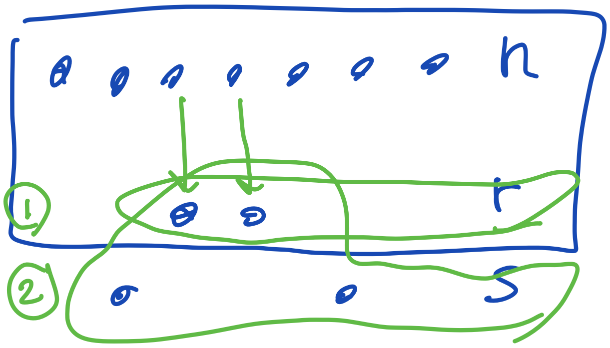
$\frac{1}{2}$ $\frac{1}{3}$ $\frac{9}{24}$

$$? \quad \boxed{\frac{\binom{n-r}{s-r}}{\binom{s}{r}} = \frac{\binom{n}{s}}{\binom{n}{r}}}$$

$\begin{pmatrix} n \\ s \\ r \end{pmatrix} =$ # ways to choose s from n and r from s

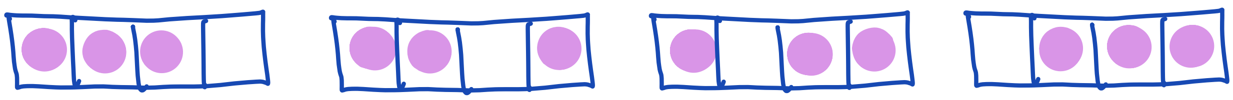
$$\begin{pmatrix} n \\ s \\ r \end{pmatrix} = \underset{1}{\binom{n}{r}}\underset{2}{\binom{n-r}{s-r}} = \underset{1}{\binom{n}{s}}\underset{2}{\binom{s}{r}}$$



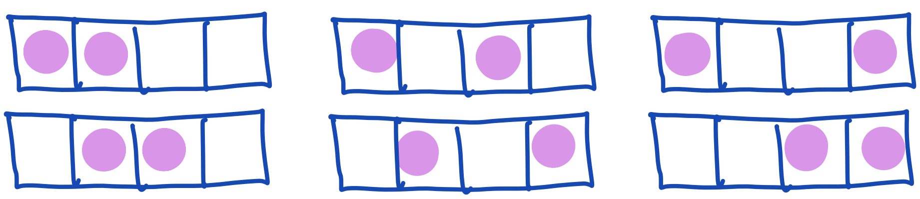


$$\underbrace{\{\cdots\}}_{r} \subset \underbrace{\{\cdots\}}_{s} \subset \underbrace{\{\quad\}}_{n}$$

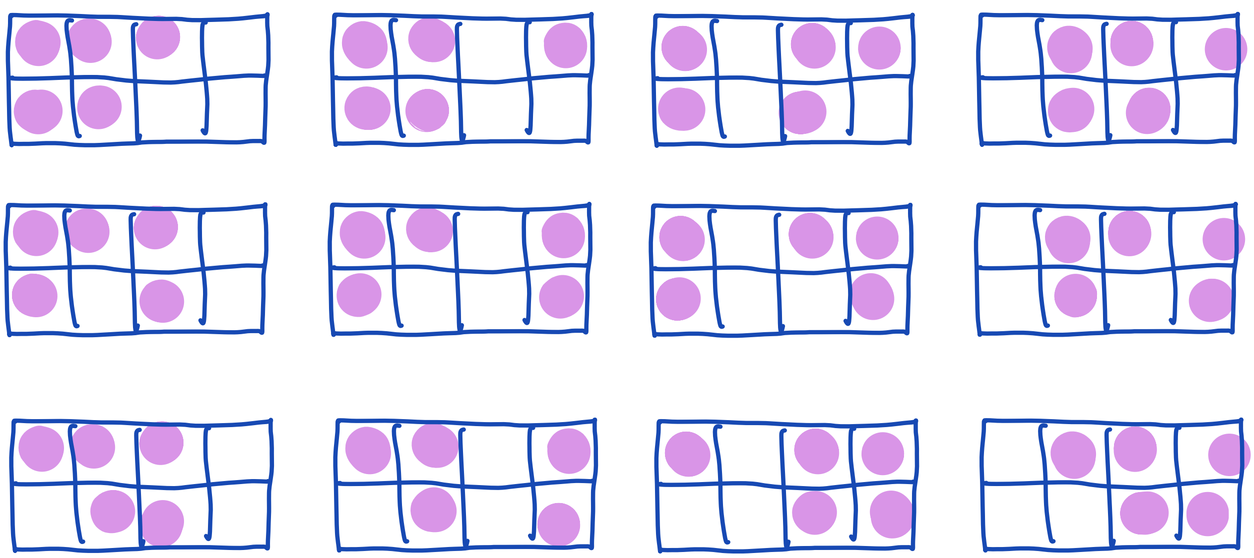
$\binom{4}{3} = 4$



$\binom{4}{2} = 6$



$$\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \binom{4}{3}\binom{3}{2} = 4 \cdot 3 = 12$$



Combinatorics #4
January 27, 2022

Derangements (hat-check problem)

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 3 1 4	3 2 1 4	4 2 1 3
1 3 4 2	2 3 4 1	3 2 4 1	4 2 3 1
1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

$n=4$ $\boxed{9}$

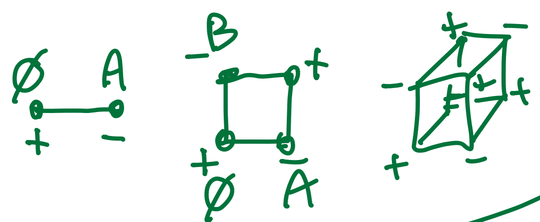
Inclusion-exclusion

Properties
A 1 in position 1
B 2 in position 2
C 3 in position 3
D 4 in position 4

$\geq\emptyset$ at least no props
$\geq A$ at least prop A
$\geq AB$ " " A and B
$\#A$ = exactly A no more
$\#\emptyset$

$$\#\emptyset = \geq\emptyset - \geq A - \geq B - \geq C - \geq D + \geq AB + \ldots$$

$$\#\emptyset = \geq\emptyset - \geq\{A,B,C,D\} + \geq\{AB, AC, AD, BC, BD, CD\} - \geq\{ABC, ABD, ACD, BCD\} + \geq ABCD$$



any n 1 property n properties

$$\binom{4}{5} = \cancel{\frac{4\cdot3\cdot2\cdot1\cdot0}{5!}}$$

$$\emptyset - \geq\{A,B,C,D\} + \geq\{AB,AC,AD,BC,BD,CD\}$$

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \binom{n}{4}(n-4)! -$$

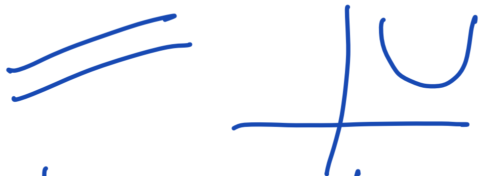
$$= n! - \frac{n}{1}(n-1)! + \frac{n(n-1)}{2\cdot1}(n-2)! - \frac{n(n-1)(n-2)}{3!}(n-3)! + \frac{(\ldots)}{4!}(n-4)! -$$

$$= n!\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - (\cdot)\right) = n!\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}$$

$$\approx \left[\frac{n!}{e}\right]$$

$$e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!} \qquad e^{-1} = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!} \qquad \frac{d}{dx}e^x = e^x$$

Algebraic Geometry



$x^2+1=0$

$\mathbb{C}$ $i, -i$

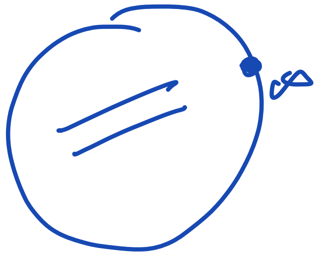
studies zerosets of polynomial equations

projective space of ratios

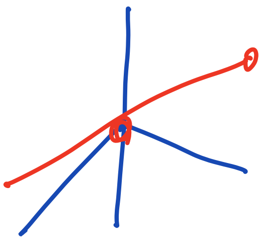
$x:y$ $\quad$ x $\quad$ ∞

$x:1$ $\quad$ $1:0$

$\mathbb{P}^2$ $\quad$ $x:y:z$



$2:3 = 4:6$



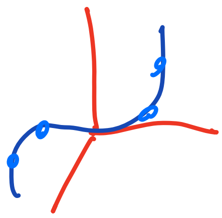
$(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for any λ

homogeneous polynomials $\quad x^2+xy+y^2 \to \lambda^2(x^2+xy+y^2)$

$x=\lambda x \quad y=\lambda y \quad z=\lambda z$

$\mathbb{R}^1 \longrightarrow \mathbb{R}^3$

$\gamma(t) = (t, t^2, t^3)$ $\quad$ twisted cubic curve



$b^2-ac=0$

$(s^2t)^2-(s^3)(st^2)=0$

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \gamma(s,t) = (s^3, s^2t, st^2, t^3) \end{array}$$

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$$\left\{ \begin{array}{l} b^2-ac=0 \\ bc-ad=0 \\ c^2-bd=0 \end{array} \right\}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbb{R}^4$ $\quad$ $4-2=2$

$3d+1$

1	$\{a,b,c,d\}$	$\{a^2,b^2,c^2,d^2,ab,ac,ad,bc,bd,cd\}$	
$\mathbb{R}^1$ $d=0$	$\mathbb{R}^4$ $d=1$	$\mathbb{R}^{10}$ $d=2$	$\mathbb{R}^{20}$ $d=3$
1	4	7	10

$\mathbb{P}^0$ • ratios in x $\qquad 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \to 2 = \frac{1}{1-\frac{1}{2}}$

$1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5 + \cdots$ $\qquad$ generating functions

$f(d) = \#$ terms deg d in our variables $\qquad f(n) = \ldots$

$\mathbb{P}^0 \quad \begin{matrix} n=1 \\ x \end{matrix} \qquad f(d)=1 \qquad \sum_{d=0}^{\infty} f(d)\,t^d = \sum_{d=0}^{\infty} t^d \qquad \sum_{n=0} f(n)x^n$

$$1+t+t^2+\cdots = \boxed{\frac{1}{1-t}}$$

$\mathbb{P}^1$ / ratios in x, y

$$\begin{matrix} 1 & x, y & x^2, xy, y^2 & x^3, x^2y, xy^2, y^3 \\ 1 & 2 & 3 & 4 \end{matrix}$$

$$f(d) = d+1 \qquad \sum_{d=0}^{\infty} f(d)\,t^d = \frac{1}{(1-t)^2}$$

$$(1+x+x^2+x^3+\cdots)(1+y+y^2+y^3+\cdots)$$
$$= 1+x+y+x^2+xy+y^2+\cdots$$

$\downarrow \begin{matrix} x=t \\ y=t \end{matrix}$

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right) \longrightarrow \frac{1}{(1-t)^2}$$

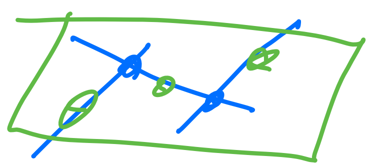
How many mononomials of degree D in a, b, c, d are not multiples of b^2 or bc or c^2?

$$\frac{1-3t^2+2t^2}{(1-t)^4} = \frac{3}{(1-t)^2} - \frac{2}{(1-t)}$$

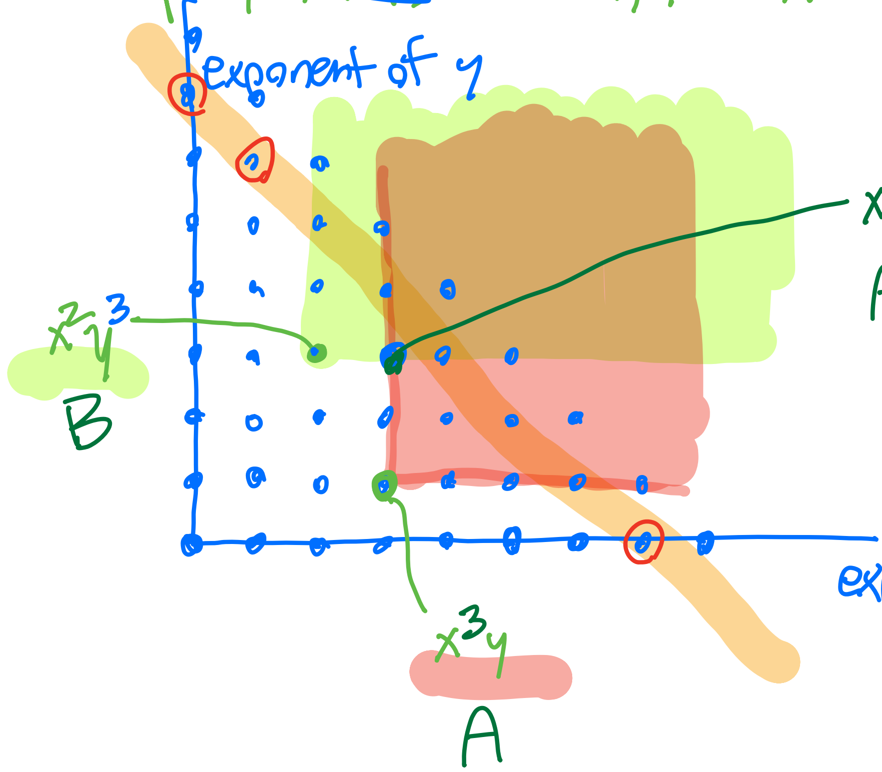
$3\mathbb{P}^1 - 2\mathbb{P}^0$

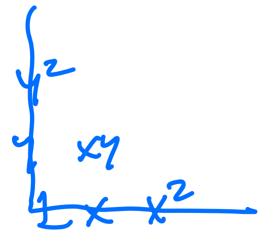
$$\begin{tikzcd}
 & & b^2c^2 \arrow[dl, dash] \arrow[dr, dash] & & \\
 & b^2c \arrow[dl, dash] \arrow[dr, dash] & & bc^2 \arrow[dl, dash] \arrow[dr, dash] & \\
b^2 \arrow[drr, dash] & & bc \arrow[d, dash] & & c^2 \arrow[dll, dash] \\
 & & 1 & &
\end{tikzcd}$$

$$\left\{ \begin{matrix} b^2 - ac = 0 \\ bc - ad = 0 \\ c^2 - bd = 0 \end{matrix} \right\}$$ how much stuff?



~~polynomials~~ terms in x, y not multiples of x^3y or x^2y^3





$$\frac{1}{(1-t)^2} - \frac{t^4}{(1-t)^2} - \frac{t^5}{(1-t)^2} + \frac{t^6}{(1-t)^2} = \frac{3}{(1-t)}$$

$\Sigma\, \emptyset$ $\qquad \underbrace{(1+x+x^2+\dots)}_{\frac{1}{1-x}}\underbrace{(1+y+y^2+\dots)}_{\frac{1}{1-y}} \xRightarrow[y=t]{x=t} \frac{1}{(1-t)^2}$

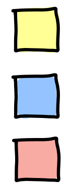
ΣA $\qquad x^3y\,\underbrace{(1+x+x^2+\dots)}_{\frac{1}{1-x}}\underbrace{(1+y+y^2+\dots)}_{\frac{1}{1-y}} \Rightarrow \frac{t^4}{(1-t)^2}$

all multiples of x^3y

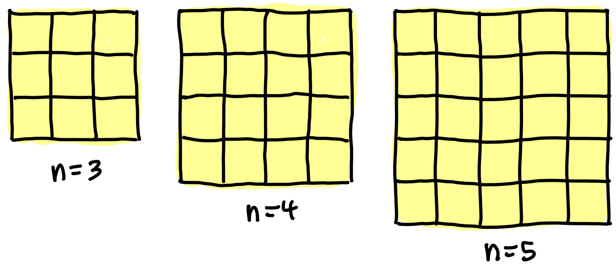
Combinatorics prep, 3/10/22

How many ways to place k markers on an n×n board up to rotations, flips?

$D_4 = \{ 1, ↷, \underbrace{⤸, ⤹}_{\text{same}}, \underbrace{↔, ↕}_{\text{same}}, \underbrace{⤢, ⤡}_{\text{same}} \}$ $\qquad |M| = \frac{1}{|G|} \sum_{g \in G} |X^g|$

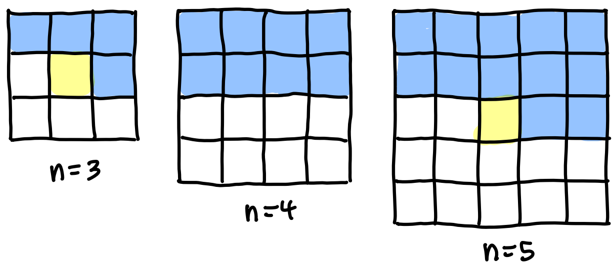


g = 1



$\binom{n^2}{k}$

g = ↷



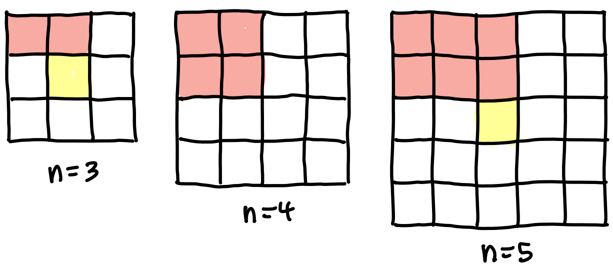
$m = \lfloor \frac{n^2}{2} \rfloor$ = #□

k=2 $\quad m$

k=3 $\quad$ [n odd?] m

k=4 $\quad \binom{m}{2}$

g = ⤸



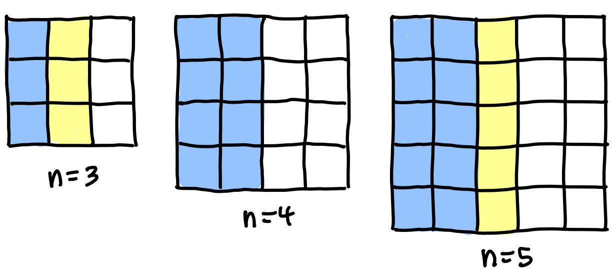
$m = \lfloor \frac{n^2}{4} \rfloor$ = #□

k=2 $\quad 0$

k=3 $\quad 0$

k=4 $\quad m$

g = ↔



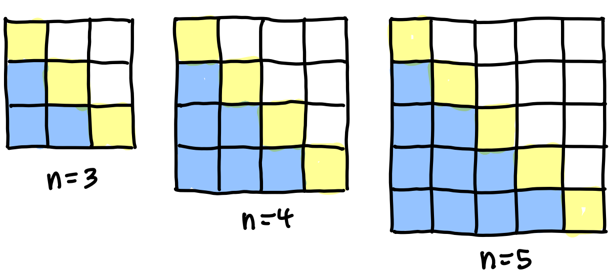
$m = n \lfloor \frac{n}{2} \rfloor$ = #□

k=2 $\quad m +$ [n odd?] $\binom{n}{2}$

k=3 $\quad$ [n odd?] mn

k=4 $\quad \binom{m}{2} +$ [n odd?] $\left(m\binom{n}{2} + \binom{n}{4}\right)$

g = ⤢



$m = \frac{n^2-n}{2}$ = #□

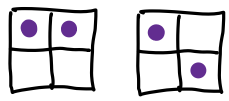
k=2 $\quad m + \binom{n}{2}$

k=3 $\quad mn$

k=4 $\quad \binom{m}{2} + m\binom{n}{2} + \binom{n}{4}$

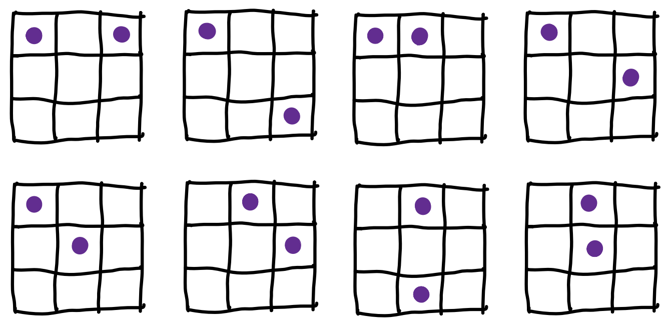
	$g=1$	$g=\curvearrowright$	$g=\curvearrowright,\curvearrowleft$	$g=\leftrightarrow,\updownarrow$	$g=\nearrow\!\!\swarrow,\nwarrow\!\!\searrow$
		$m=\left\lfloor \frac{n^2}{2} \right\rfloor$	$m=\left\lfloor \frac{n^2}{4} \right\rfloor$	$m=n\left\lfloor \frac{n}{2} \right\rfloor$	$m=\frac{n^2-n}{2}$
$k=2$	$\binom{n^2}{2}$	m	0	$m+[n \text{ odd?}]\binom{n}{2}$	$m+\binom{n}{2}$
$n=2$	$\binom{4}{2}=6$	$\left\lfloor \frac{4}{2} \right\rfloor=2$	0	$2\left\lfloor \frac{2}{2} \right\rfloor=2$	$\frac{4-2}{2}+\binom{2}{2}=2$

$$\frac{1}{8}\Big(\underbrace{6+2}_{8}+2\cdot 0+\underbrace{2\cdot 2+2\cdot 2}_{8}\Big)=\frac{16}{8}=2$$



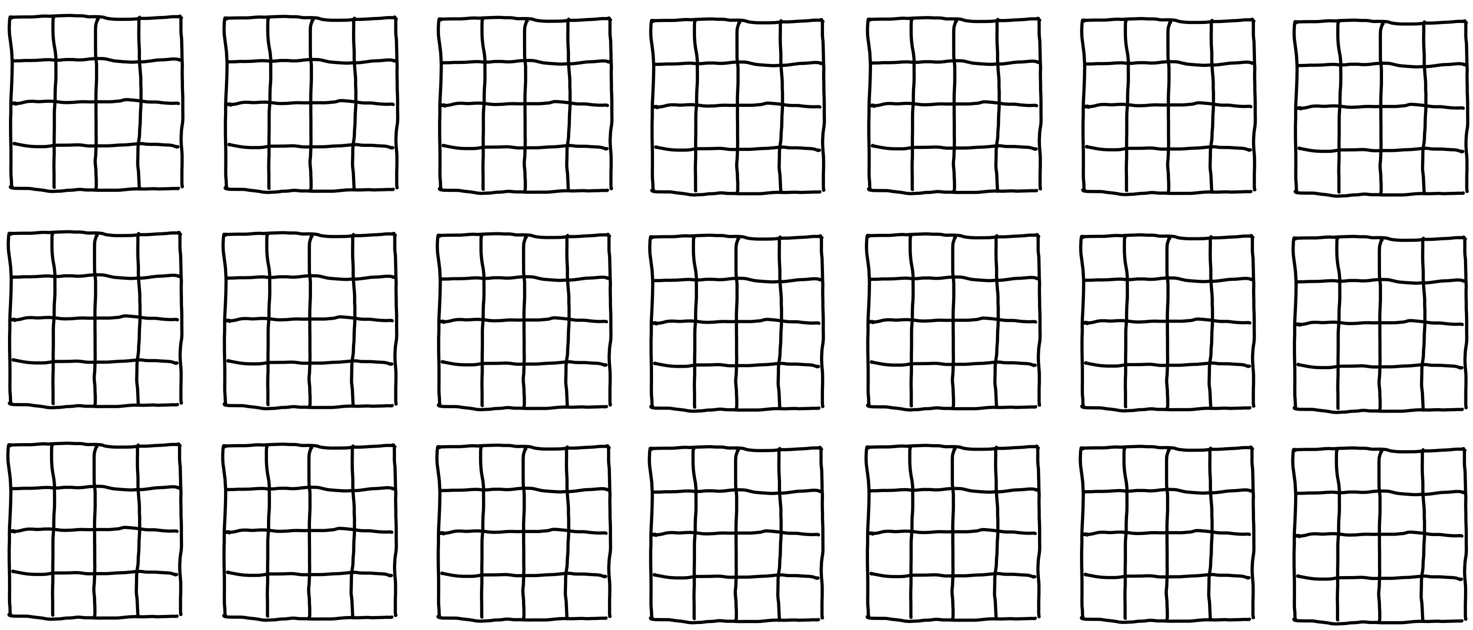
$n=3$	$\binom{9}{2}=36$	$\left\lfloor \frac{9}{2} \right\rfloor=4$	0	$3\left\lfloor \frac{3}{2} \right\rfloor+\binom{3}{2}=6$	$\frac{9-3}{2}+\binom{3}{2}=6$

$$\frac{1}{8}\Big(\underbrace{36+4}_{40}+2\cdot 0+\underbrace{2\cdot 6+2\cdot 6}_{24}\Big)=\frac{64}{8}=8$$



$n=4$	$\binom{16}{2}=120$	$\left\lfloor \frac{16}{2} \right\rfloor=8$	0	$4\left\lfloor \frac{4}{2} \right\rfloor=8$	$\frac{16-4}{2}+\binom{4}{2}=12$

$$\frac{1}{8}\Big(\underbrace{120+8}_{128}+2\cdot 0+\underbrace{2\cdot 8+2\cdot 12}_{40}\Big)=\frac{168}{8}=21$$



	$g=1$	$g=$ $m=\left\lfloor \frac{n^2}{2} \right\rfloor$	$g=$, $m=\left\lfloor \frac{n^2}{4} \right\rfloor$	$g=$, $m=n\left\lfloor \frac{n}{2} \right\rfloor$	$g=$, $m=\frac{n^2-n}{2}$
$K=3$	$\binom{n^2}{3}$	$[n \text{ odd?}]\, m$	0	$[n \text{ odd?}]\, mn$	mn
$K=4$	$\binom{n^2}{4}$	$\binom{m}{2}$	m	$\binom{m}{2}+[n \text{ odd?}]\left(m\binom{n}{2}+\binom{n}{4}\right)$	$\binom{m}{2}+m\binom{n}{2}+\binom{n}{4}$

Combinatorics March 1, 2022

Burnside's lemma

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|$$

$$\sum_{g\in G} |X^g| = |\{(g,x) \mid gx = x\}| = \sum_{x\in X} |G_x|$$

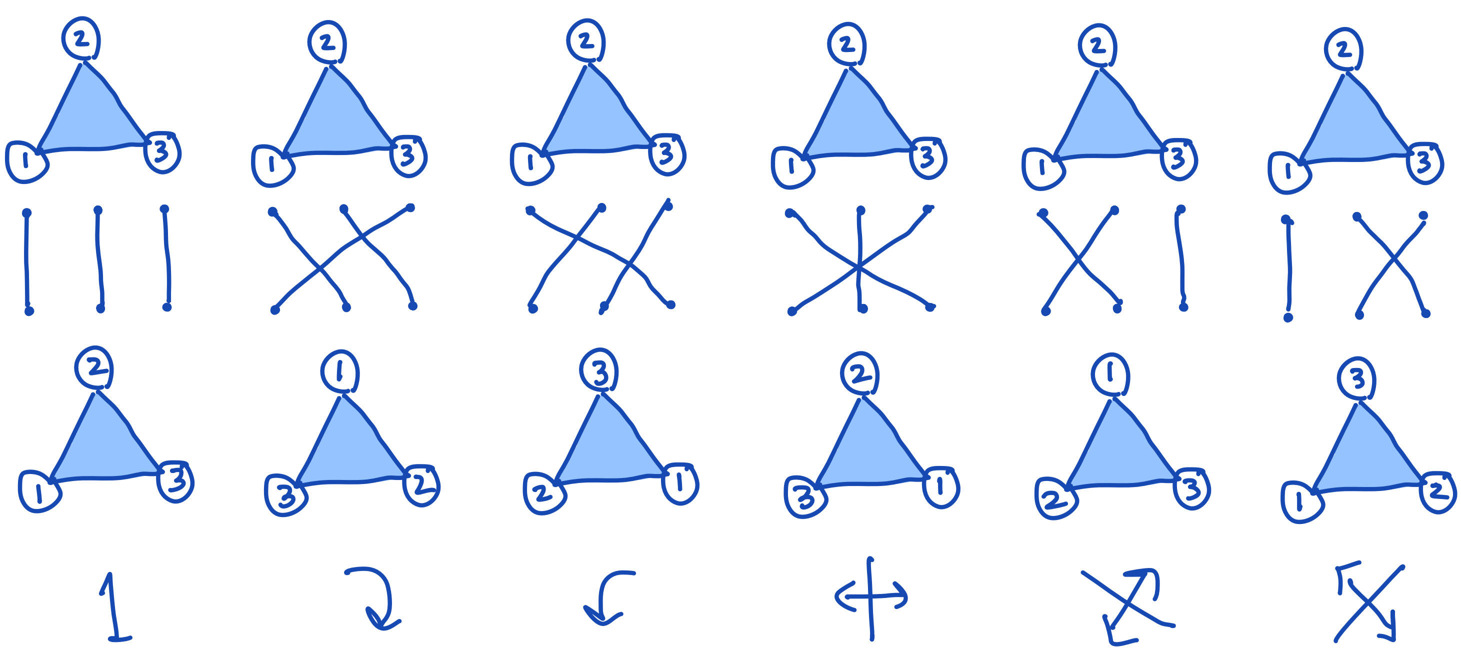
$$= \sum_{x\in X} \frac{|G|}{|Gx|} = |G| \sum_{x\in X} \frac{1}{|Gx|} = \sum_{A\in X/G} 1 = |X/G|$$

Or, every orbit of G acting on X has $|G|$ fixed points

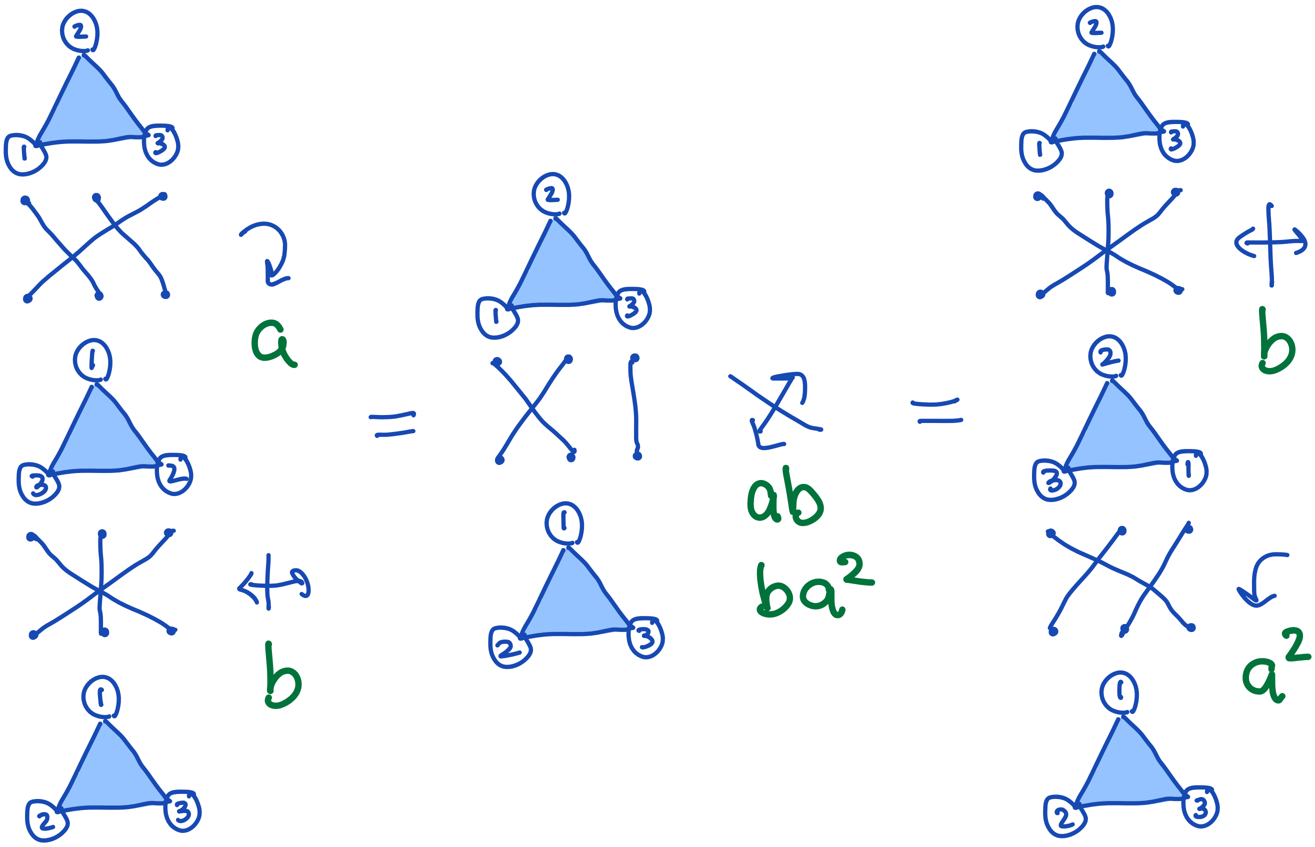
$$|X/G| = \#\text{orbits} = \frac{1}{|G|}(\text{total \# fixed points})$$

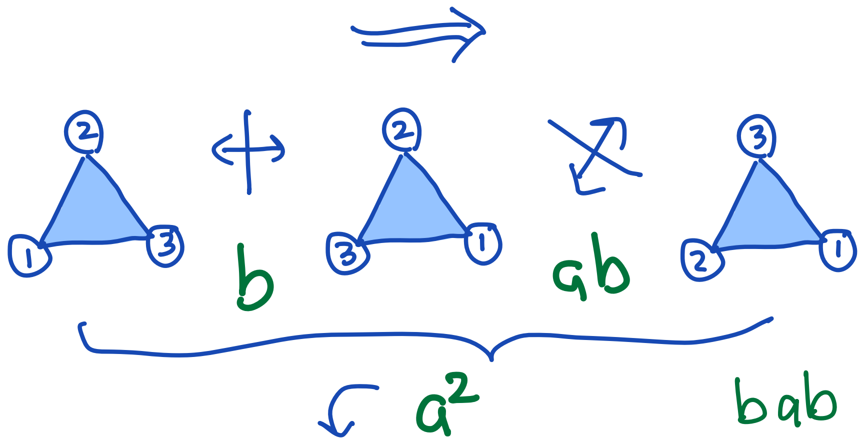
$G = S_3$ permutations of $\{1,2,3\}$

symmetries of Δ (rotations, flips)



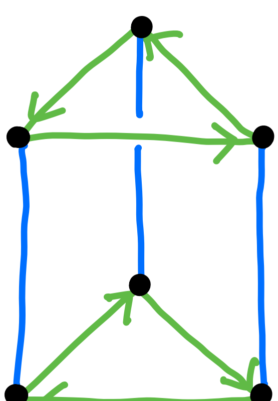
1 $\quad a \quad a^2 \quad b \quad ab \quad a^2b$



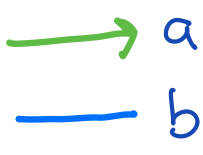


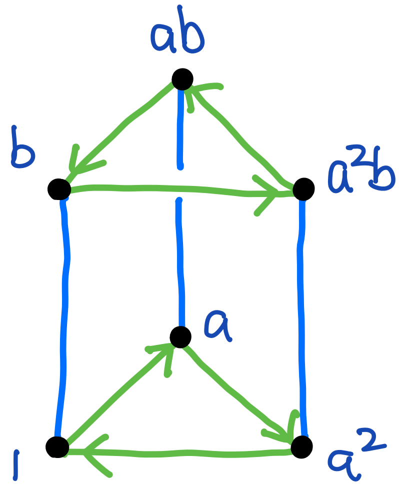
$bab = bba^2 = a^2$

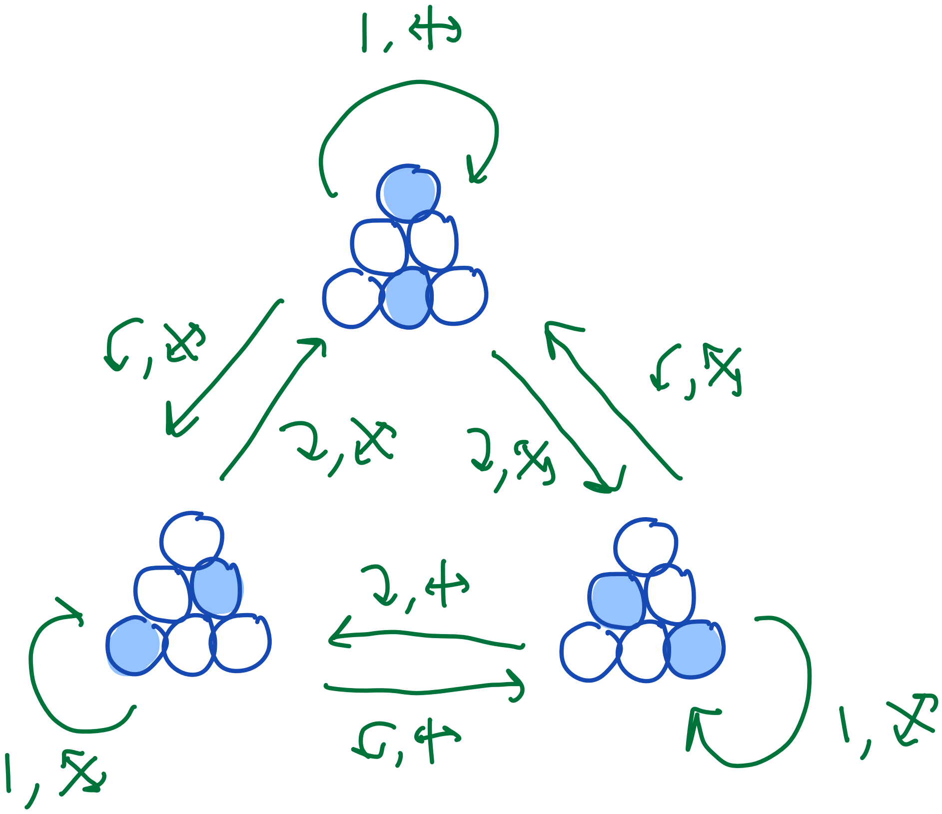
$$S_3 = \langle a, b \mid a^3 = b^2 = 1,\ ab = ba^2 \rangle$$

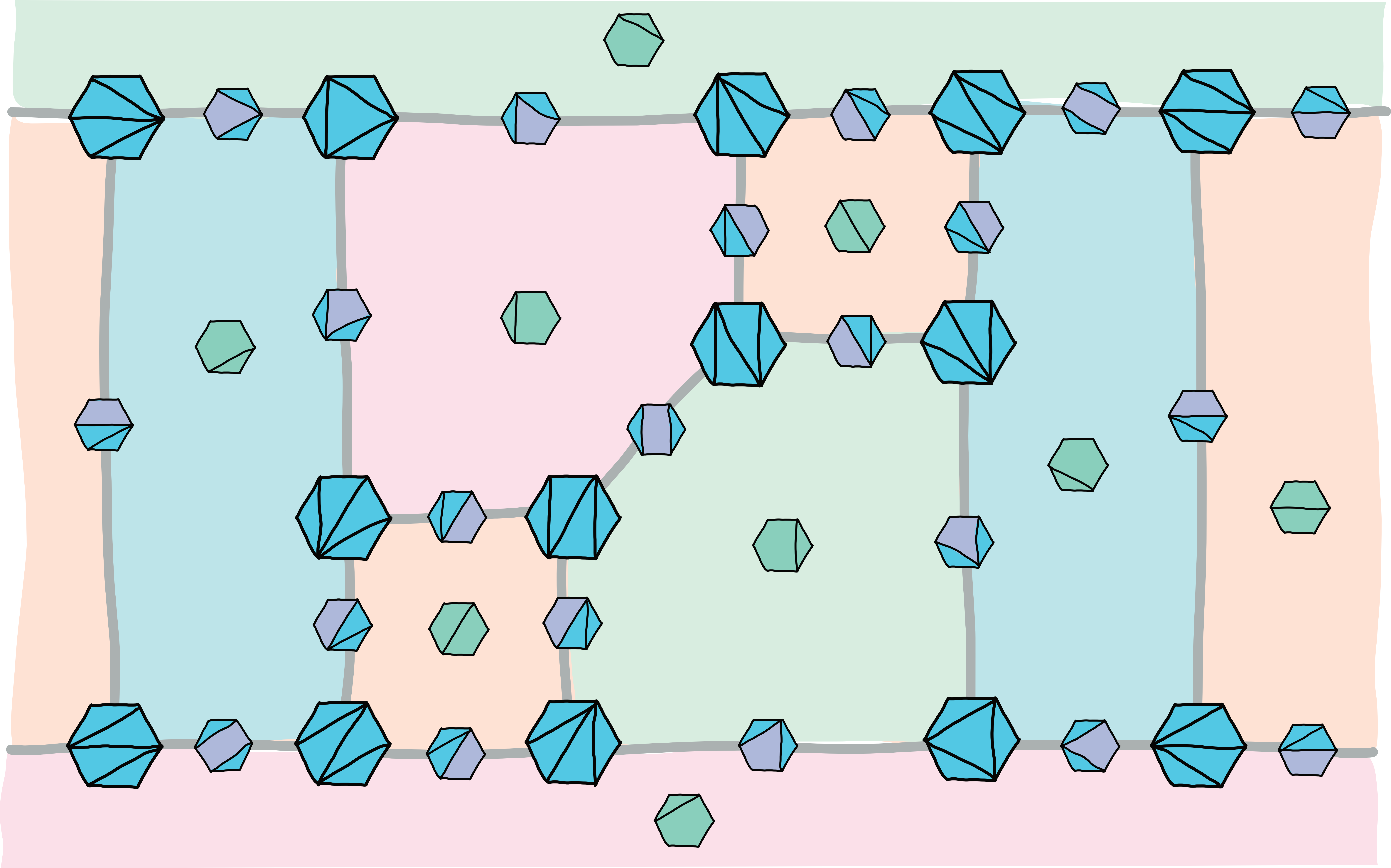


Cayley diagram



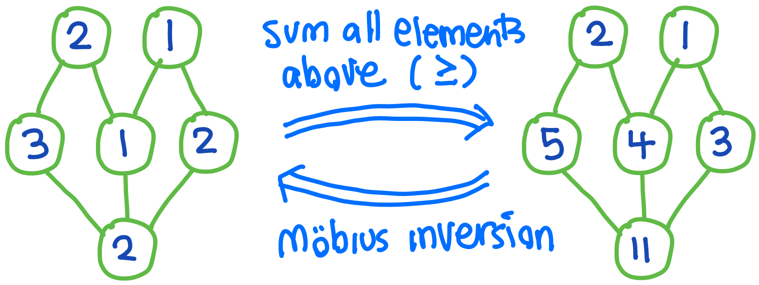




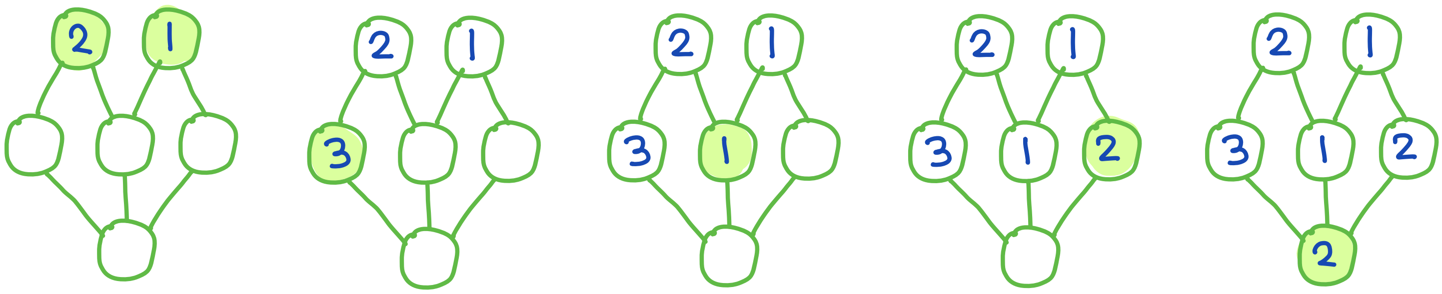


Möbius inversion and cyclotomic polynomials

For any poset, Möbius inversion is inverse to partial sums:

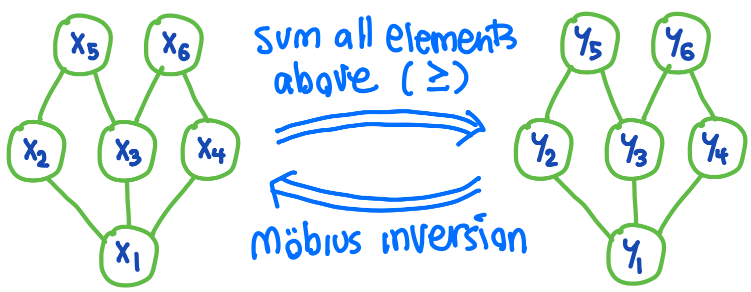


One can carry out the inversion incrementally, correcting each partial sum by summing the values strictly above:



(The theory is identical if one is instead summing below.)

Partial sums is a linear map; Möbius inversion is its inverse:



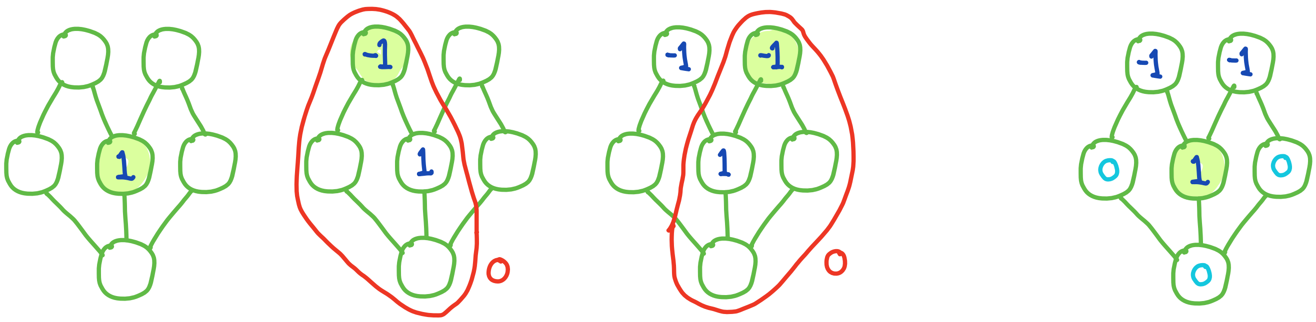
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 0 \\ & & 1 & 0 & 1 & 1 \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

sum all elements above (≥)

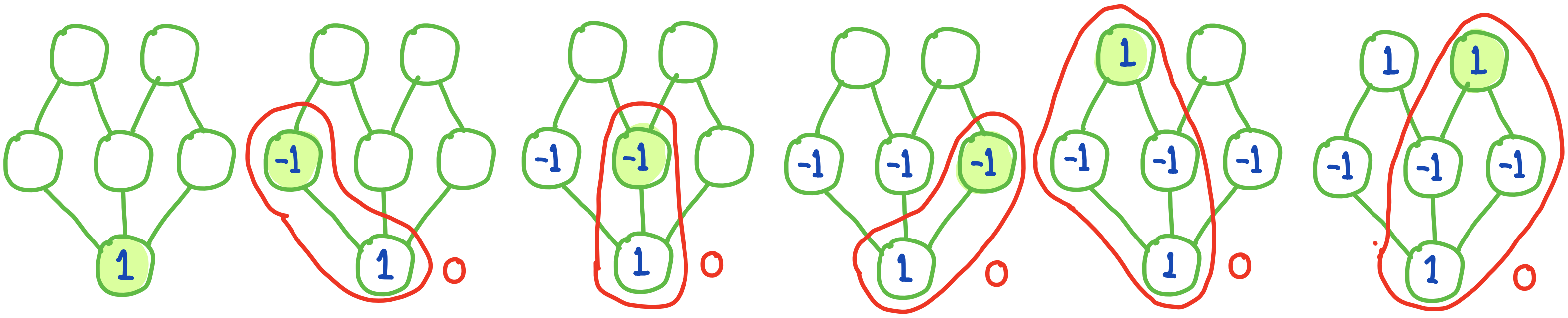
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 \\ & 1 & 0 & 0 & -1 & 0 \\ & & 1 & 0 & -1 & -1 \\ & & & 1 & 0 & -1 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

Möbius inversion

Each row of this inverse can be computed incrementally. Start with a 1 at the desired entry, then work up the poset so all coefficient partial sums below are zero.

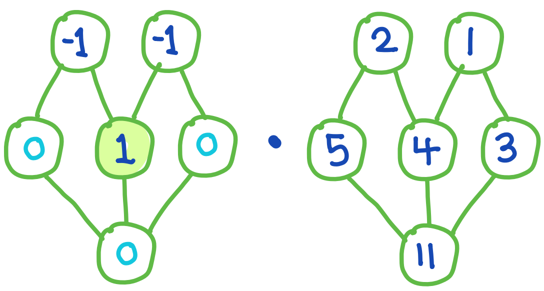


Coefficients for x_4 are $[0\ 0\ 1\ 0\ -1\ -1]$

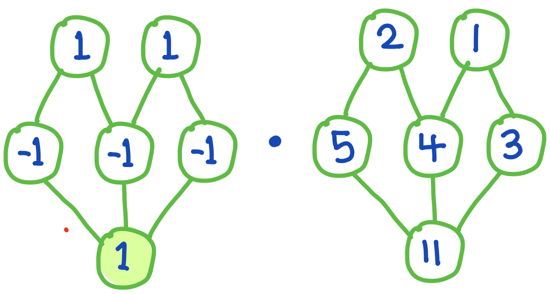


Coefficients for x_1 are $[1\ -1\ -1\ -1\ 1\ 1]$

We can use these coefficients to directly invert the partial sums:



$$= [0\ 0\ 1\ 0\ -1\ -1]\begin{bmatrix}11\\5\\4\\3\\2\\1\end{bmatrix} = 1 = x_4$$



$$= [1\ -1\ -1\ -1\ 1\ 1]\begin{bmatrix}11\\5\\4\\3\\2\\1\end{bmatrix} = 2 = x_1$$

In many applications one can find these coefficients by other means than direct matrix inversion.

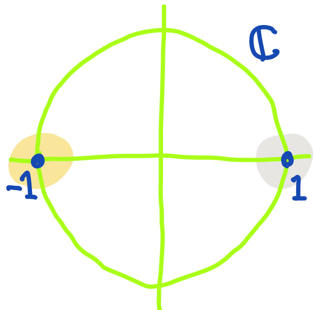
For example, there are often homological* methods.

* algebraic topology

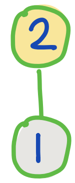
Cyclotomic polynomials

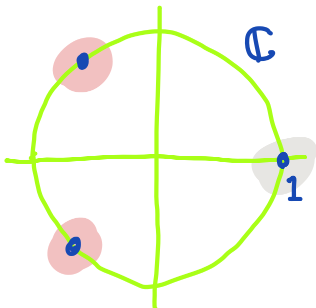
The n^{th} roots of unity are the roots of $x^n - 1 = 0$.

Each $x^n - 1$ has a factorization into distinct irreducible polynomials with integer coefficients, called cyclotomic polynomials.

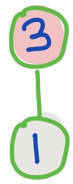


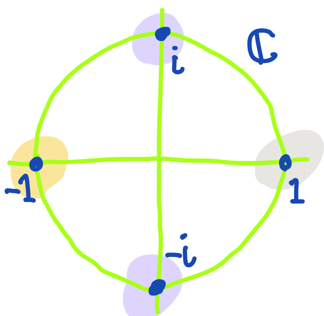
$$x^2 - 1 = (x-1)(x+1)$$



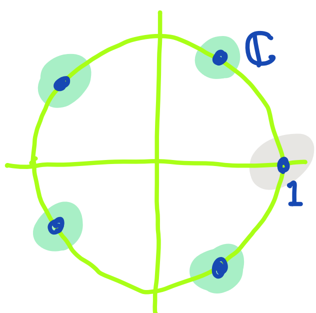


$$x^3 - 1 = (x-1)(x^2+x+1)$$

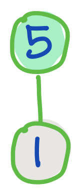


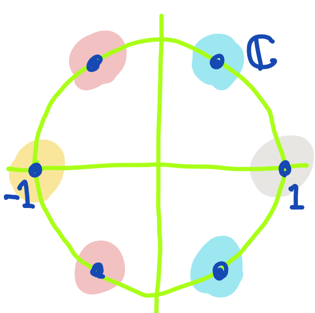


$$x^4 - 1 = (x-1)(x+1)(x^2+1)$$

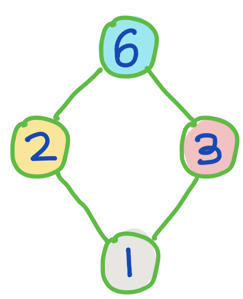


$$x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$$



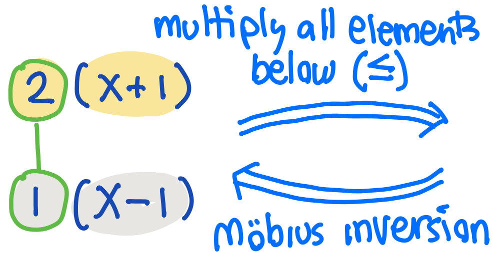


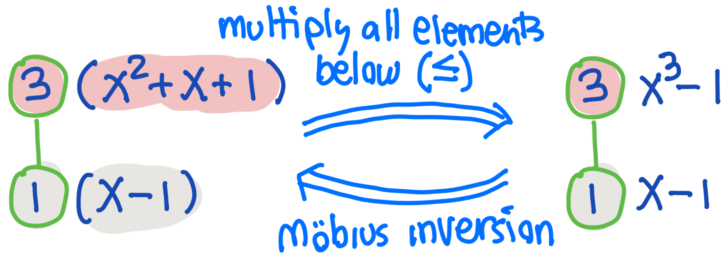
$$x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$$

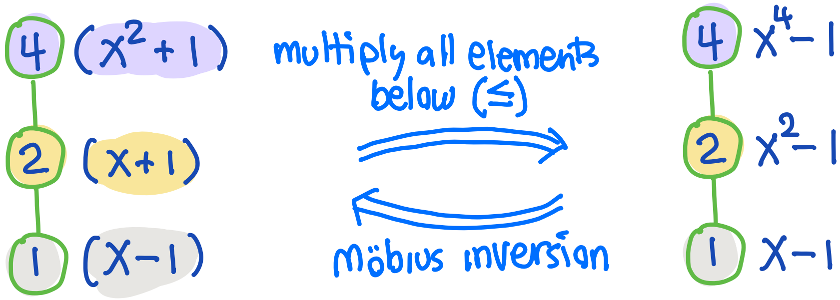


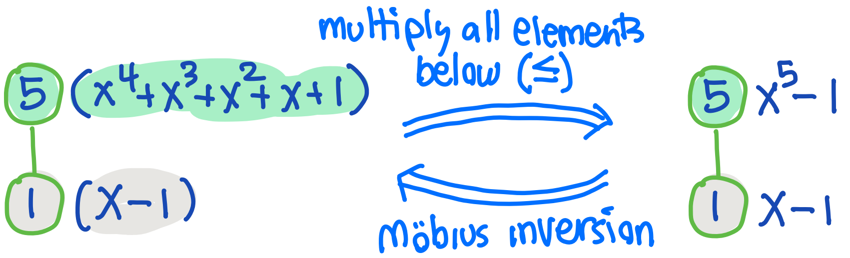
The divisor lattice of n is the poset of integer factors of n, partially ordered by divisibility.

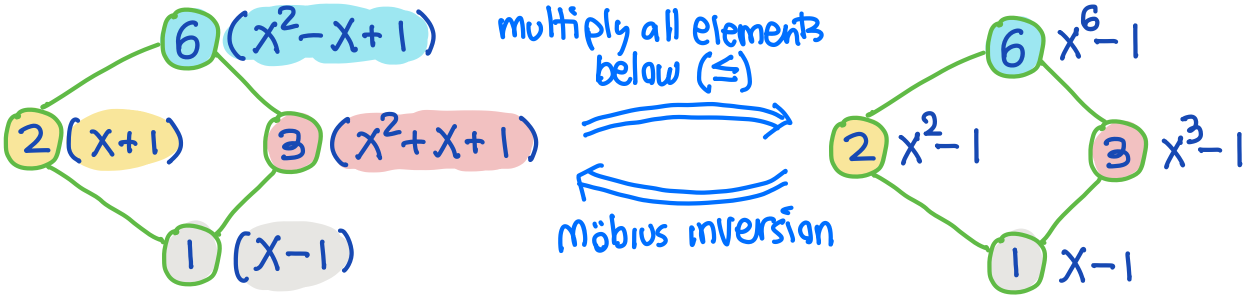
As seen above, the cyclotomic polynomials that are factors of $x^n - 1$ correspond to the elements of the divisor lattice. The polynomials $x^d - 1$ for each factor d of n appear as partial products of the cyclotomic polynomials for each factor e of d. We recognize this as a form of Möbius inversion, with sum replaced by product, and ascending rather than descending partial products:









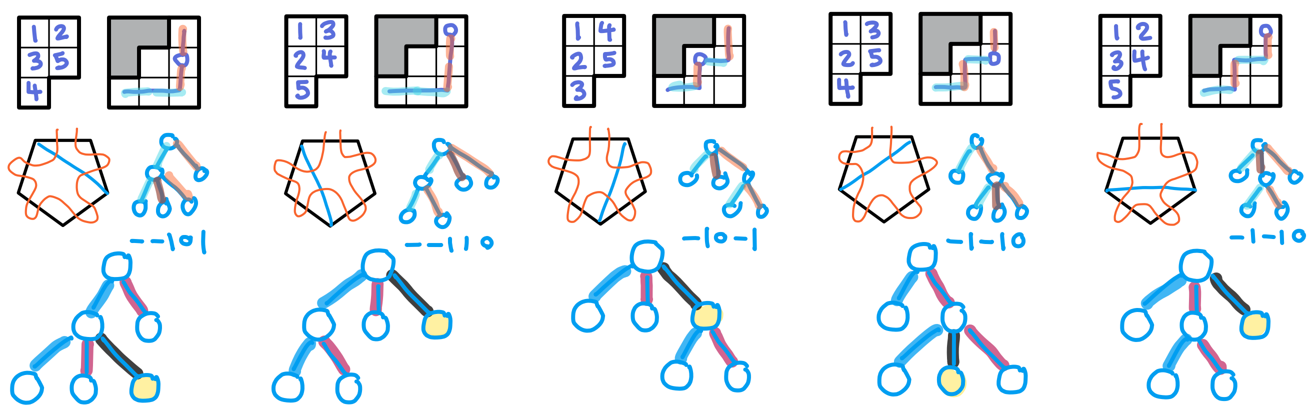


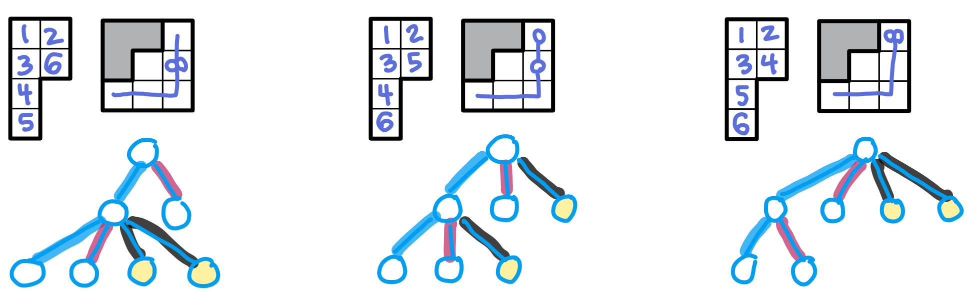
This gives effective algorithms for computing cyclotomic polynomials.

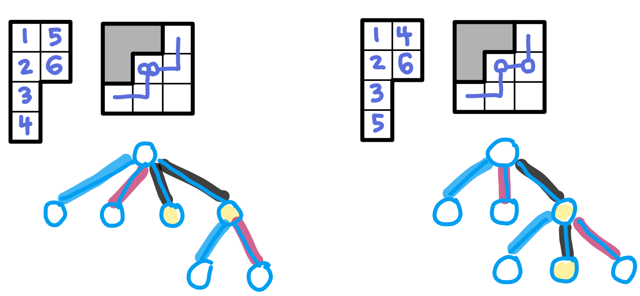
March 3, 2021

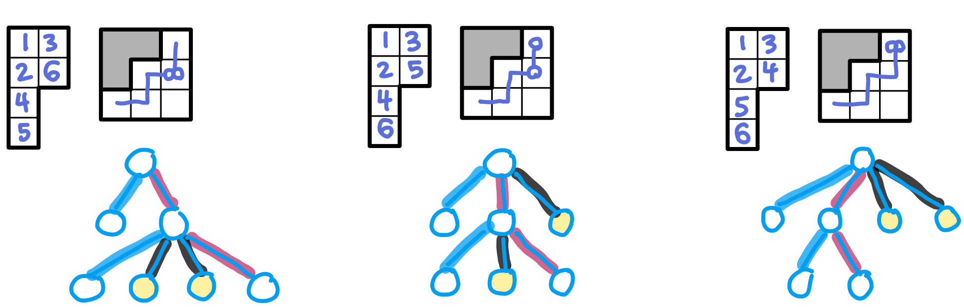
These are my notes as I look for an alternative construction.

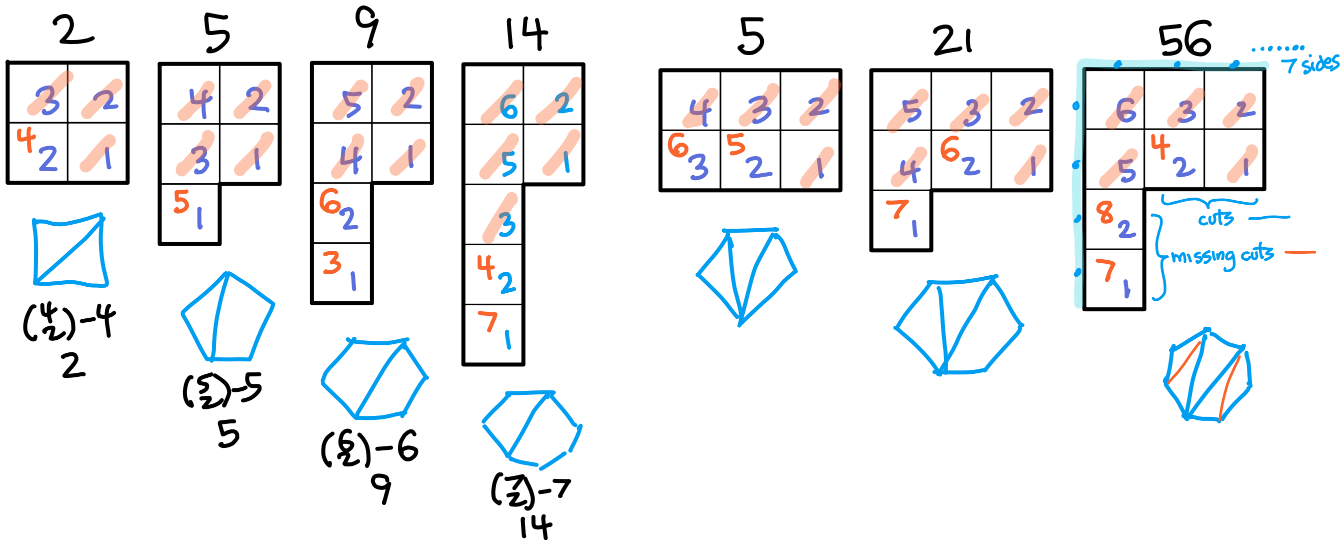
Thinking of Stanley correspondence as lattice walks with pauses

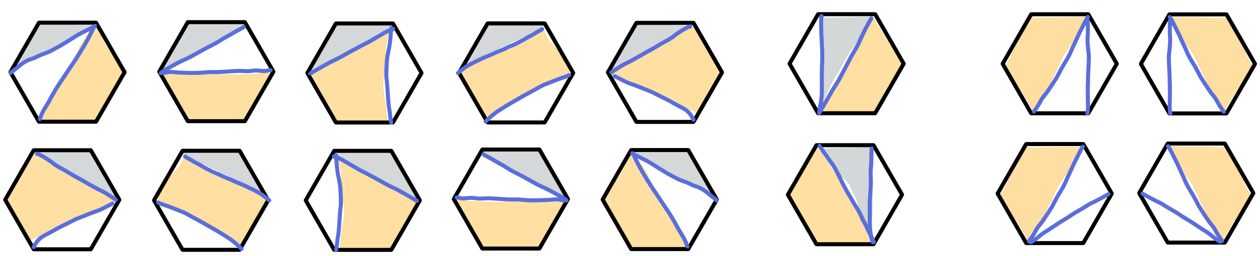




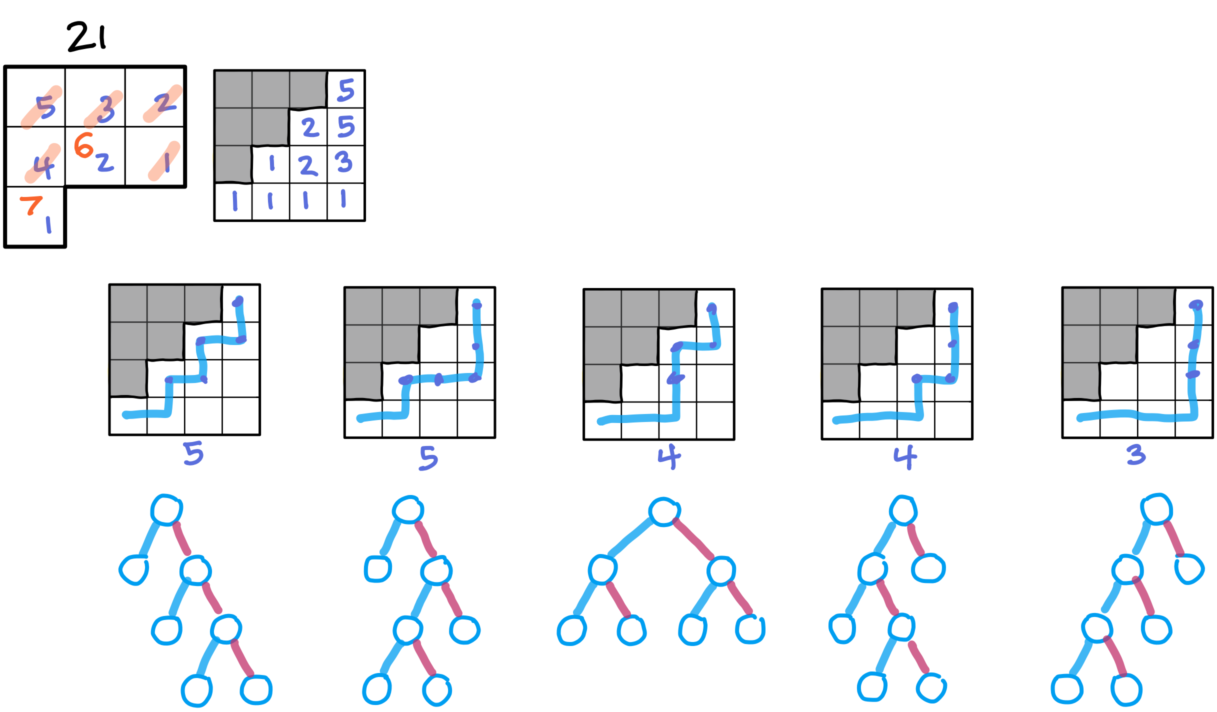


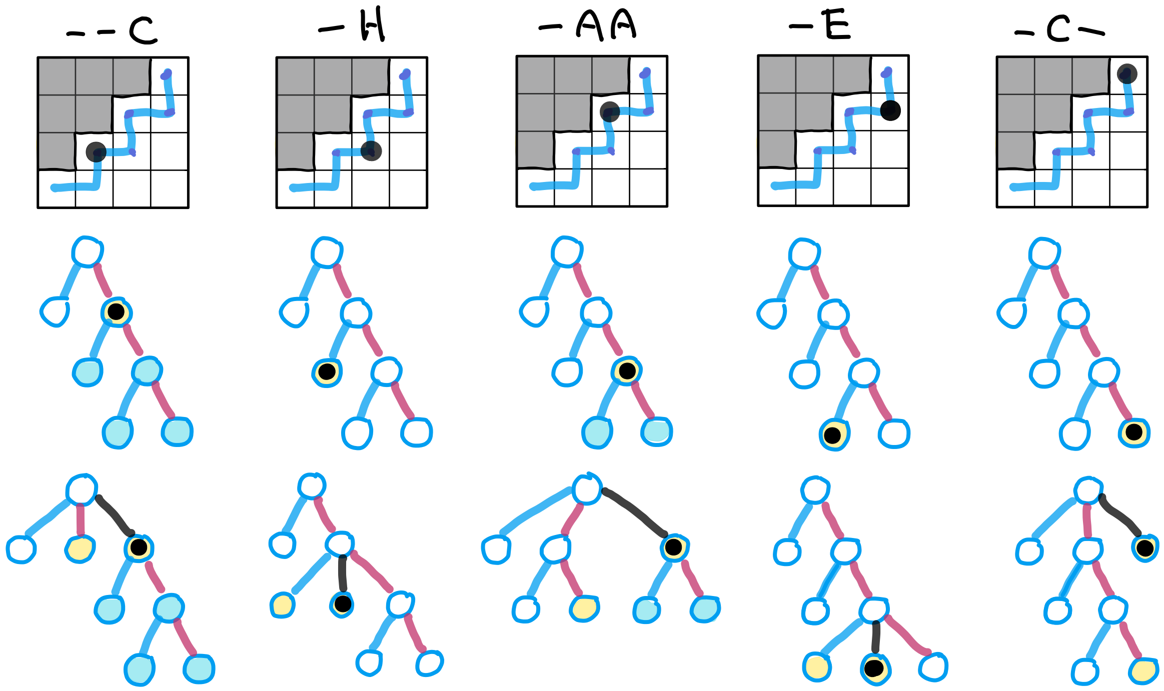


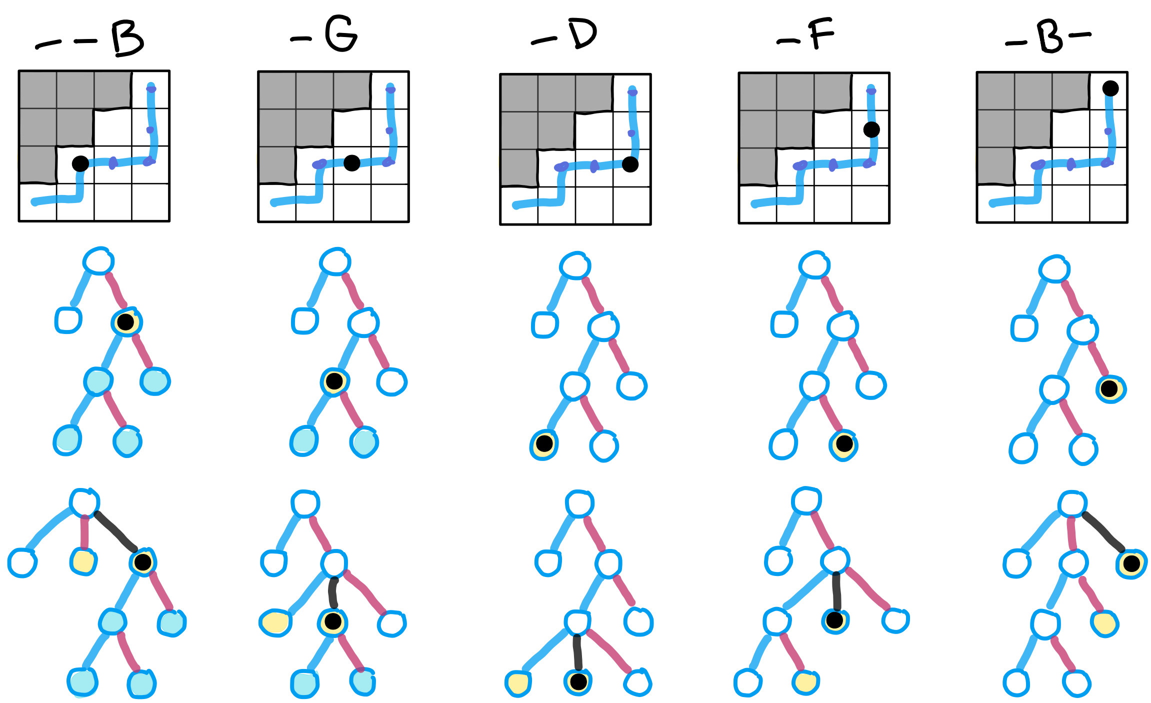




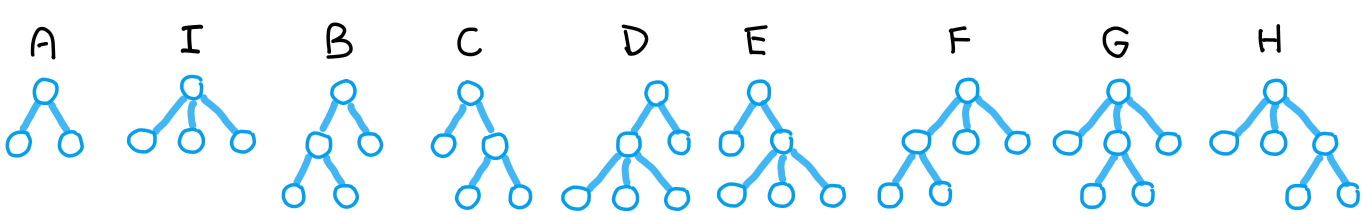
March 5, Friday

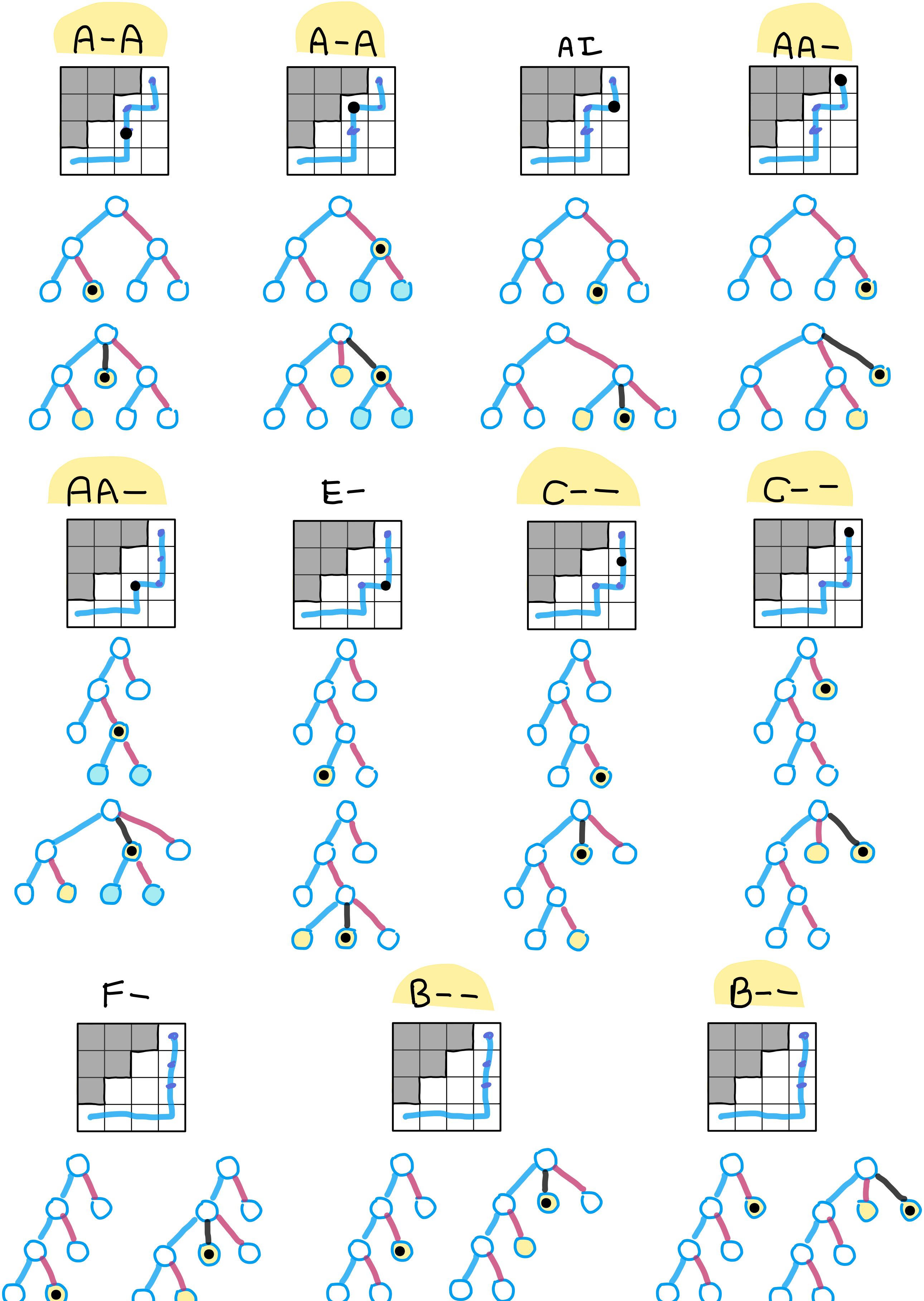




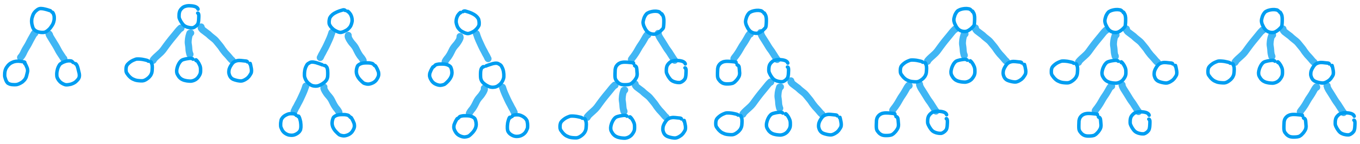


(First try is wrong)

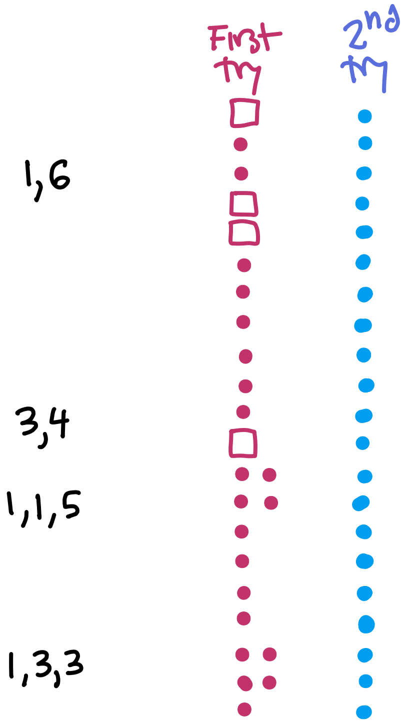



A-A
A-A
AI
AA-
AA-
E-
C--
C--
F-
B--
B--

3,2 A   4,3 I   5,3 B   5,3 C   6,4 D   6,4 E   6,4 F   6,4 G   6,4 H



8-root = 7 nodes, 5 leaves   10+2+6+3 = 21 ✓

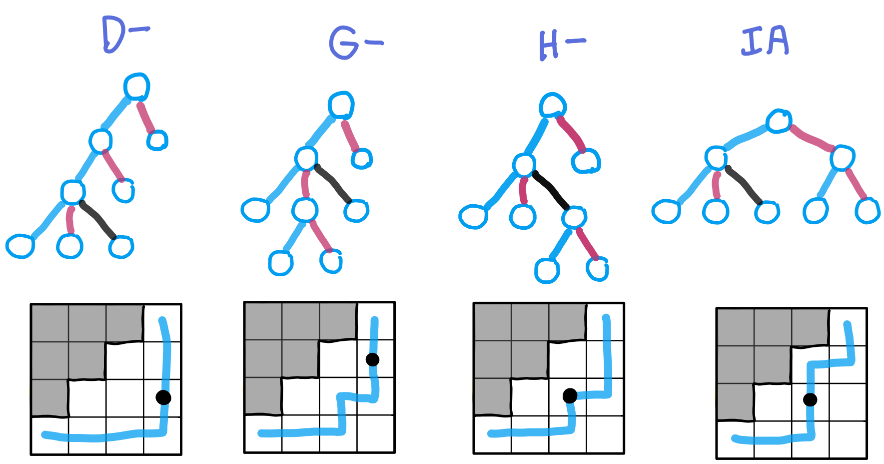


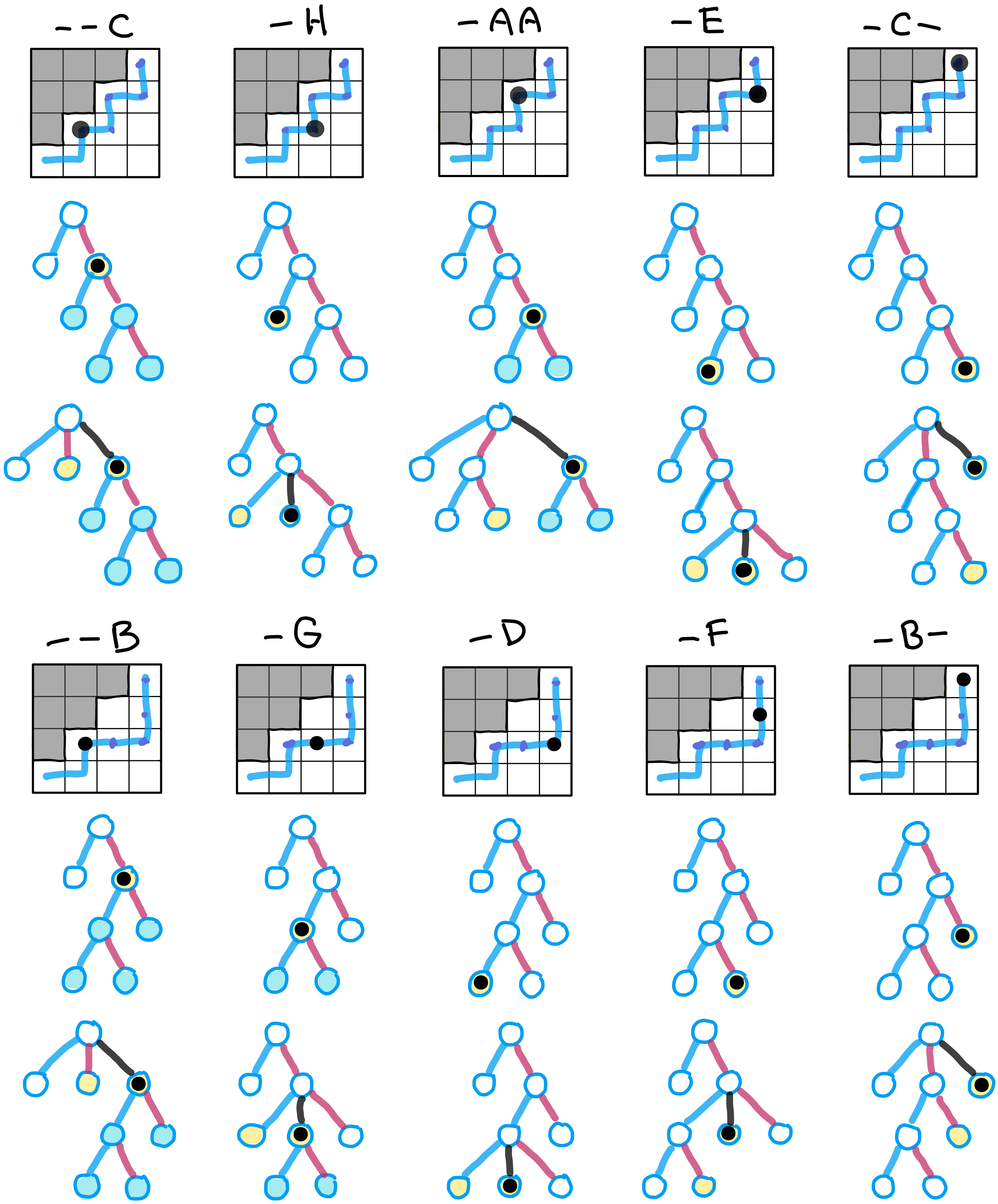
First try, wrong

--C   -H   -AA   -E   -C-
--B   -G   -D   -F   -B-

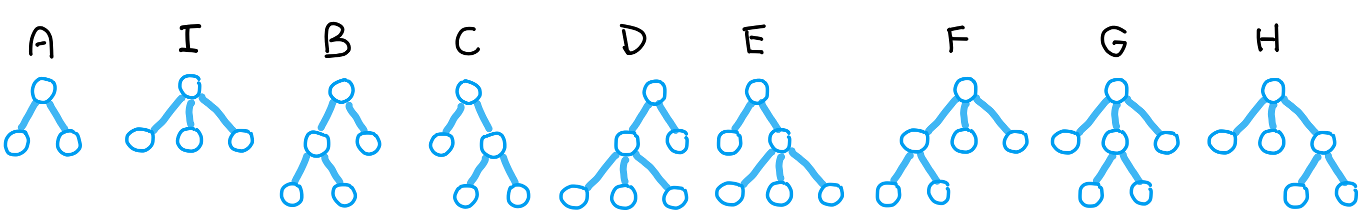
A-A   A-A   AI   AA-
AA-   E-   C--   C--
F-   B--   B--

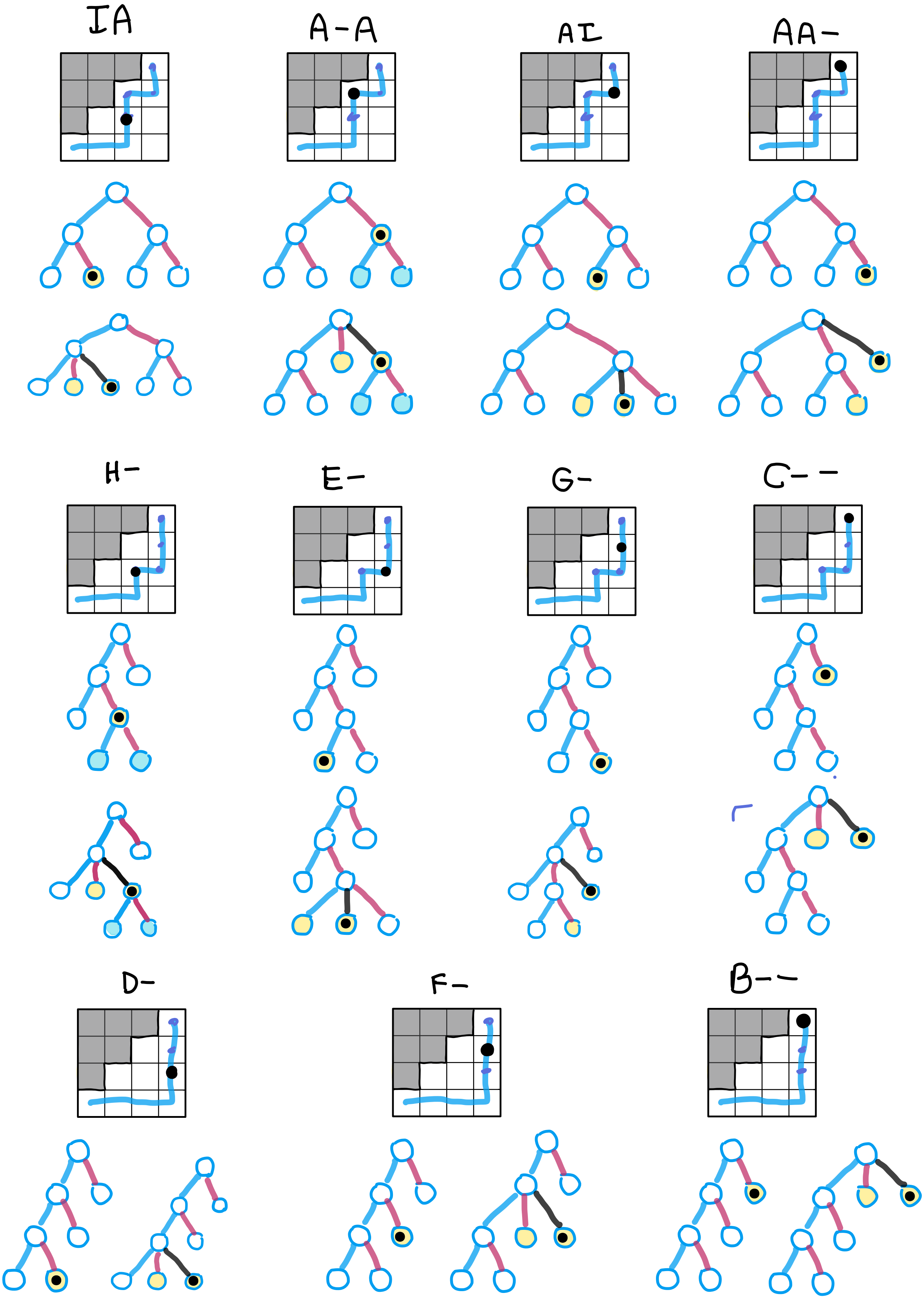
So black floats up red, but not blue
We need a harder example to see
black on black rules

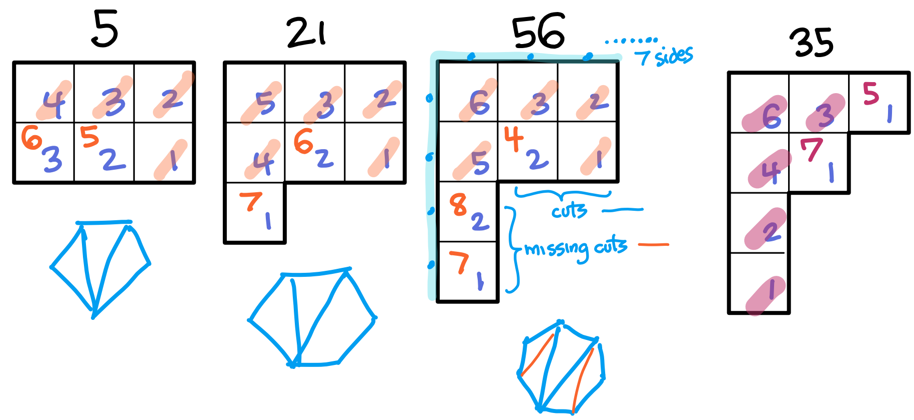




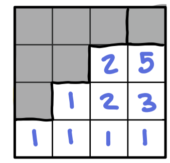
(second try, correct)

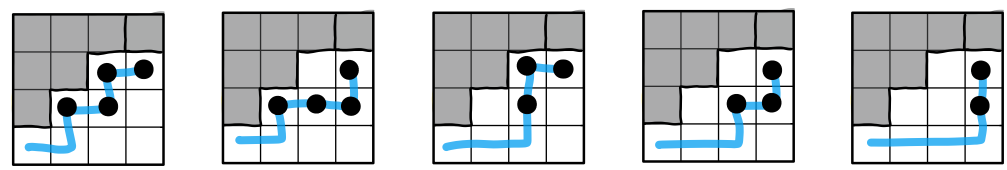



IA
A-A
AI
AA-
H-
E-
G-
C--
D-
F-
B--



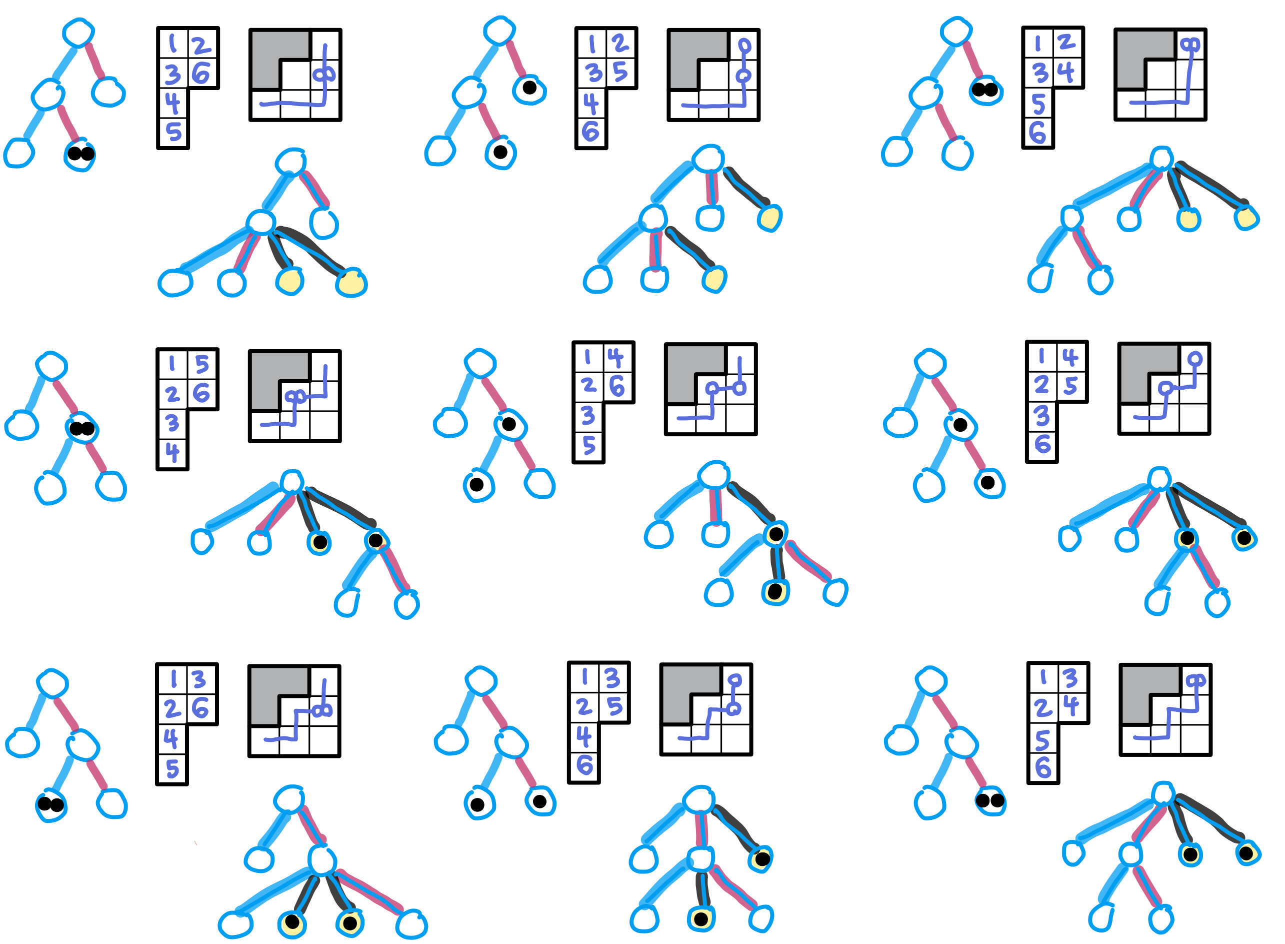
We now understand broken tableaus as incomplete trees.

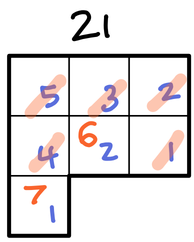




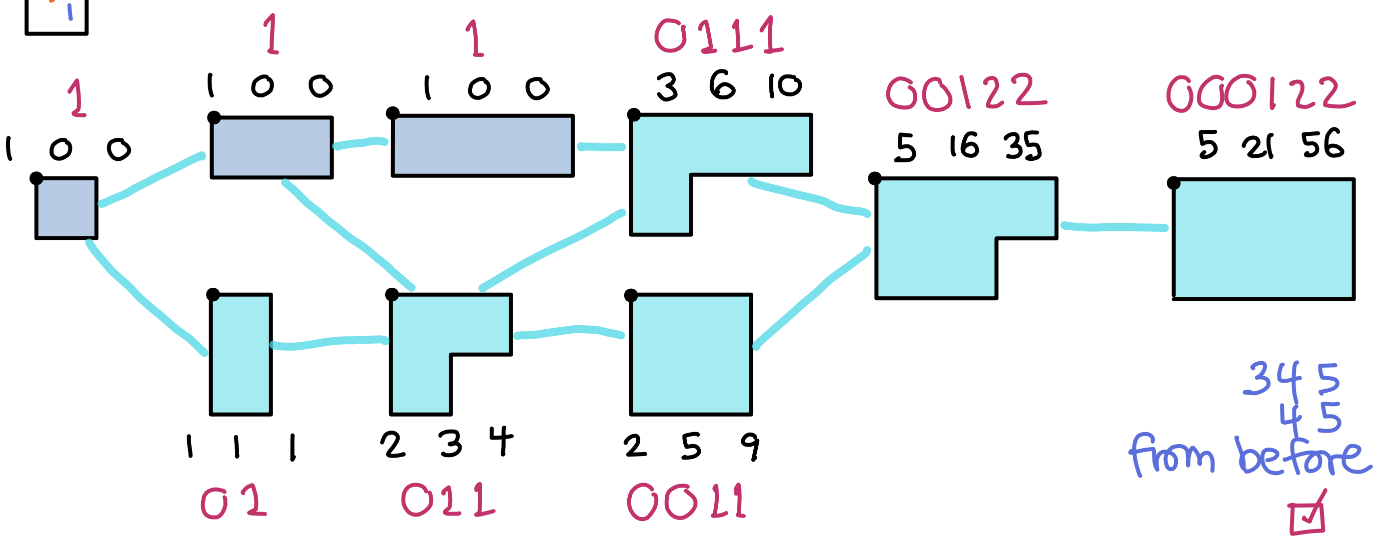
$4+\binom{4}{2}=10$ $\quad 4+\binom{4}{2}=10$ $\quad 3+\binom{3}{2}=6$ $\quad 3+\binom{3}{2}=6$ $\quad 2+\binom{2}{2}=3$

Process multiple dots in sequence as encountered, depth first search





General principle: Allow pauses in all walks.
Here, only once two rows



$g(t)=\sum a_n t^n$, n pauses

$$g(t) = a(t)+b(t)+tg(t)$$
$$(1-t)g(t) = a(t)+b(t)$$

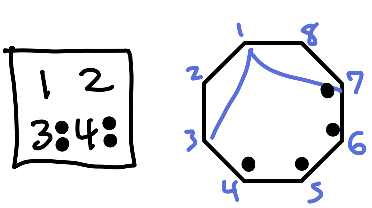
write $a + \frac{b}{1-t} + \frac{c}{(1-t)^2} + \ldots$ as a b c

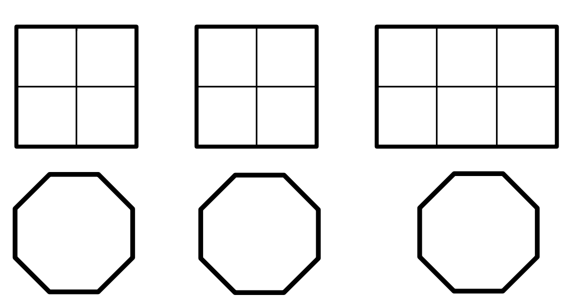
There should be a generalized hook length formula here.

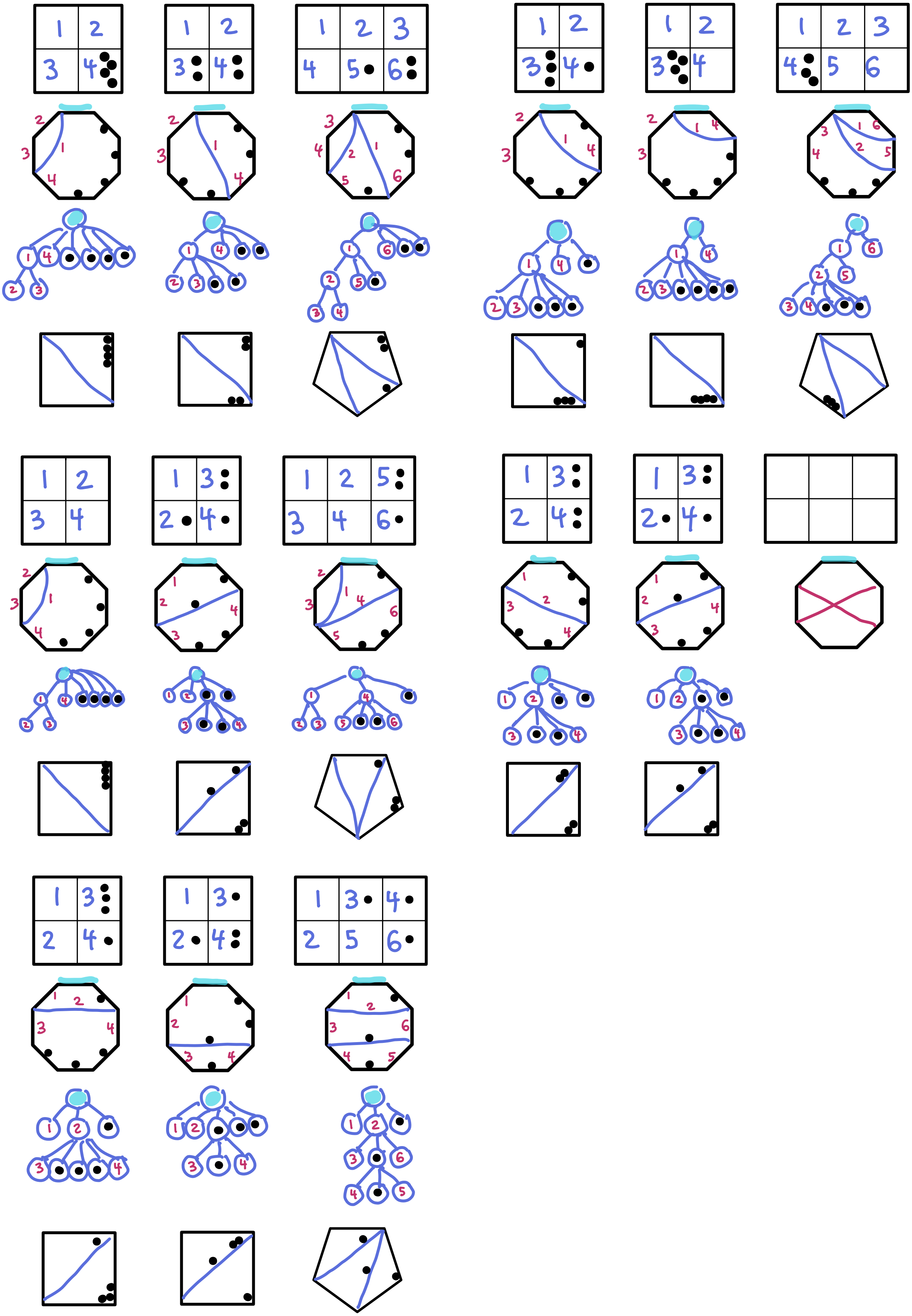
Sunday March 7

Want entire associahedron polytope to map through correspondence

When are two ab3 noncrossing?





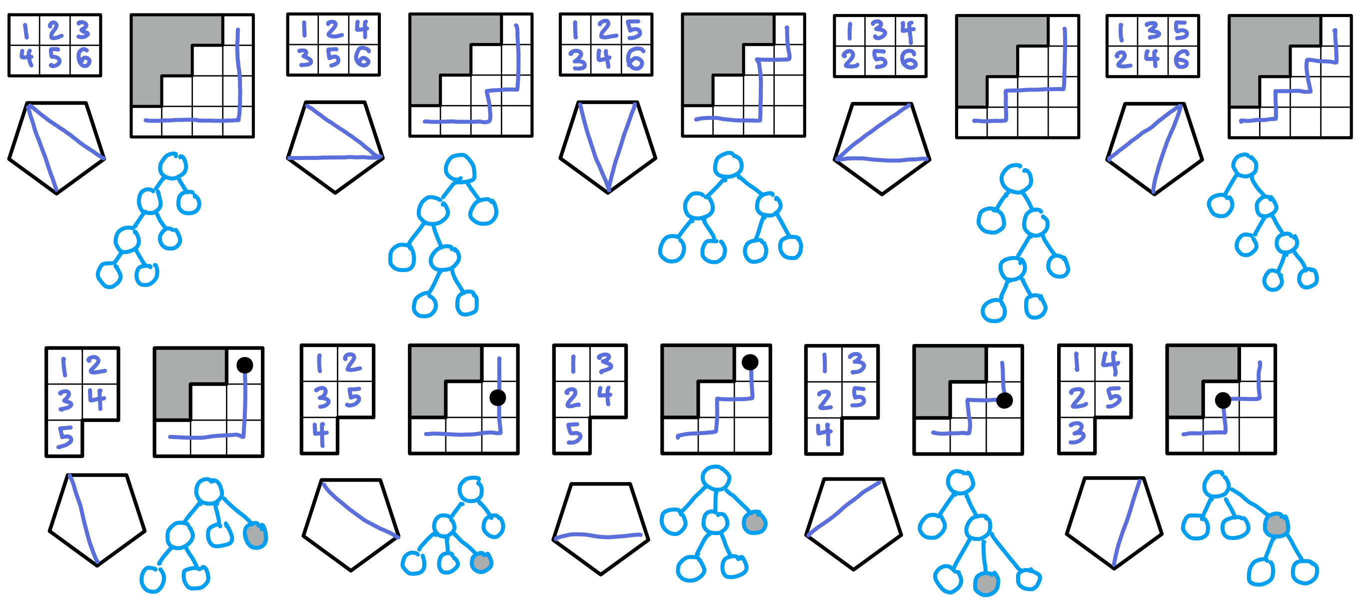


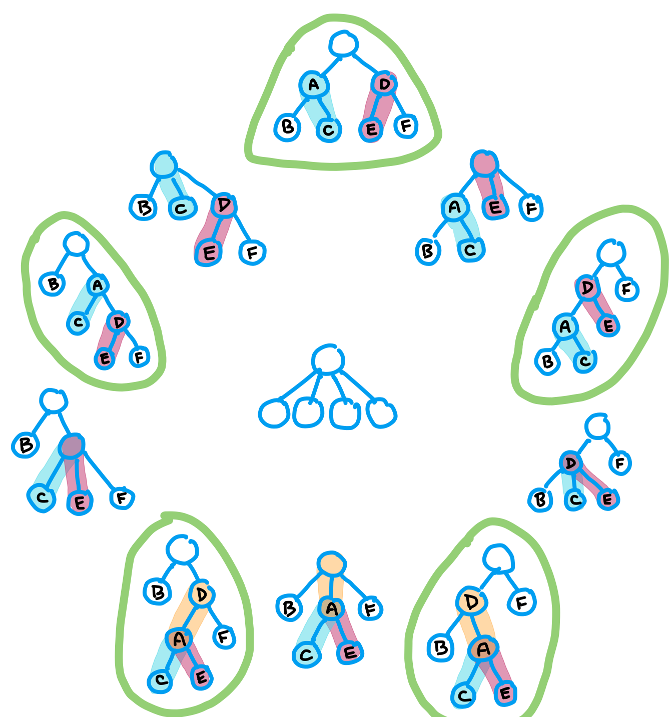
Monday, March 8

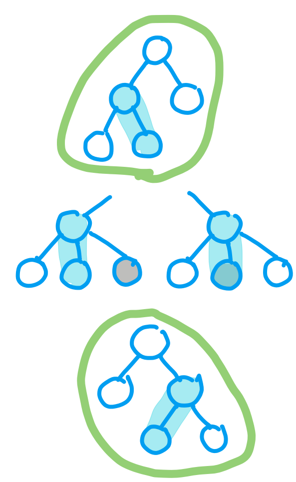
Need to study squarefree ideal on variables = interior edges,
monomials = crossing pairs.

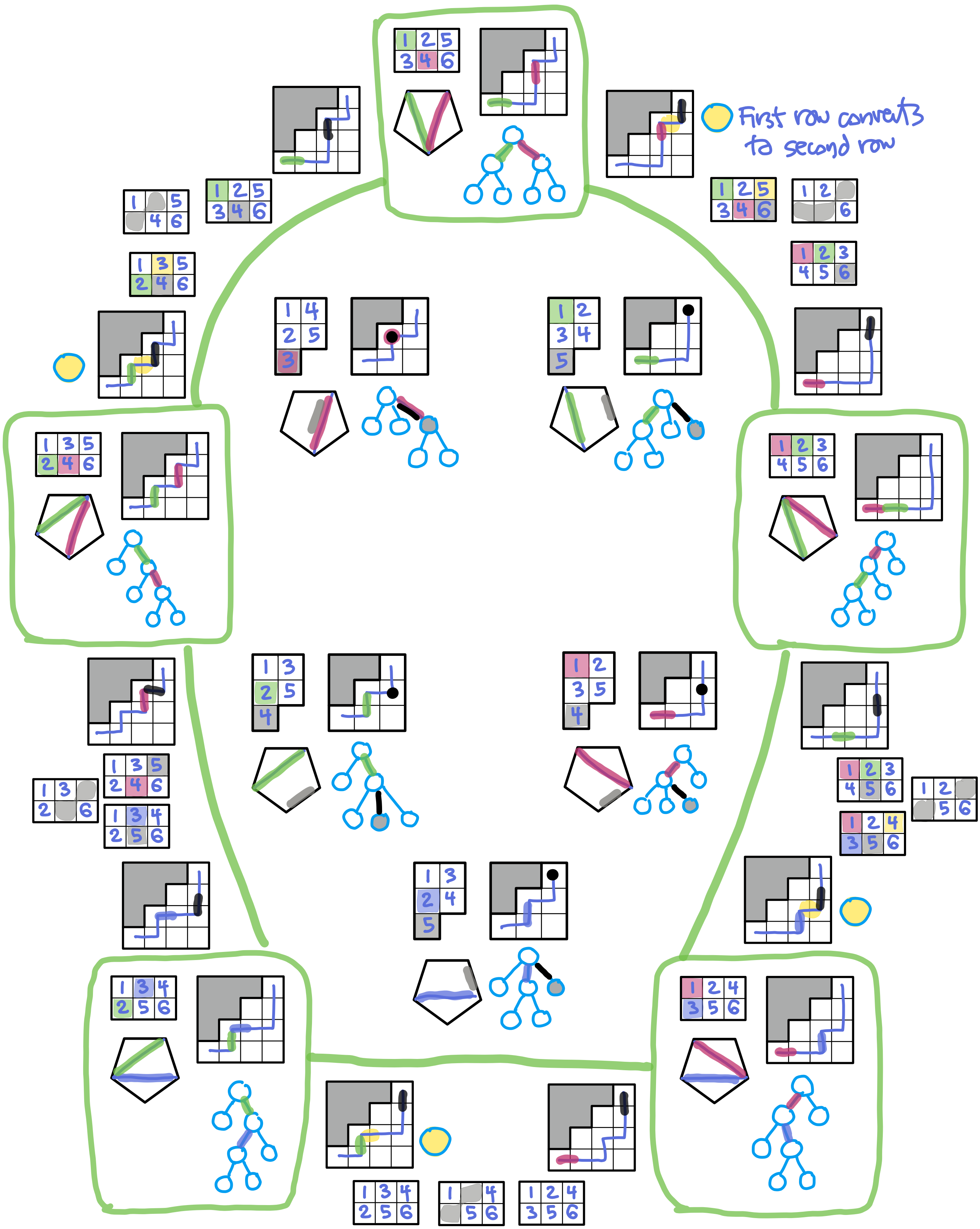
This should have the dual associahedron as minimal resolution.

Meanwhile, start instead trying to understand edges on polytope.



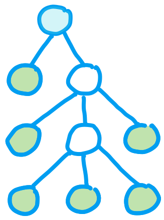
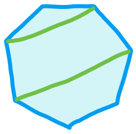





First row converts to second row

Tuesday, March 9, 2021

Want to confirm rule in general, for nested extra edges.

7-gon

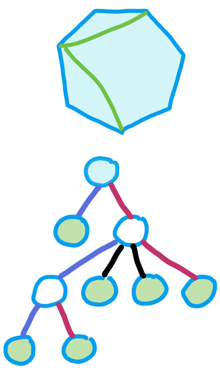
$\binom{7}{2} - 7 = 21 - 7 = 14$ interior edges.

$\binom{14}{2} = \frac{14\cdot 13}{2\cdot 1} = 7\cdot 13 = 91$ pairs

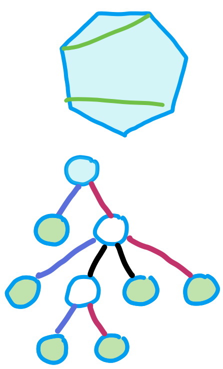
$\binom{7}{4} = \frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 35$ crossing pairs

$91 - 35 = 56$ too many to write all out?

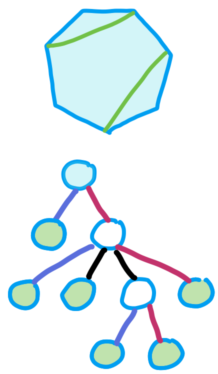
Compare with neighboring trees. $\binom{6}{2} - 6 = 9$ cases with this start



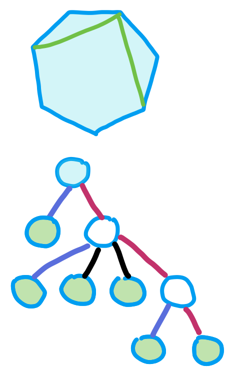
1	3	4
2	5	8
6		
7		



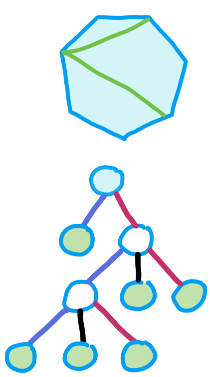
1	3	5
2	6	8
4		
7		



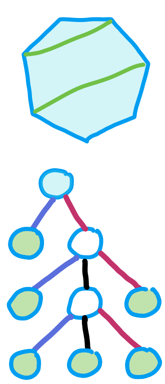
1	3	6
2	7	8
4		
5		



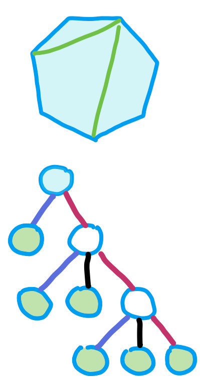
1	3	7
2	6	8
4		
5		



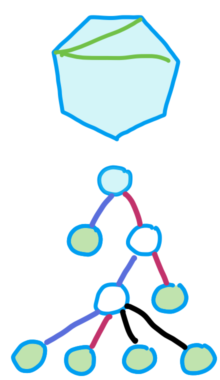
1	3	4
2	6	8
5		
7		



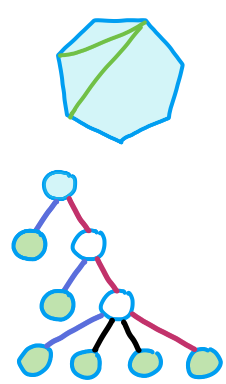
1	3	5
2	7	8
4		
6		



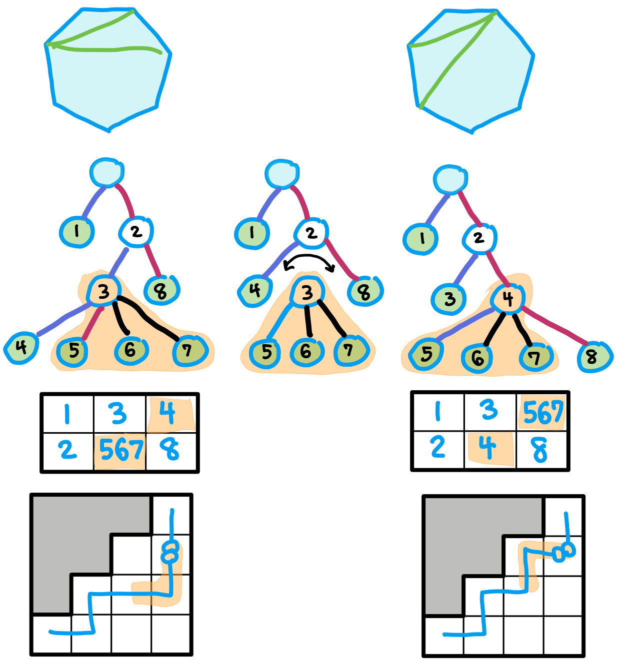
1	3	6
2	5	8
4		
7		

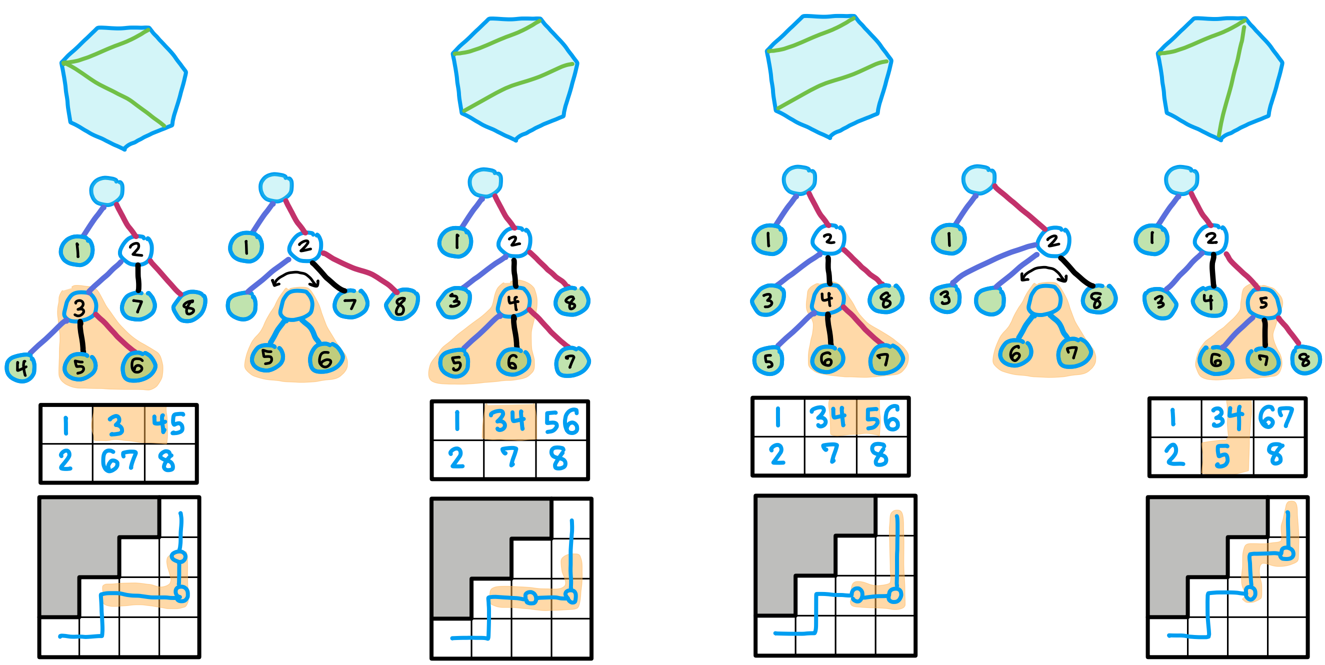


1	3	4
2	5	8
6		
7		

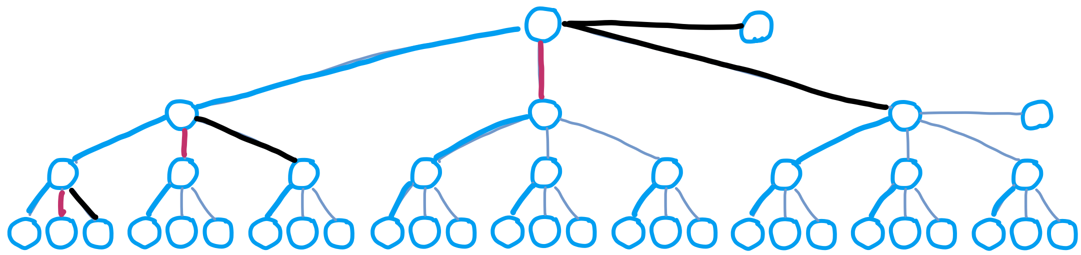


1	3	5
2	4	8
6		
7		



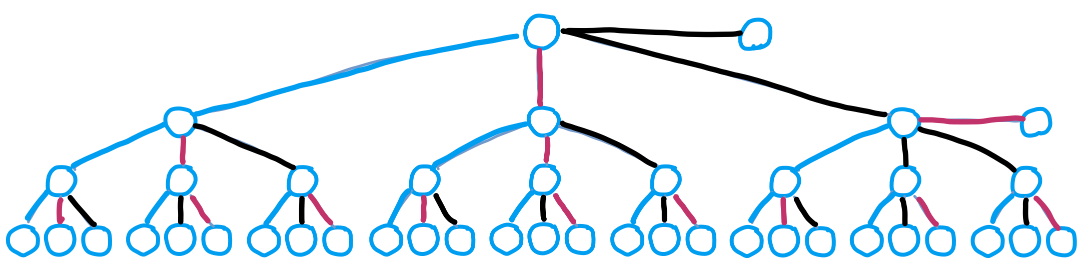


Thursday March 11 What are rules in general?

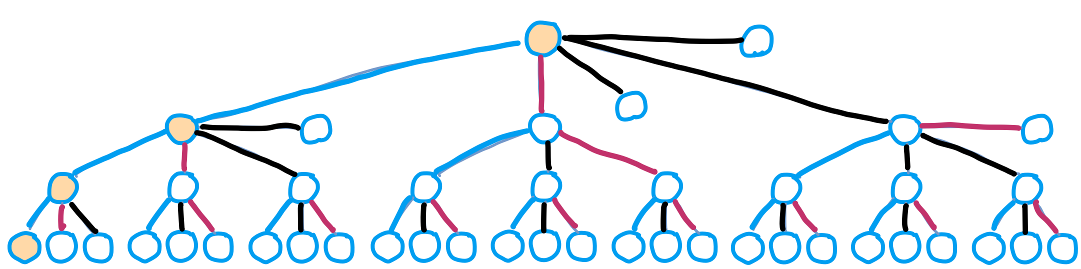


/ Left (blue) always descends to create new node
Use is forced, no question when to use

| \ Right (red), Extra (black) have variant rules based on context
\ Right always rises to attach to nearest available host
| Rules for black need to be pinned down.
For standard Young tableau correspondence, no black till first red
Given this, need red to rise, so goes after blue again
This sets rules for leftmost blue spine

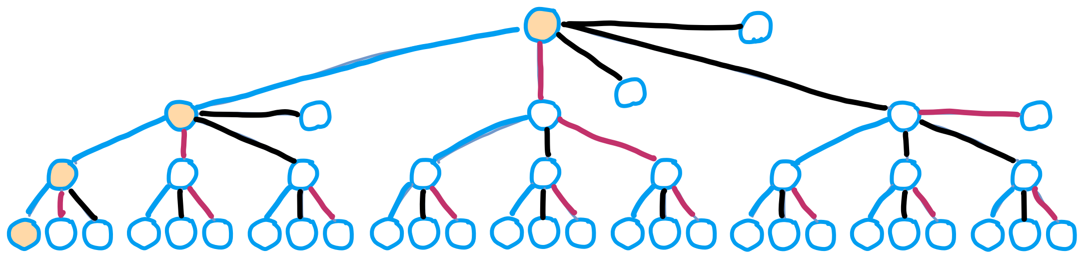


Perhaps rules are local, in which case locally leftmost is same
Starting in middle, always ambiguity whether black rises
Need red as divider between cases
Same issues in middle, so leftmost spine can't be local?



Off the leftmost spine, red closes the level.
Black climbs to find an open level.

To find a potential flaw, what changes if leftmost spine isn't special?

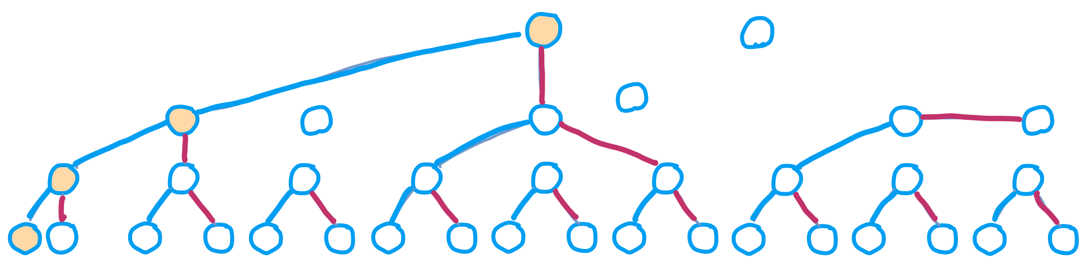


Reason the other way, down from root
How will we ever get black edges last in traversal?
Now, how do we avoid two reds in a row, down spine?
Leftmost spine has to be special.

The SYT (Standard Young Tableau) is just a fancy way of writing words in {L, R, E} so initial segments never have more R than L, and no E can appear before first R.

So we want bijection between words and our trees.
They are essentially the same thing.
Tree traversal is extra structure on words, strip tree to get words.

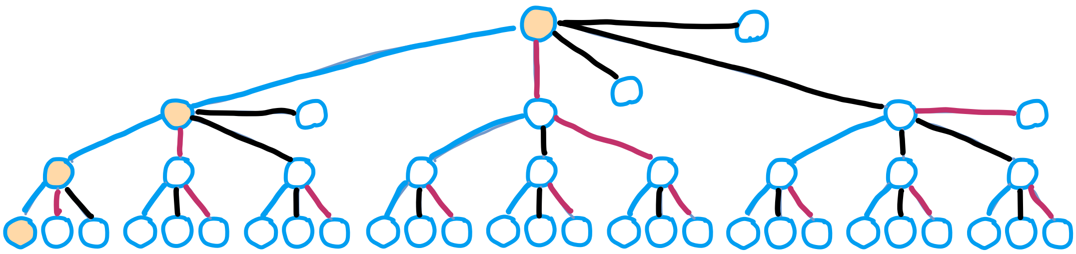
Need to see there is a unique tree for each word.



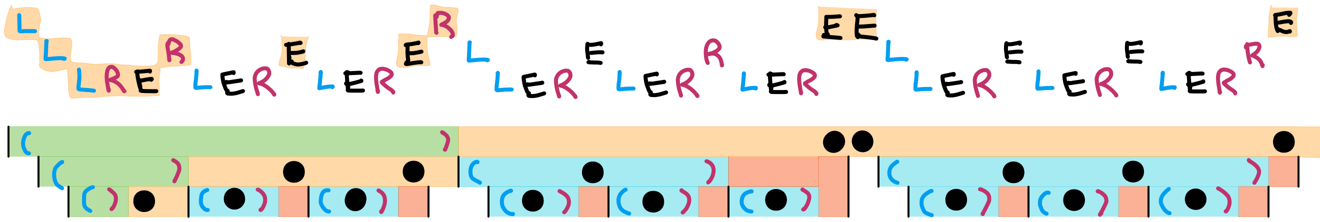
Remove black E edges, left with forest.
Enough to see that words lift to tree in order, and every E has unambiguous position in tree.

Seems to work, still feels unexpected.
Need code to check bijection for many cases.

Because trees are valence ≥ 2, leftmost doesn't need to climb multiple steps ever. Anywhere else, need possibility.



For each black edge or cluster, explain exactly its position.



Each layer has a state, given by color bar
Each new symbol amends these states

(always creates a new layer, green from spine else blue

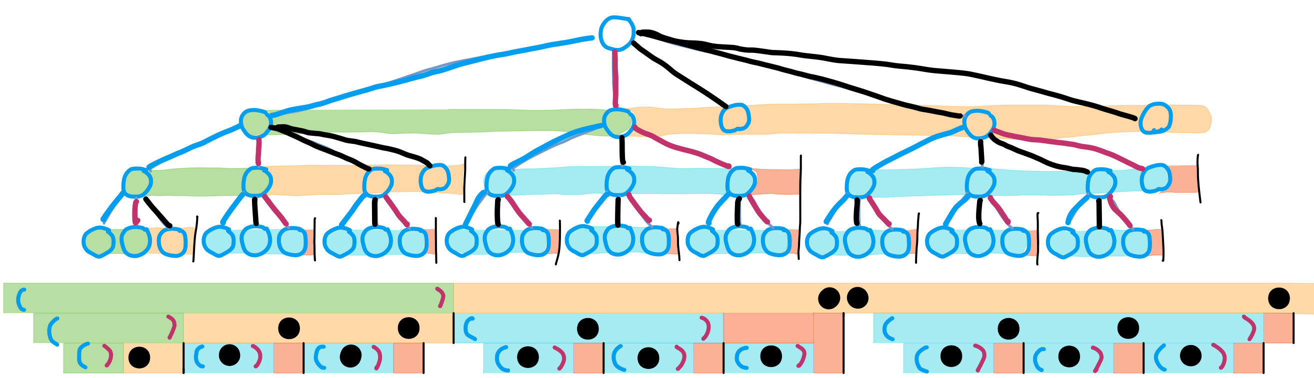
) turns green to yellow, rises through yellow turning it off, and turns off blue

● parks in first available layer, without changing state
not allowed in green

This is a push-down automata, one could imagine general rules for similar trees.

We could generalize this to include all standard Young tableaus? Or more?
Gist of theorem now is that color scheme is unique per tree.

Can a standard PDA automata pop several symbols? Why not?
Is the general enumeration solvable? Hook length formula?

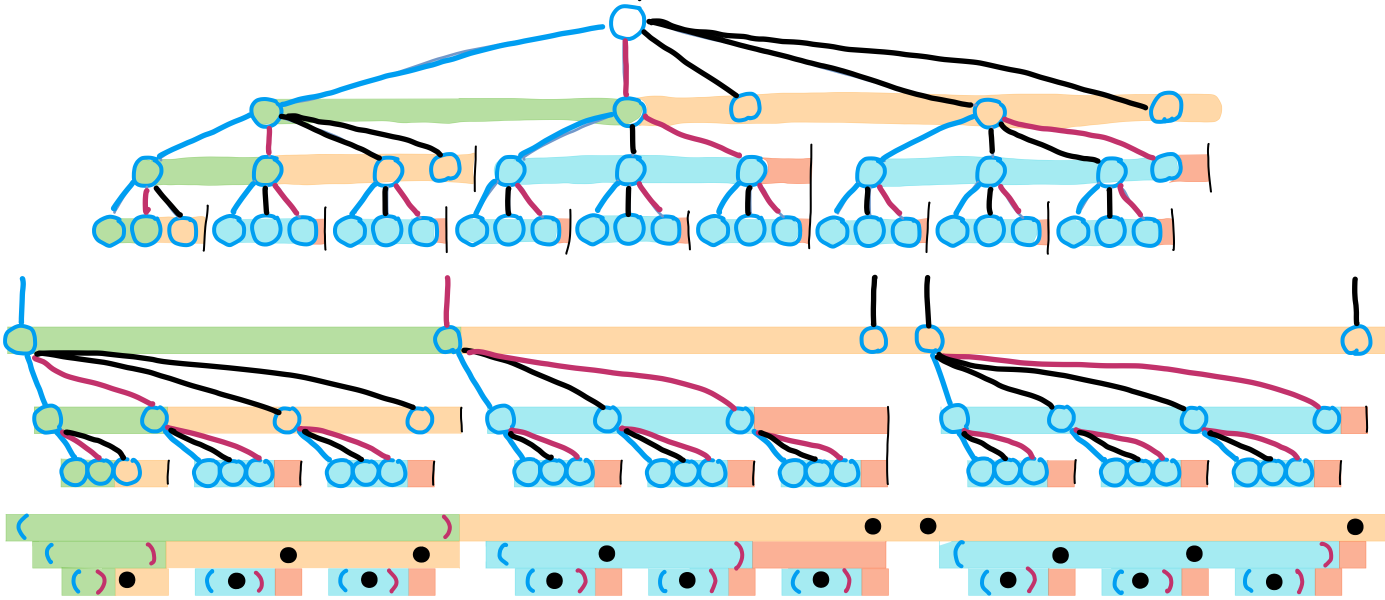


With colors as states, drawing editor could directly write machines such as PDAs.
Could machine learning guess "programs" to solve counting problems?
In this view, every PDA is a tree generator.

Would any PDA that recognizes language work here?

What does it mean that words can be recovered from their state sequences?

Redo to make traversal order and gaps clear.



Second form makes the tree clear, don't need tree edges.

	∅				
(	↓	↓	↓	↓	↓
)		—	↑	—	↑
●			—	—	↑

↓ pushes new state onto stack

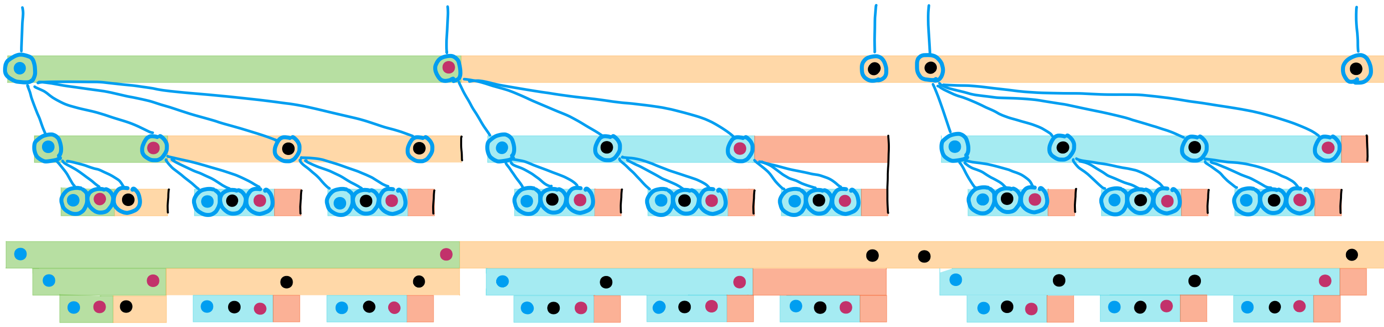
— sets current state

↑ pops state, tries again

\# end of string always pops one state, accept if empty

We could/should program this in this generality.

Under what circumstances does the language represent a SYT?



In a way, this is an attempt to pull a "Hilbert basis theorem" on various counting correspondences. Make finding and verifying the correspondence a routine computation.

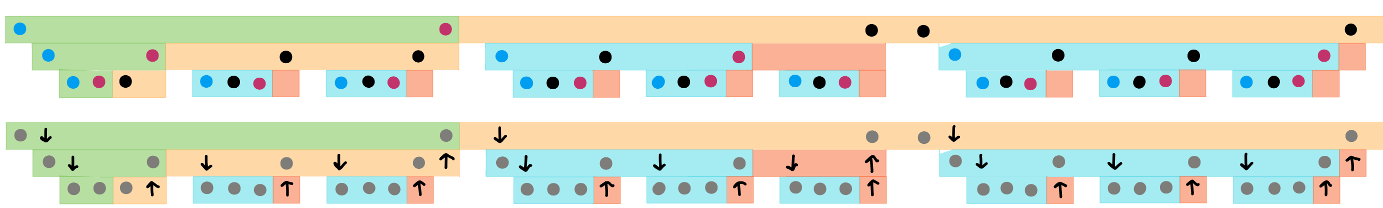
Given a word or a tree there's a word tree connecting them. What's the theory?

	∅				
● (blue)	↓	↓	↓	↓	↓
● (pink)		—	↑	—	↑
● (black)			—	—	↑

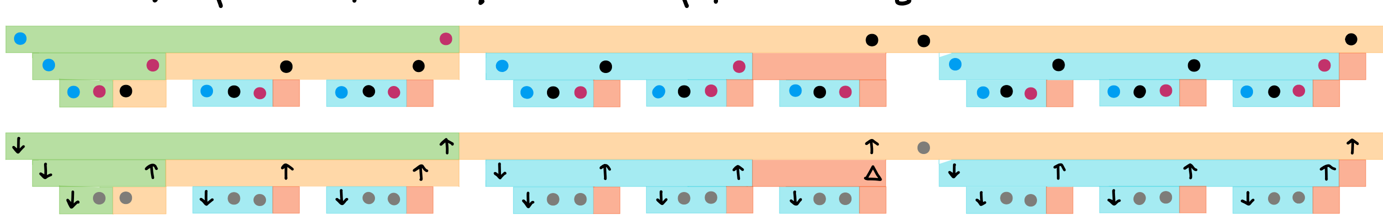
Any PDA table of this form lifts words that it recognizes to uniquely determined trees, nodes labeled by alphabet and levels retaining states of the PDA. (or not

$$w \nearrow wT \searrow T$$

Lift word to word tree, drop labels to project to tree. What machine does the inverse?



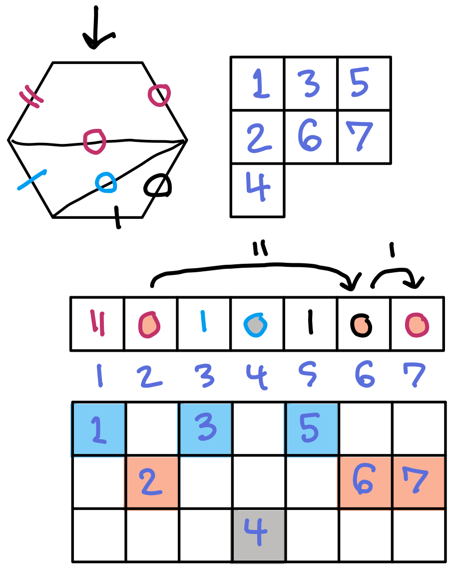
So with one kind of node stamper, need separate symbols for navigation?
Harder to define lift without backtracking.
Easy to have a symbol also go down one level. Multistep ↑↑ is the problem.
Like my Lisp notation, need symbol for empty rise. Augment words. △

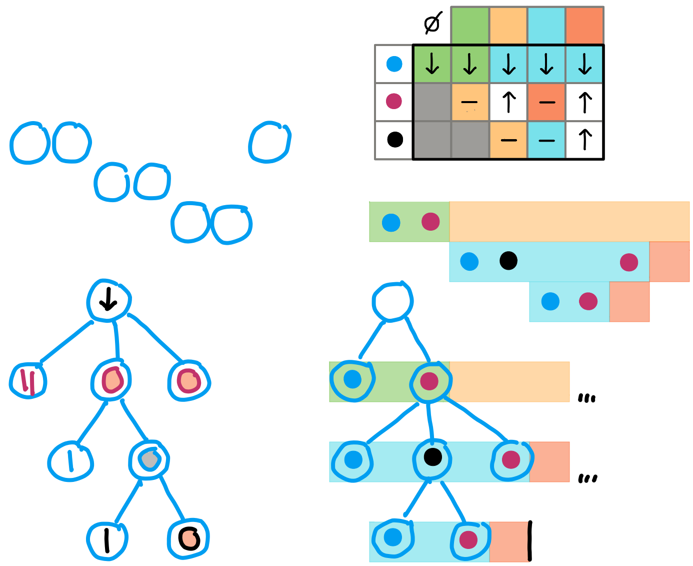


This reveals language issues. Marking a node as last seems to use lookahead. Remove/fix this. Redefine the machine. Ideally we want transducers, not hardwired so one side is a tree. Sequence is traversal but machine has stack access to last...

Sunday March 14

Review Stanley's construction.





So yes, different construction.
Unlikely to be a PDA.

The machine I want doesn't label nodes in each level till next node.
Reasonable question: Is this transducer algebraic?
We need an end of word symbol to clear stack, and fix last symbols. #
Need more states for a lag transducer like this.
Or, output options, symbol if level extends, another for pop?