Exam 1

Combinatorics, Dave Bayer, February 16-20, 2022

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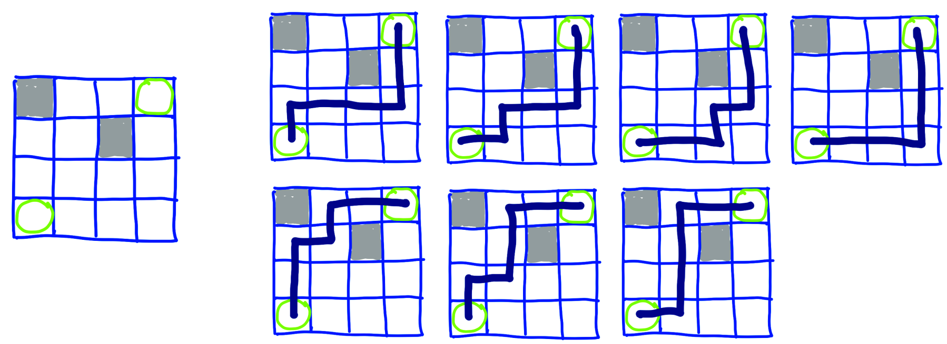
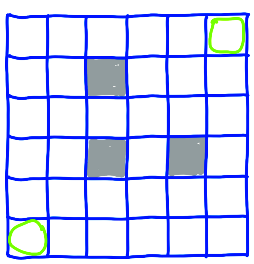
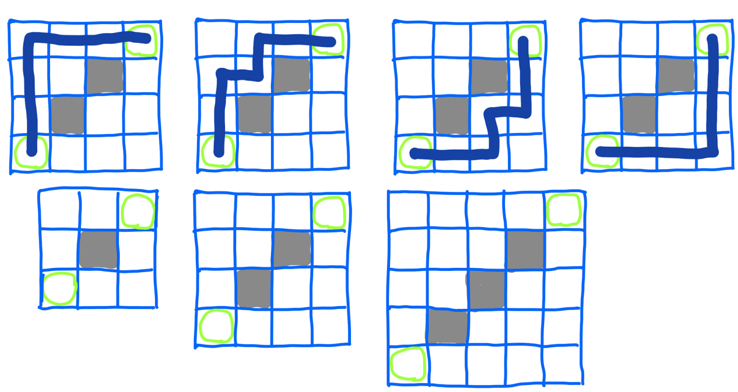
Please match your understanding of my words with the examples, and contact me if you're concerned about any ambiguity.

[1] Shown are two grids with shaded obstacles. We are counting paths that start in the lower left square, end in the upper right square, and step either up or to the right, avoiding the obstacles. For the smaller grid there are 7 paths. How many paths are there, for the larger gird?

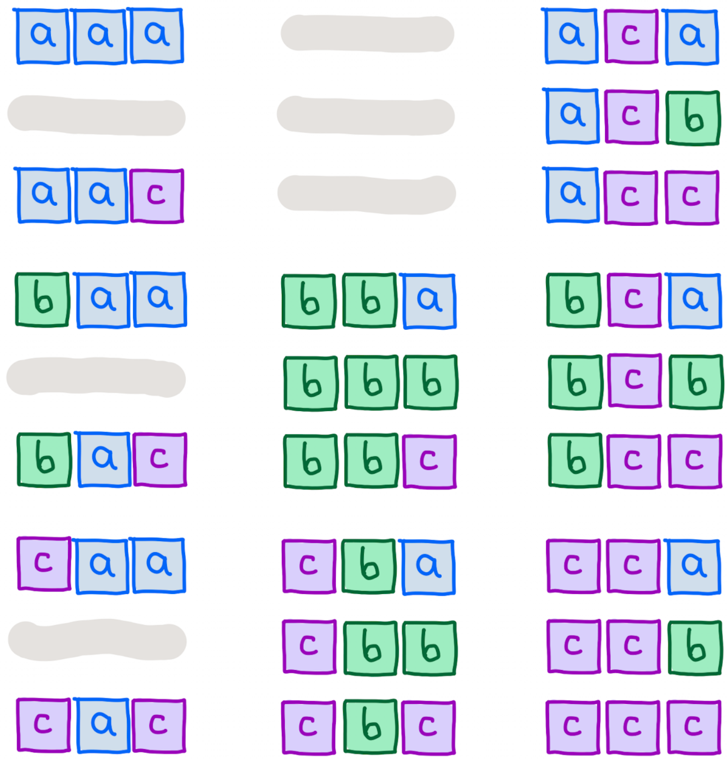




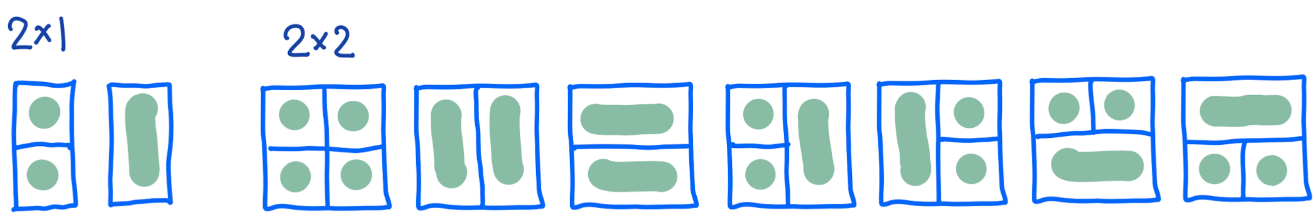
[2] Count paths as before. Let $f(n)$ be the number of paths that avoid the diagonal squares on an $n \times n$ grid, except at the start and the end. As shown below, $f(4) = 4$. Find $f(5)$ and $f(6)$. What can you say about $f(n)$?



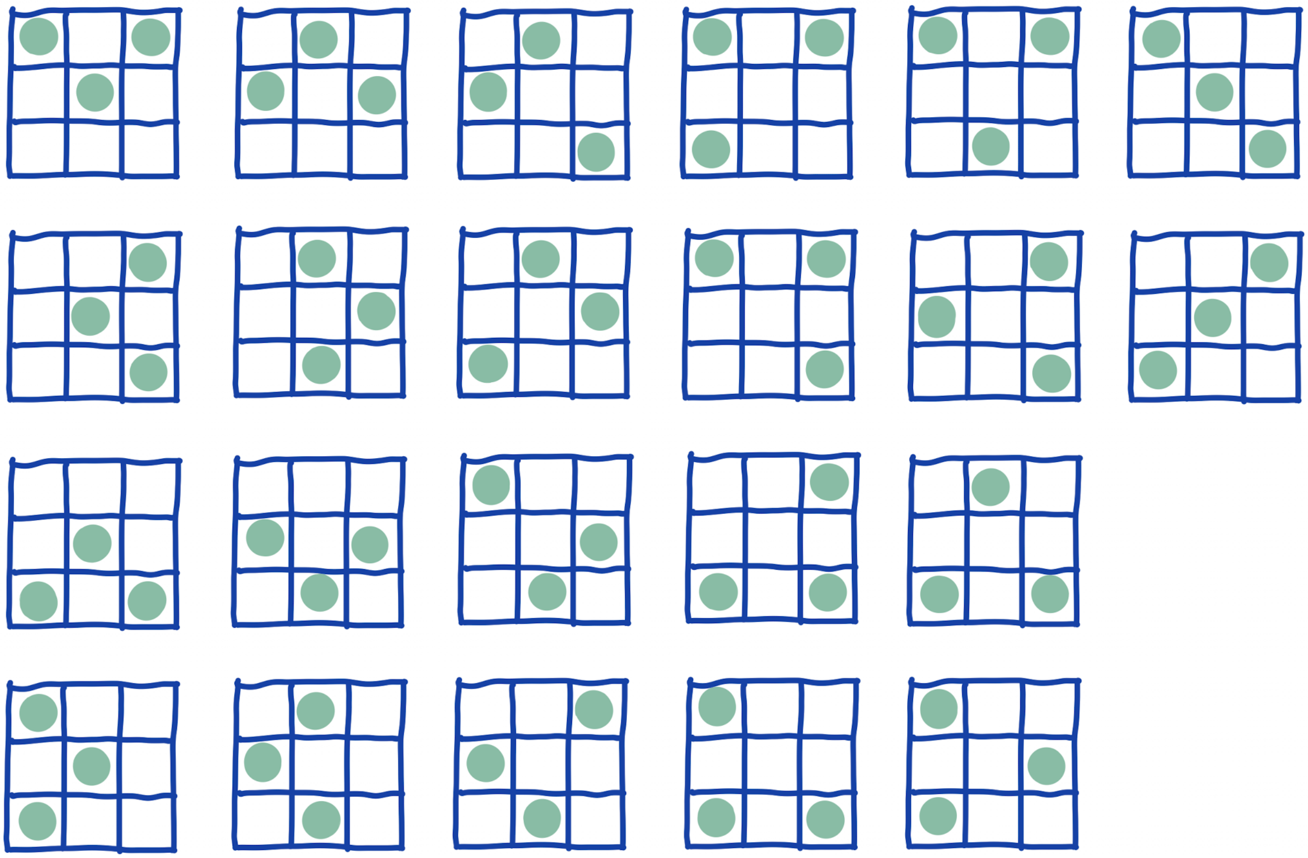
[3] Let $f(n)$ be the number of words of length n in the alphabet $\{a, b, c\}$ with the property that b never immediately follows a. As shown below, $f(3) = 21$. Find $f(4)$ and $f(5)$. What can you say about $f(n)$?



[4] Let $f(n)$ be the number of ways of arranging 1×1 tiles and 1×2 tiles in a $2 \times n$ grid. As shown below, $f(1) = 2$ and $f(2) = 7$. Find $f(3)$ and $f(4)$. What can you say about $f(n)$?



[5] Let $f(n)$ be the number of ways of placing three markers on an $n \times n$ board so no two markers are side by side, either vertically or horizontally. As shown below, $f(3) = 22$. Find $f(4)$. What can you say about $f(n)$?



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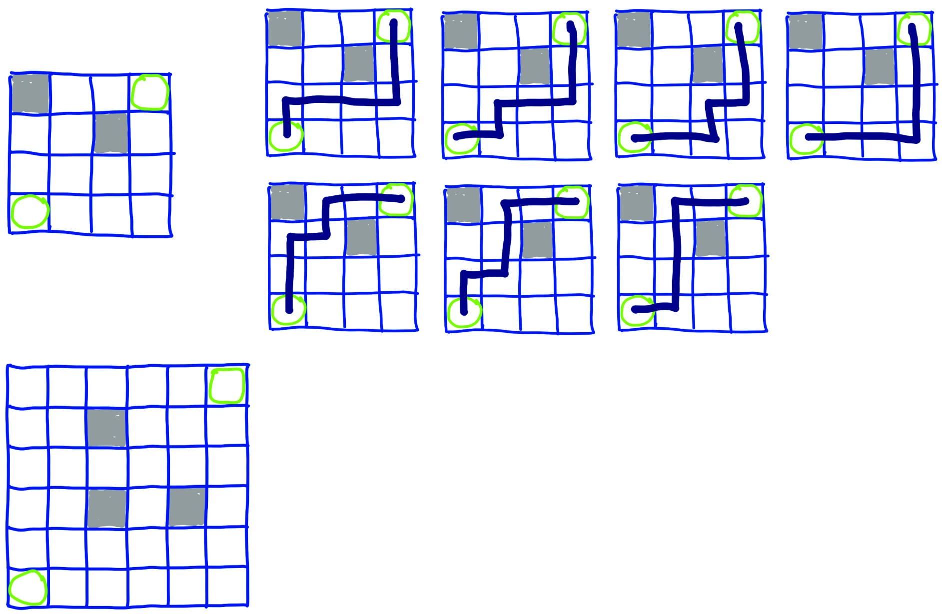

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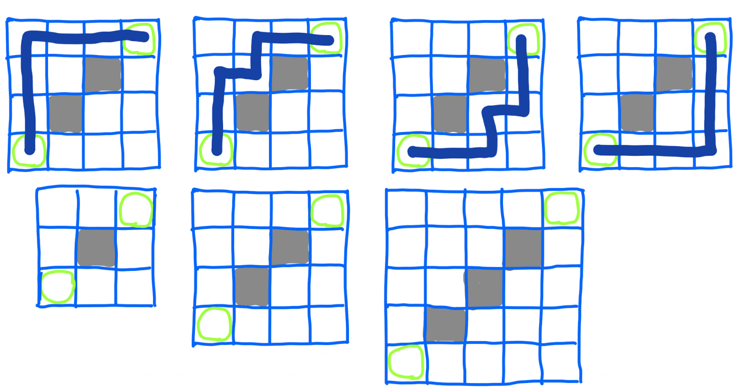
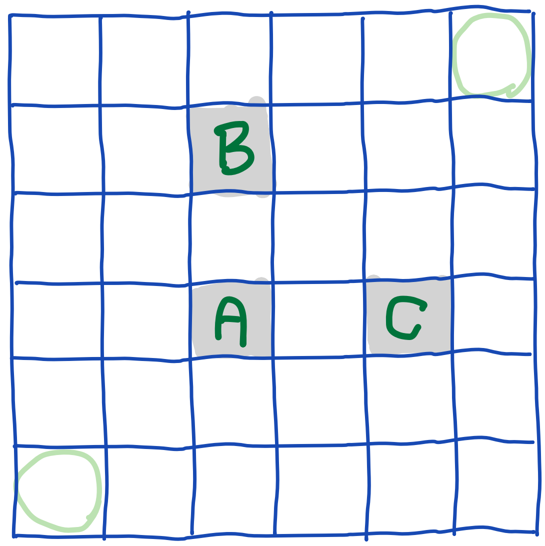


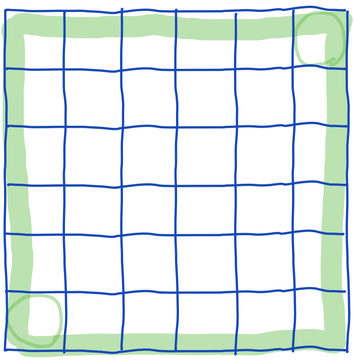
[1]

60 paths

1	6	6	14	30	60
1	5		8	16	30
1	4	4	8	8	14
1	3		4		6
1	2	3	4	5	6
1	1	1	1	1	1

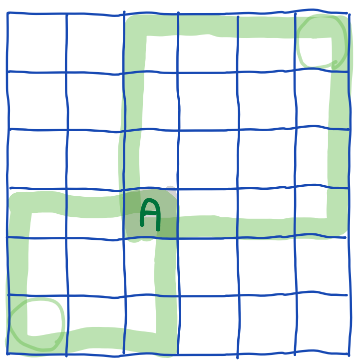


Or by inclusion-exclusion counting:



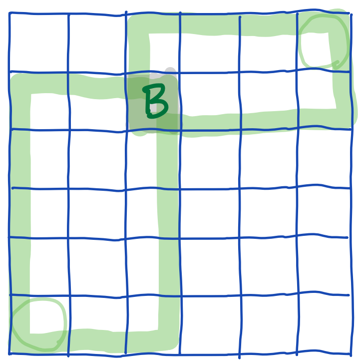
$\binom{10}{5} = \frac{10\cdot9\cdot8\cdot7\cdot6}{5\cdot4\cdot3\cdot2\cdot1}$

$4\cdot9\cdot7 = 252$



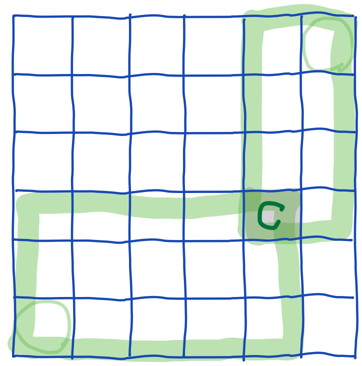
$\binom{4}{2}\binom{6}{3} = 6\cdot20$

-120



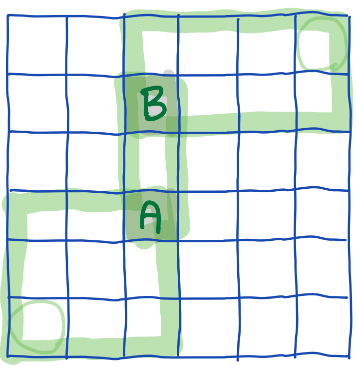
$\binom{6}{2}\binom{4}{3} = 15\cdot4$

-60



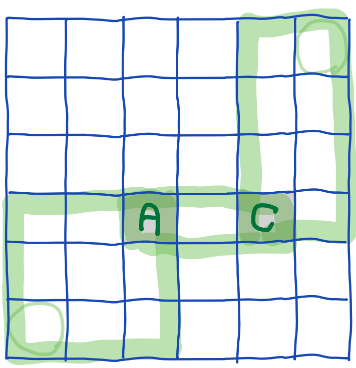
by symmetry

-60



$\binom{4}{2}\binom{2}{0}\binom{4}{3} = 6\cdot1\cdot4$

$+24$



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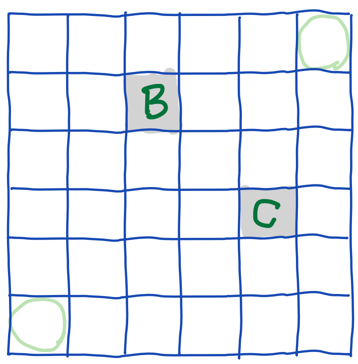
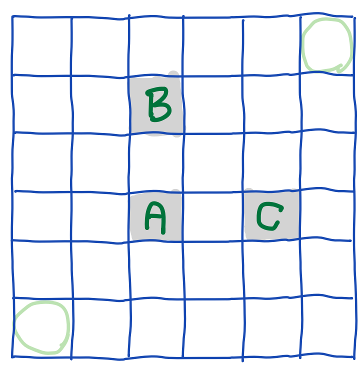


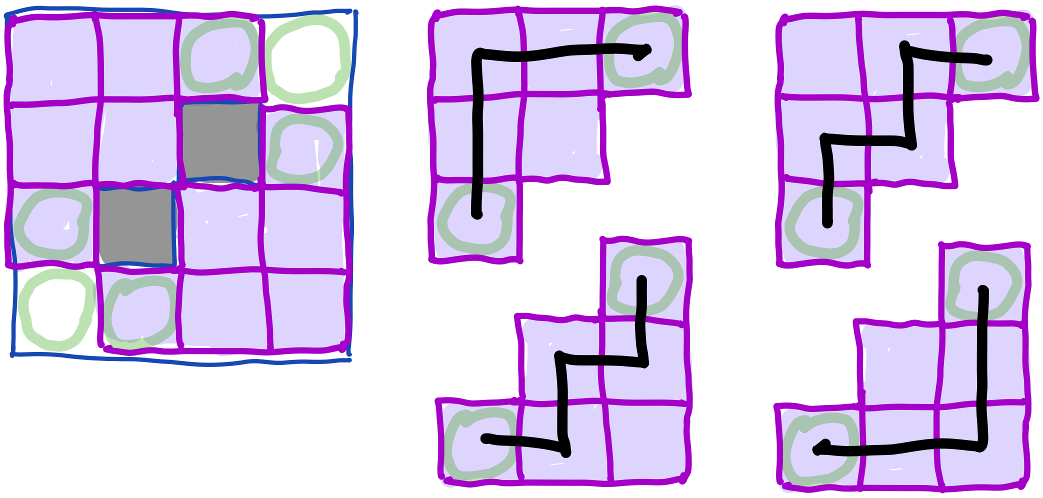
0 not possible



0 not possible

$252 - 120 - 60 - 60 + 24 + 24 = 60$ ✓

[2] $F(4) = 4$



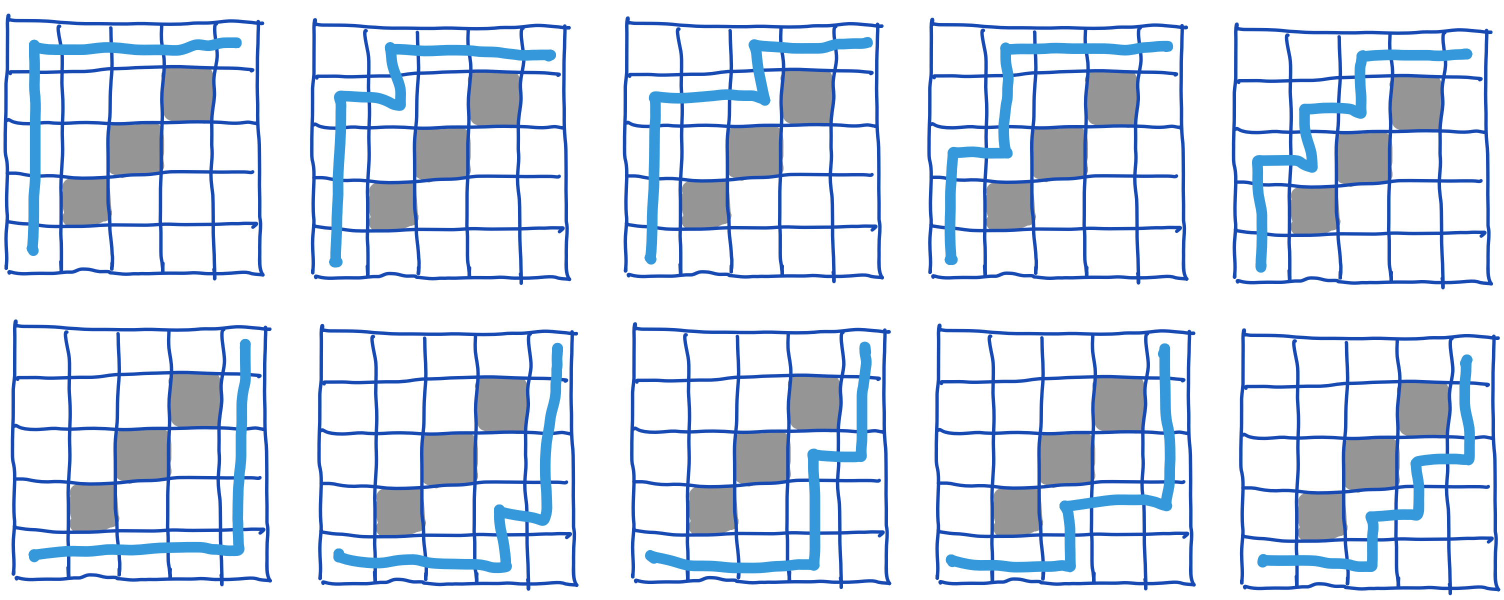
The paths on either side of diagonal are Catalan paths. So twice Catalan numbers.

$F(5) = 10$
$F(6) = 28$

$(1,1,2)\cdot(2,1,1) = 5$
$(1,1,2,5)\cdot(5,2,1,1) = 14$

OEIS A284016

Check $F(5)=10$:



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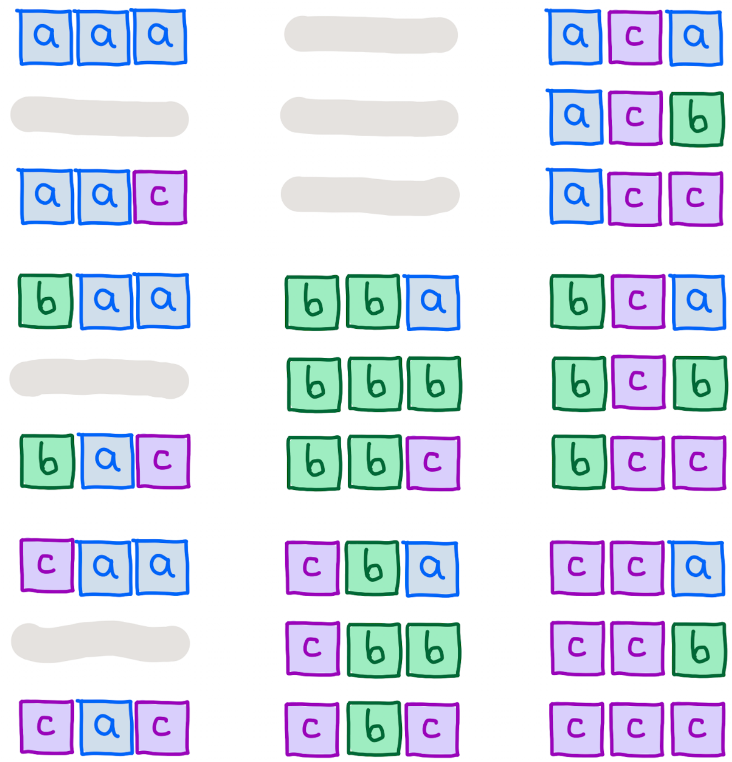
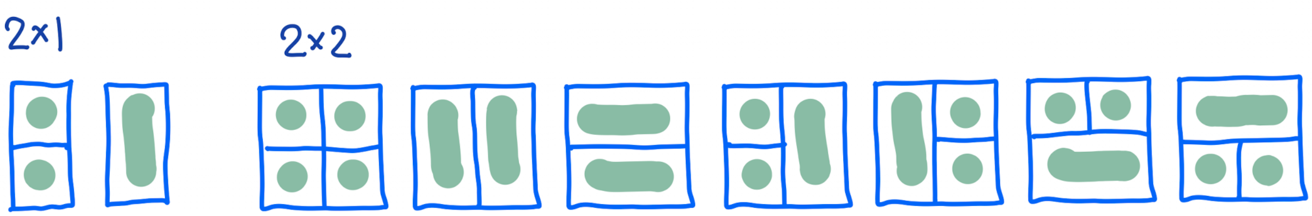
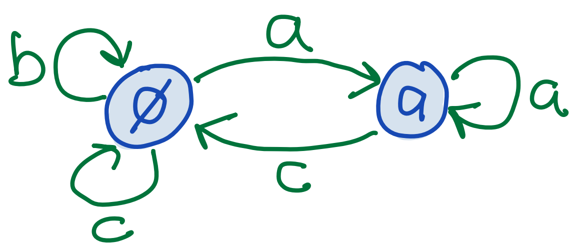


[4] Let $f(n)$ be the number of ways of arranging 1×1 tiles and 1×2 tiles in a $2 \times n$ grid. As shown below, $f(1) = 2$ and $f(2) = 7$. Find $f(3)$ and $f(4)$. What can you say about $f(n)$?



[3] Model as finite state machine



We want to count the total # paths of length n starting at ∅ and ending at ∅ or a.

to \ from	∅	a
∅	2	1
a	1	1

$n=1$

$$f(n) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$[1 \;\; 1]$: ending at ∅ or a; $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n$: n steps; $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$: starting at ∅

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \qquad f(2) = 3+5 = 8$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} \qquad f(3) = 8+13 = 21$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} \qquad f(4) = 21+34 = 55$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} = \begin{bmatrix} 89 & 55 \\ 55 & 34 \end{bmatrix} \qquad f(5) = 55+89 = 144$$

OEIS A001906

Fibonacci 1 1 2 3 5 8 13 21 34 55 89

$\underbrace{1\ 2}_{F(1)}\ \underbrace{3\ 5}_{F(2)}\ \underbrace{8\ 13}_{F(3)}\ \underbrace{21\ 34}_{F(4)}\ \underbrace{55\ 89}_{F(5)}$

(Nice puzzle: Find a bijection between these words and an example of Fibonacci numbers.)

Find recurrence via generating function:

$$F(t) = \sum_{n=0}^{\infty} F(n)t^n = \sum_{n=0}^{\infty} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n t^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\sum_{n=0}^{\infty} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n t^n = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} t \right)^{-1}$$

$$= \begin{bmatrix} 1-2t & -t \\ -t & 1-t \end{bmatrix}^{-1} = \begin{bmatrix} 1-t & t \\ t & 1-2t \end{bmatrix} \Big/ (1-3t+t^2)$$

$$(1-2t)(1-t)-t^2 = 1-3t+t^2$$

So $\boxed{F(t) = 1/(1-3t+t^2)}$

We read this as
$$F(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 3F(n-1)-F(n-2), & n>0 \end{cases}$$

Why? $(1-3t+t^2)\,F(t) = 1$

*	1	3t	8t²	21t³	55t⁴
1	1	3t	8t²	21t³	55t⁴ ...
−3t	−3t	−9t²	−24t³	−61t⁴ ...	
t²	t²	3t³	8t⁴ ...		

⇒

*	1	3	8	21	55
1	1	3	8	21	55 ...
−3	−3	−9	−24	−61 ...	
1	1	3	8 ...		

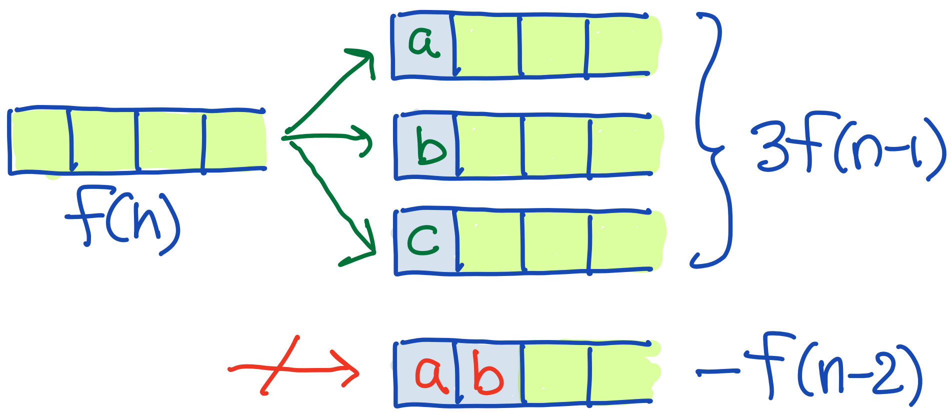
$f(0) = 1, \quad f(n) - 3f(n-1) + f(n-2) = 0$

$1 \quad -3t \quad +t^2$

We can understand $f(n) = 3f(n-1) - f(n-2)$ as

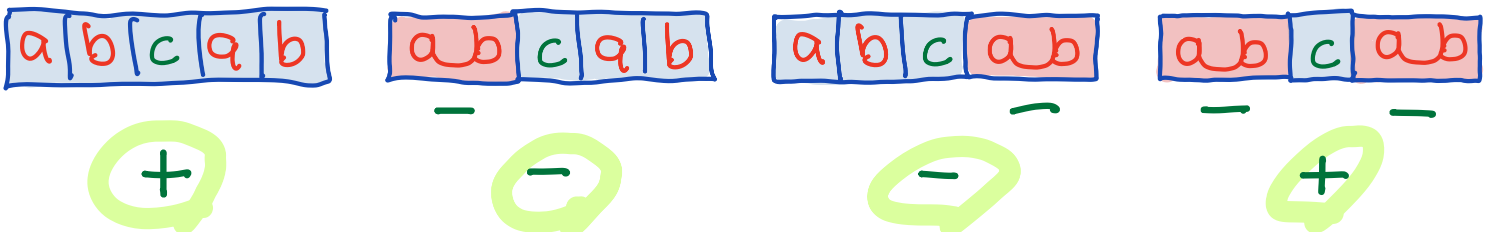
allow everything

but subtract off forbidden ab



Another approach: Consider all words in a, b, c and the length 2 ligature ab

There are 2^k ways to write a word with k copies of ab:



We can get these all to cancel out by making each ab minus, leaving only allowed words.

A generating function that writes out all allowed words, and cancels out all forbidden words, is the geometric series

$$1 + (a+b+c-ab) + (a+b+c-ab)^2 + \ldots$$

1 a b c ~~−ab~~ aa ~~ab~~ ac −aab ba bb ...

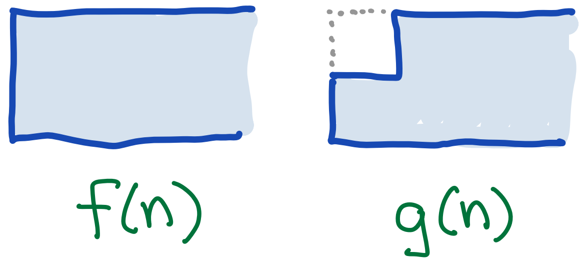
$$= \sum_{m=0}^{\infty} (a+b+c-ab)^m = 1/(1-a+b+c+ab)$$

Setting $a=b=c=t$ and $ab = t^2$ we get

$$F(t) = 1/(1-3t+t^2)$$

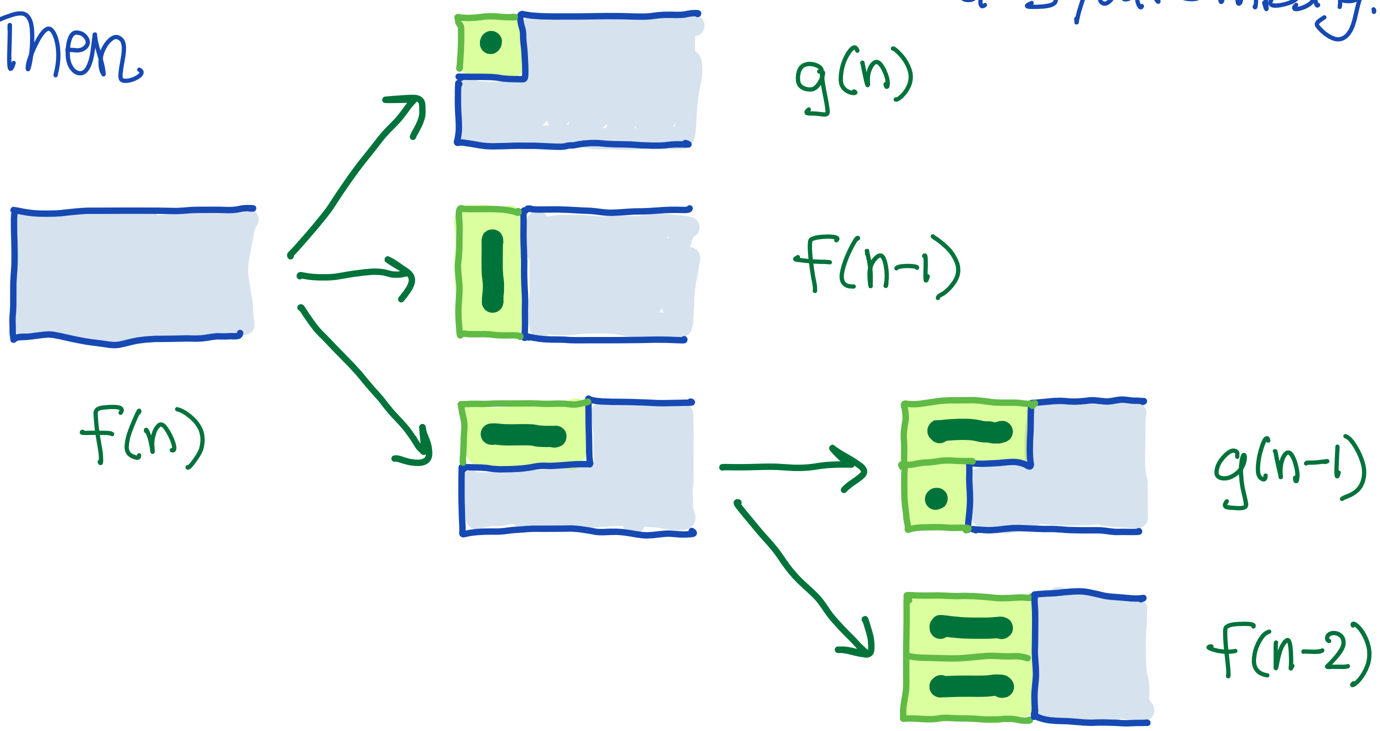
as before.

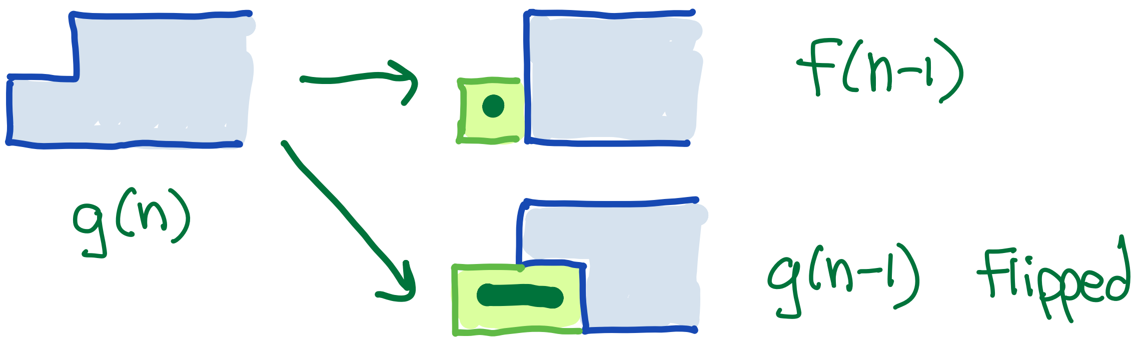
[4]



Let f(n) count tilings of the 2×n grid, and let g(n) count tilings of the helper grid with a square missing.

Then

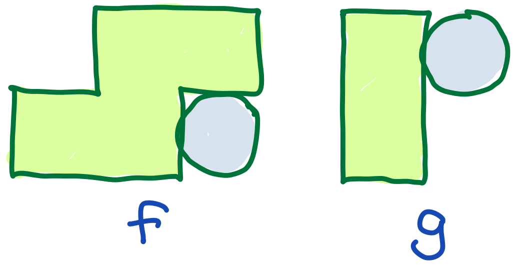


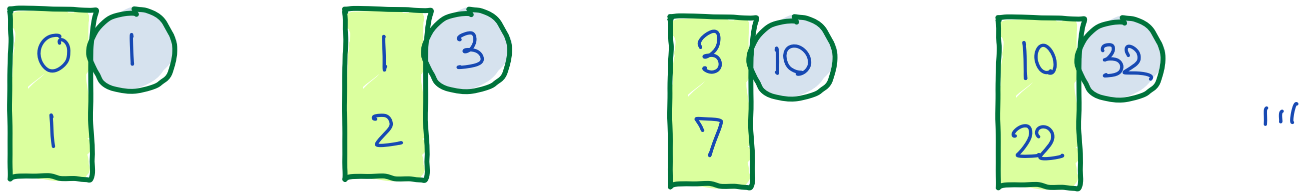


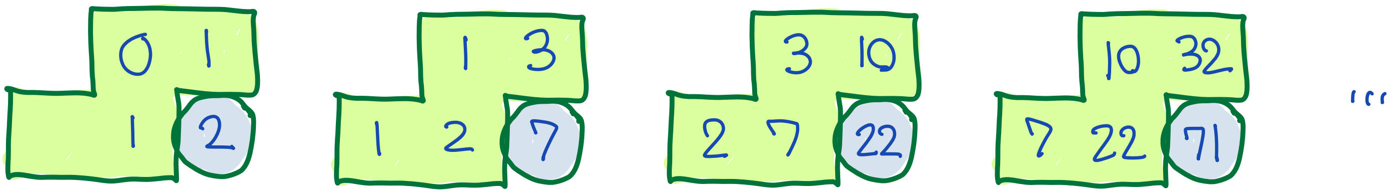
So

$$f(n) = f(n-1) + f(n-2) + g(n) + g(n-1)$$
$$g(n) = f(n-1) + g(n-1)$$

n	0	1	2	3	4
$g(n)$	0	1	3	10	32
$f(n)$	1	2	7	22	71







Now rewrite as generating functions:

$$F(t) = 1 + tF(t) + t^2F(t) + G(t) + tG(t)$$
$$G(t) = tF(t) + tG(t)$$

$$\begin{bmatrix} 1-t-t^2 & -1-t \\ -t & 1-t \end{bmatrix} \begin{bmatrix} F(t) \\ G(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F(t) \\ G(t) \end{bmatrix} = \begin{bmatrix} 1-t & 1+t \\ t & 1-t-t^2 \end{bmatrix} \Big/ (1-3t-t^2+t^3) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(1-t-t^2)(1-t) - (-t)(-1-t) = 1-3t-t^2+t^3$$

$1-6-4+8 = -1$ ✓

1	t	t^2	t^3
+	−	−	
	−	+	+
	−	−	

$$\left| \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix} \right| = -1$$

check $t=2$

So $$F(t) = \frac{1-t}{1-3t-t^2+t^3}$$

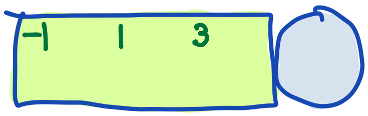
or $$(1-3t-t^2+t^3)\,F(t) = 1-t$$

	1	2	7	22	71
1	1	2	7	22	71
−3	−3	−6	−21	−66	
−1	−1	−2	−7		
1	1	2			

$F(0) = 1$

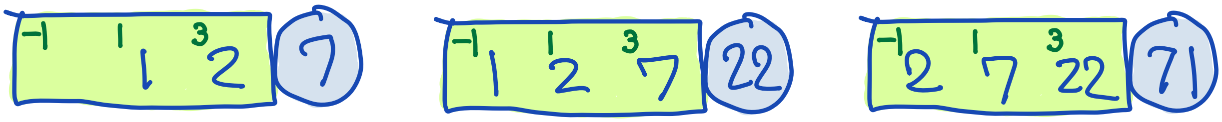
$F(1) = 2$

$F(n) = 3F(n-1) + F(n-2) - F(n-3)$



F

n	0	1	2	3	4
F(n)	1	2	7	22	71

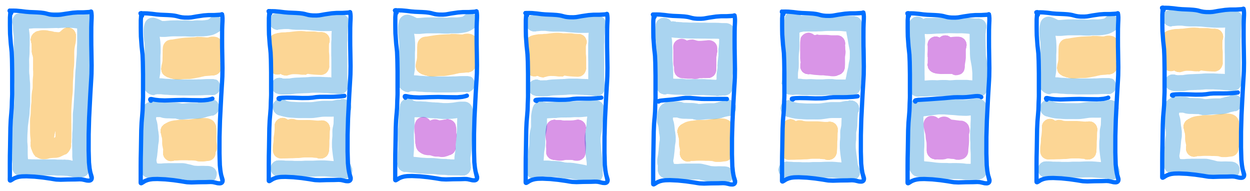


F(2) F(3) F(4)

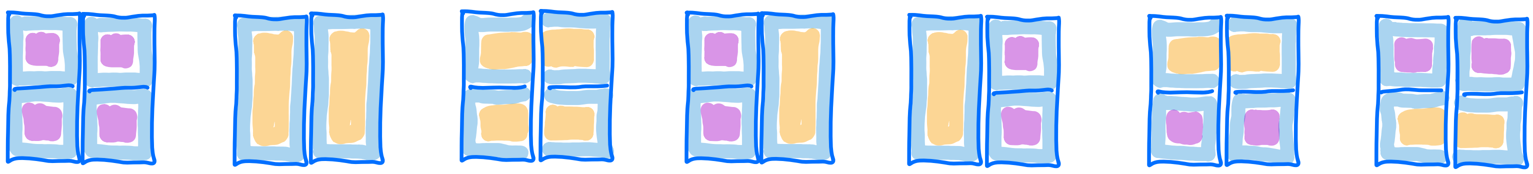
OEIS A030186

Alternate approach:

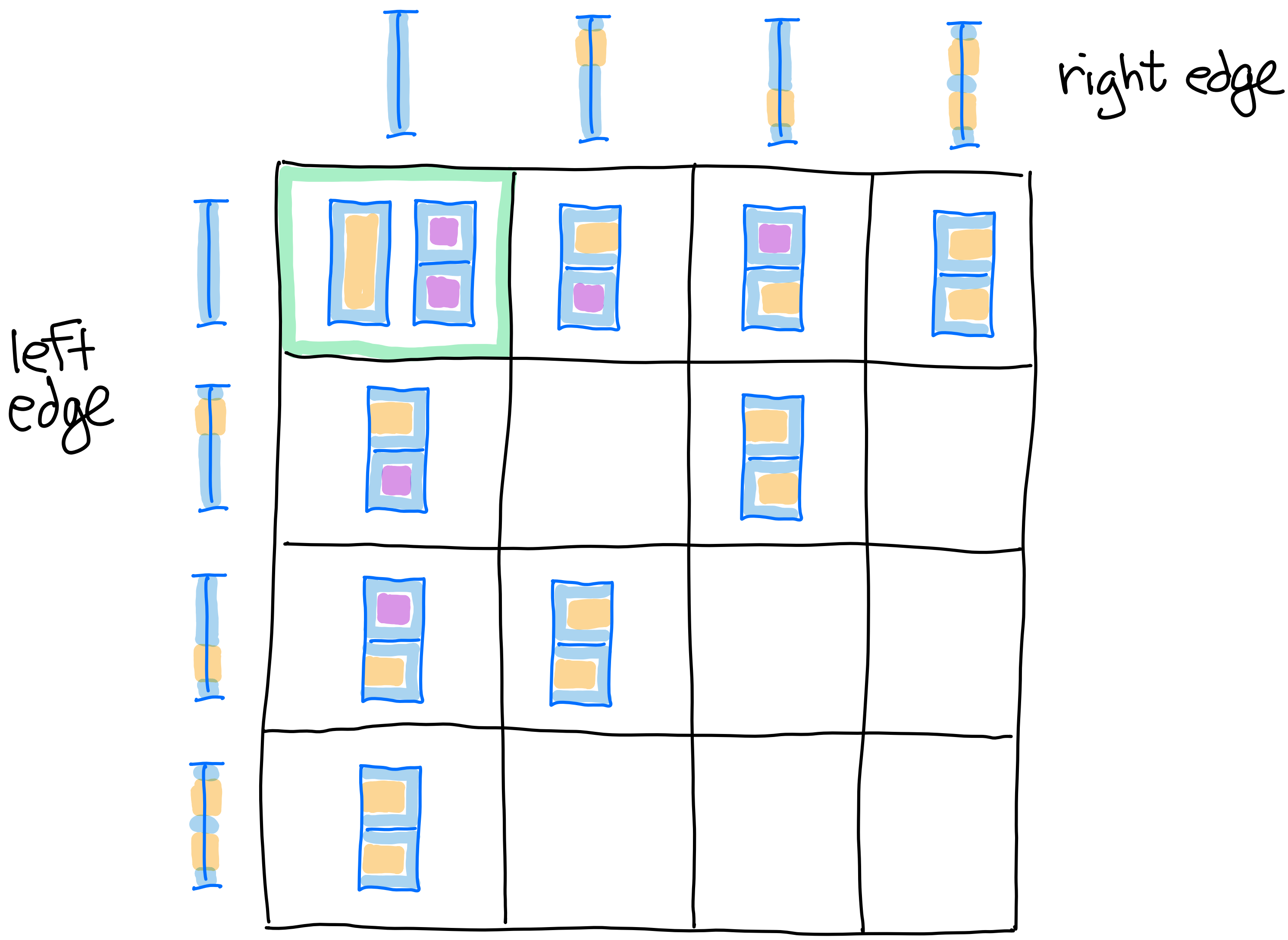
These are length n "words" using the alphabet



The letters have to match up to form valid words:



We get a transition matrix



We can get the diagrams themselves by the $(1,1)$ entry in powers of this matrix of drawings.

For just the counts $F(n)$,

$$\underset{A}{\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{A^2}{\begin{bmatrix} 7 & 3 & 3 & 2 \\ 3 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}} \quad \underset{A^3}{\begin{bmatrix} 22 & 10 & 10 & 7 \\ 10 & & & \\ 10 & & \text{skip} & \\ 7 & & & \end{bmatrix}} \quad \underset{A^4}{\begin{bmatrix} 71 & & & \\ & & \text{skip} & \\ & & & \\ & & & \end{bmatrix}}$$

For the generating function $F(t) = \sum_{n=0}^{\infty} F(n) t^n$,

$$\sum_{n=0}^{\infty} A^n t^n = (I - At)^{-1}$$

$$\begin{bmatrix} 1-2t & -t & -t & -t \\ -t & 1 & -t & 0 \\ -t & -t & 1 & 0 \\ -t & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} F(t) & & & \\ & & \text{skip} & \\ & & & \\ & & & \end{bmatrix}$$

use formula for inverse

(or use a computer algebra system!)

$$\begin{vmatrix} 1-2t & -t & -t & -t \\ -t & 1 & -t & 0 \\ -t & -t & 1 & 0 \\ -t & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1-2t-t^2 & -t & -t & 0 \\ -t & 1 & -t & 0 \\ -t & -t & 1 & 0 \\ -t & 0 & 0 & 1 \end{vmatrix}$$

Adding $t*(\text{row } 4)$ to $(\text{row } 1)$ doesn't change determinant

This reduces us to 3x3, keep playing this game:

$$\begin{vmatrix} 1-2t-t^2 & -t & -t \\ -t & 1 & -t \\ -t & -t & 1 \end{vmatrix} = \begin{vmatrix} 1-2t-2t^2 & -t-t^2 & 0 \\ -t-t^2 & 1-t^2 & 0 \\ -t & -t & 1 \end{vmatrix}$$

This reduces us to 2x2

$$\begin{vmatrix} 1-2t-2t^2 & -t-t^2 \\ -t-t^2 & 1-t^2 \end{vmatrix} = (1-2t-2t^2)(1-t^2) - (-t-t^2)(-t-t^2)$$

$$= 1-2t-4t^2+t^4$$

1	t	t^2	t^3	t^4
1	-2	-2		
		-1	2	2
		-1	-2	-1
1	-2	-4	0	1

Now the numerator:
$$\begin{vmatrix} 1 & -t & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1-t^2$$

So

$$F(t) = \frac{1-t^2}{1-2t-4t^2+t^4} = \frac{\cancel{(1+t)}(1-t)}{\cancel{(1+t)}(1-3t-t^2+t^3)}$$

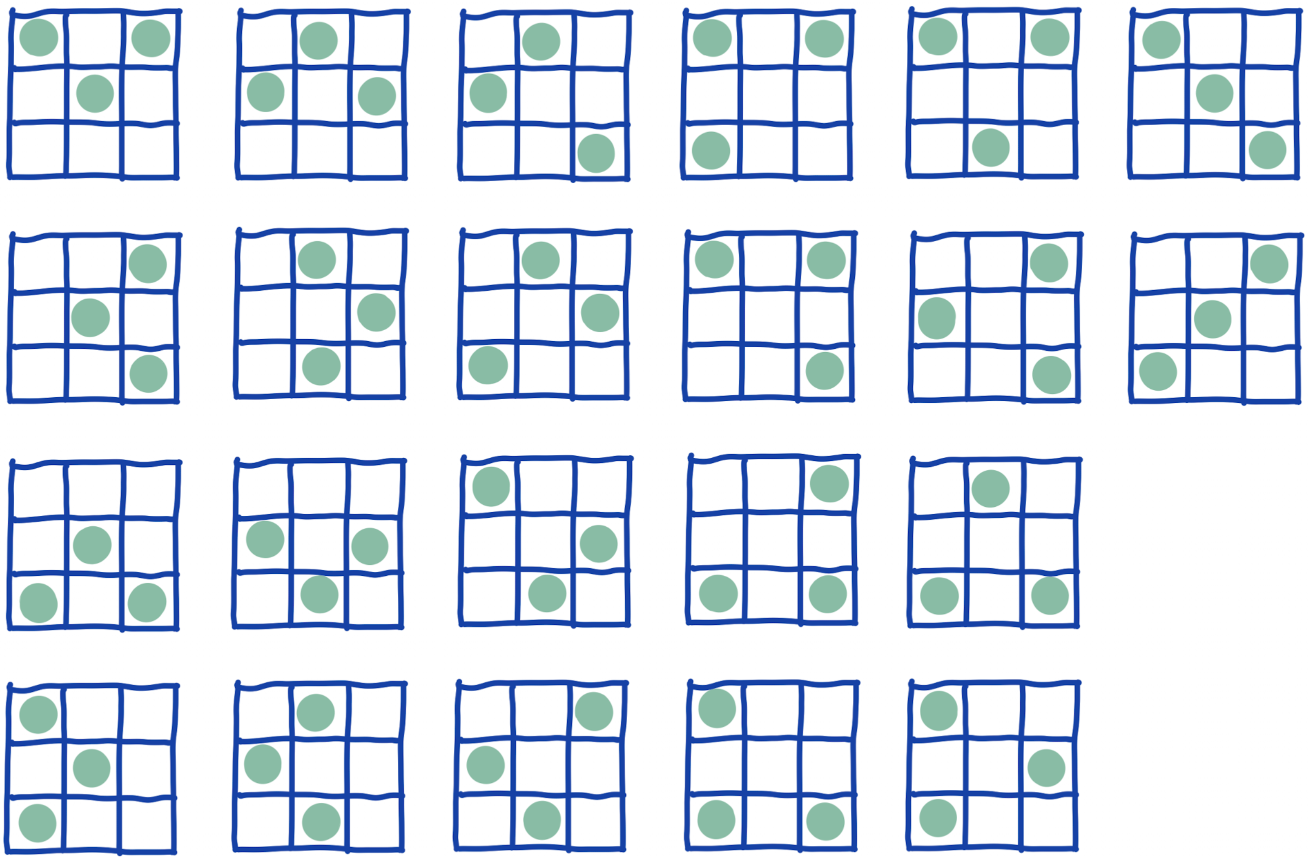
	1	-3	-1	1
1	1	-3	-1	1
1	1	-3	-1	1

1, -2, -4, 0, 1

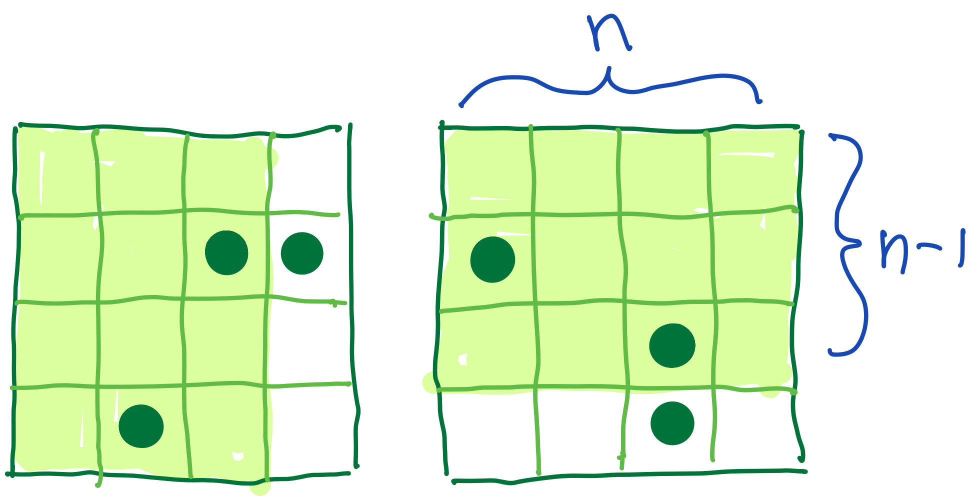
$$= \boxed{\frac{1-t}{1-3t-t^2+t^3}}$$

as before

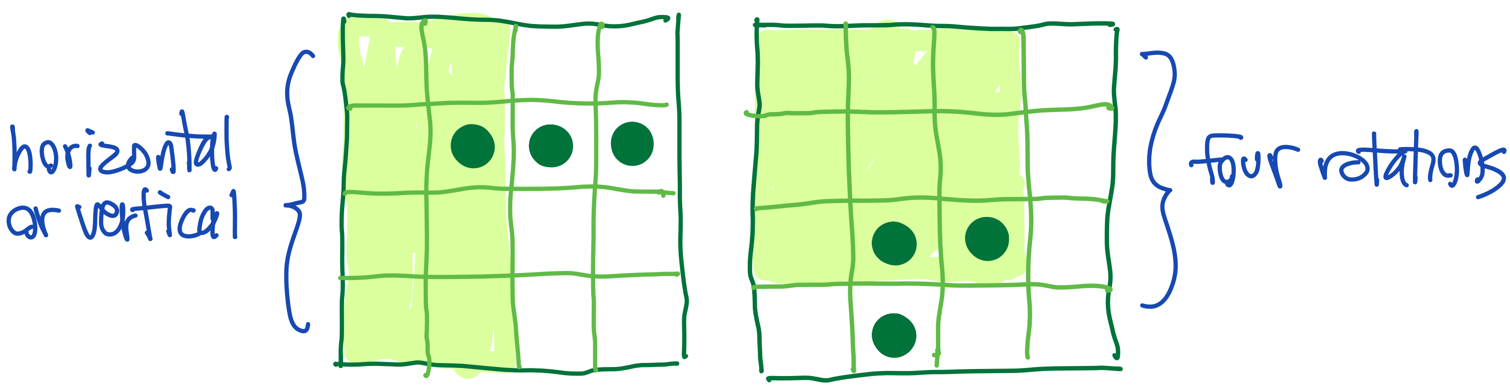
[5] Let $f(n)$ be the number of ways of placing three markers on an $n \times n$ board so no two markers are side by side, either vertically or horizontally. As shown below, $f(3) = 22$. Find $f(4)$. What can you say about $f(n)$?



[5] There are $\binom{n^2}{3}$ ways to place 3 markers on an $n \times n$ board.



There are $2n(n-1)(n^2-2)$ ways to place 3 markers so (at least) one pair is adjacent.



$2n(n-2)$ $\qquad$ $4(n-1)^2$

There are two configurations that get subtracted twice, and need to be added back in.

$$f(n) = \binom{n^2}{3} - 2n(n-1)(n^2-2) + 2n(n-2) + 4(n-1)^2$$

$n=2$

$$\binom{4}{3} - 2\cdot 2\cdot 1\cdot 2 + 2\cdot 2\cdot 0 + 4\cdot 1^2$$

$$4 \quad -8 \qquad\qquad +4 \quad = 0 \ ✓$$

$n=3$

$$\binom{9}{3} - 2\cdot 3\cdot 2\cdot 7 + 2\cdot 3 + 4\cdot 2^2$$

$$\frac{\overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot 7}{\cancel{3}\cdot \cancel{2}\cdot 1} \qquad 84 - 84 + 6 + 16 = 22 \ ✓$$

$n=4$

$$\binom{16}{3} - 2\cdot 4\cdot 3\cdot 14 + 2\cdot 4\cdot 2 + 4\cdot 3^2$$

$$\frac{\overset{8}{\cancel{16}}\cdot \overset{5}{\cancel{15}}\cdot 14}{\cancel{3}\cdot \cancel{2}\cdot 1} \qquad (8\cdot 5 - 2\cdot 4\cdot 3)14 + 16 + 36$$

$$40 - 24$$

$$16\cdot 14 + 16 + 36$$

$$16\cdot 15 + 36$$

$$276$$

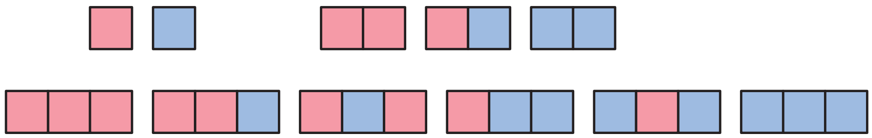
OEIS A172226

Exam 2

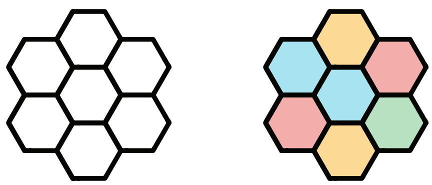
Combinatorics, Dave Bayer, April 6-10, 2022

Please show all of your work. You will be graded for both your answers and your explanations. You need not complete the entire exam; the questions vary in difficulty.

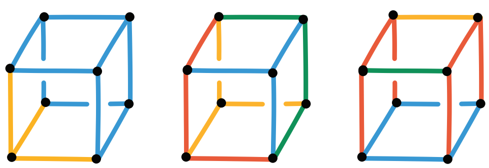
[1] How many ways can we color the cells of a strip of n squares using at most k colors, counting two patterns as the same if one is a reversal of the other?



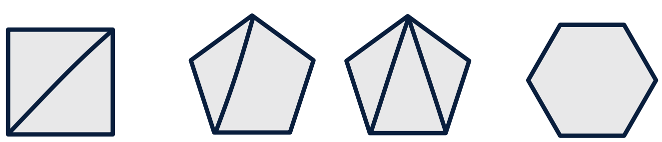
[2] How many ways can we color the cells of this beehive using at most k colors, up to the dihedral group of rotations and flips? Confirm your answer for $k = 2$, by finding all patterns up to symmetry.



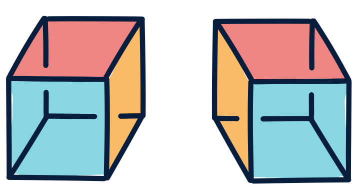
[3] How many ways can we color the edges of a cube using at most k colors, up to the group of rotational symmetries? Can you check your answer for $k = 2$?



[4] Let $f(n)$ be the number of ways of dissecting an n-gon by at least one cut, up to the dihedral group of rotations and flips. As shown, $f(4) = 1$ and $f(5) = 2$. Find $f(6)$ two ways, by drawing the cases by hand and by using Burnside's lemma.



[5] How many ways can we color the faces of a cube using at most k colors, up to the group of symmetries generated by rotations and reflections ("look in the mirror")?



Exam 2

Combinatorics, Dave Bayer, April 6-10, 2022

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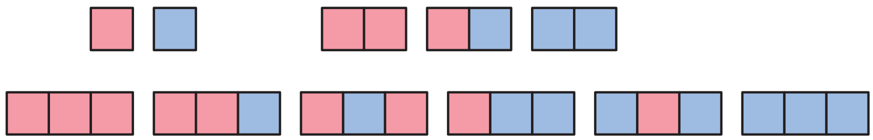
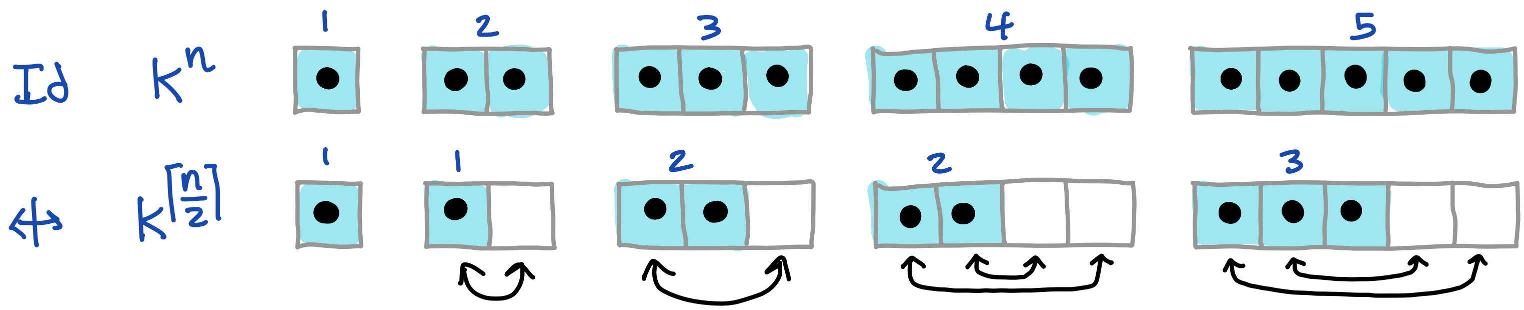


$G = \{Id, \leftrightarrow\}$ $|G| = 2$

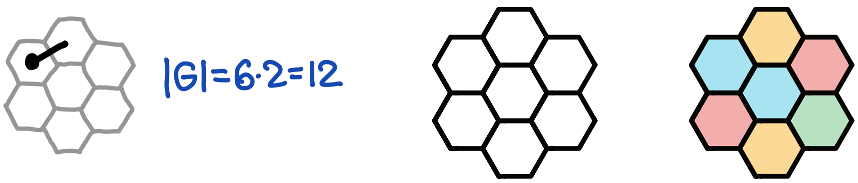


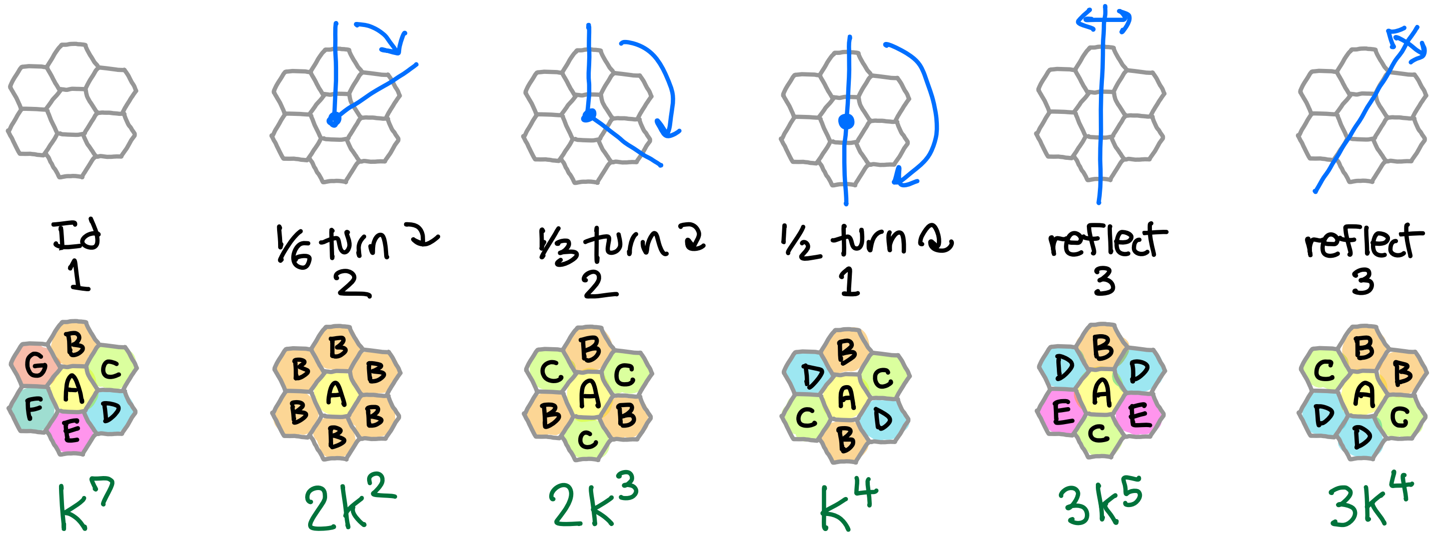
$$\left(k^n + k^{\lceil \frac{n}{2} \rceil}\right)/2$$

k	n	k^n	$k^{\lceil \frac{n}{2} \rceil}$	$\left(k^n + k^{\lceil \frac{n}{2} \rceil}\right)/2$
2	1	2	2	2
2	2	4	2	3
2	3	8	4	6

☑

[2] How many ways can we color the cells of this beehive using at most k colors, up to the dihedral group of rotations and flips? Confirm your answer for $k = 2$, by finding all patterns up to symmetry.



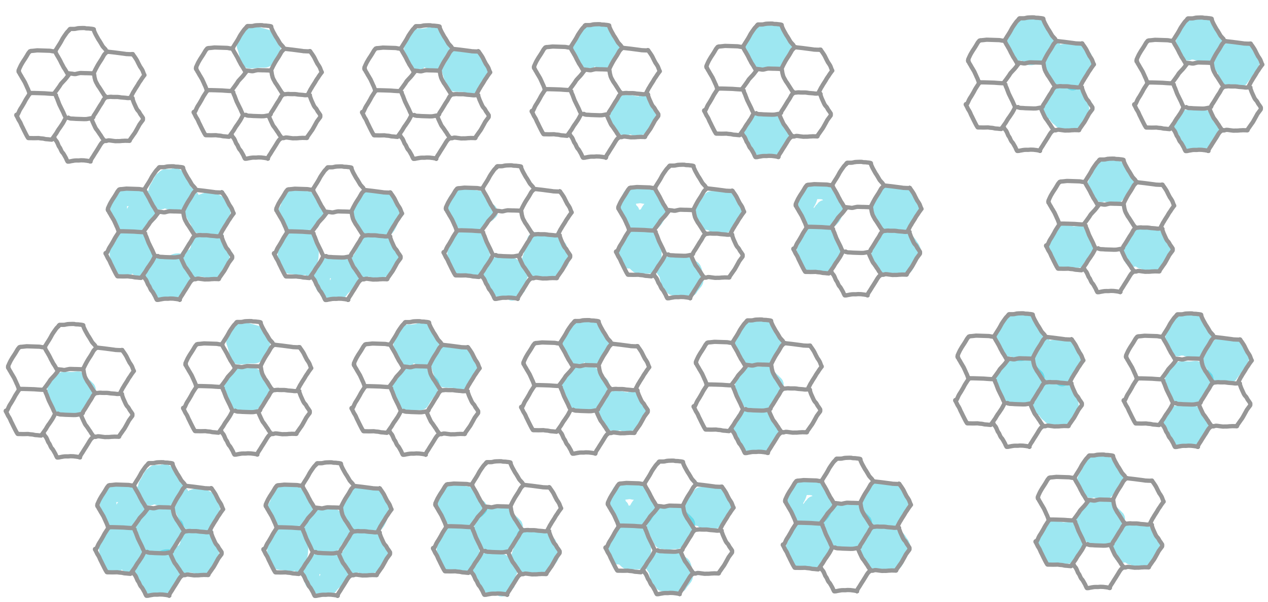


$$\left(2k^2 + 2k^3 + 4k^4 + 3k^5 + k^7\right)/12$$

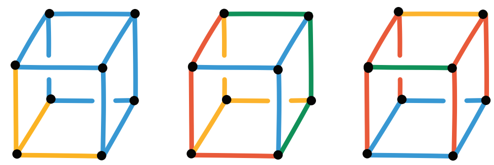
	2	3	4	5	7		12	
	2	2	4	3	1	total	count	/k
1	2	2	4	3	1	12	1	1
2	8	16	64	96	128	312	26	13
3	18	54	324	729	2,187	3,312	276	92
4	32	128	1,024	3,072	16,384	20,640	1,720	430
5	50	250	2,500	9,375	78,125	90,300	7,525	1,505

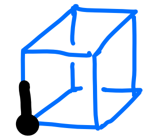
check $k=2$ 26 ✓

https://oeis.org/A027670



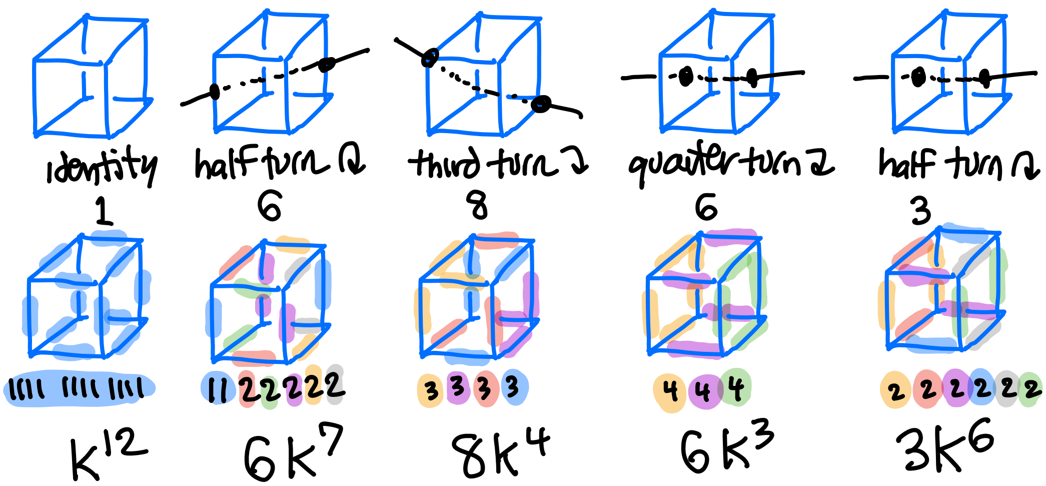
[3] How many ways can we color the edges of a cube using at most k colors, up to the group of rotational symmetries? Can you check your answer for k = 2?





$|G| = 8 \cdot 3 = 24$

$1+6+8+6+3 = 24$ ✓

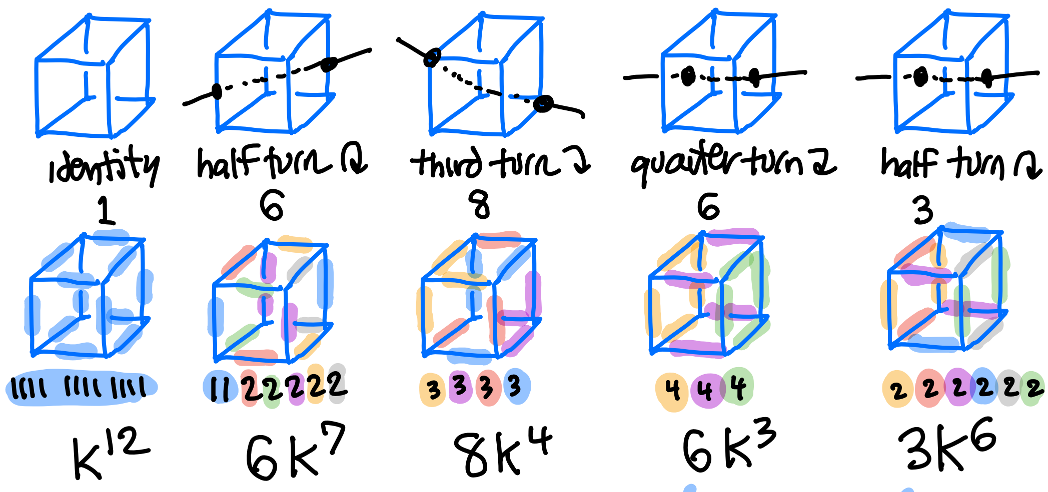


$$(6k^3 + 8k^4 + 3k^6 + 6k^7 + k^{12})/24$$

	3	4	6	7	12		24
	6	8	3	6	1	total	count
1	6	8	3	6	1	24	1
2	48	128	192	768	4,096	5,232	218
3	162	648	2,187	13,122	531,441	547,560	22,815

https://oeis.org/A060530

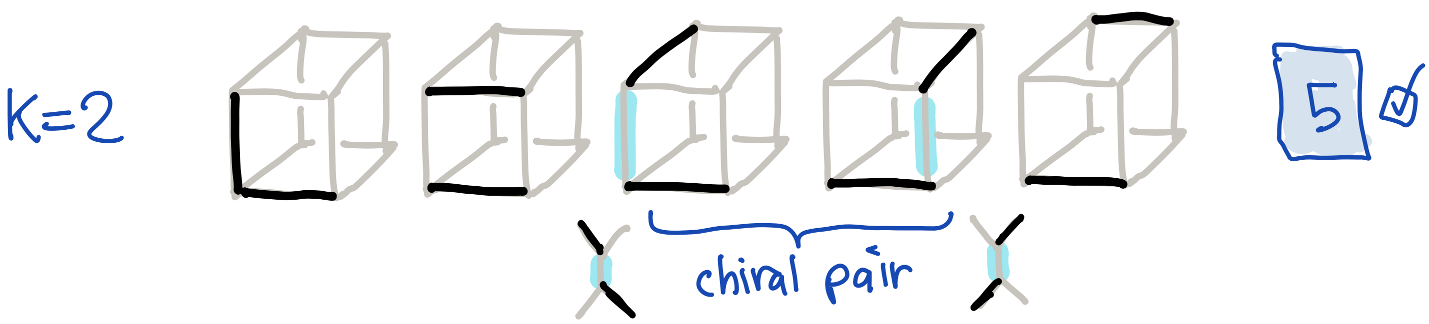
One way to check k=2 is to count subsets of each size, up to symmetry. We can confirm the smaller counts by hand, and see they add up.



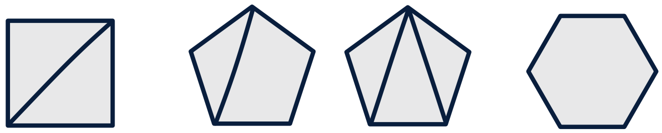
0	1	1			1	1	1
1	12		2				
2	66	5		1			6
3	220		10		4		
4	495	10		5		3	15
5	792		20				
6	924	10		10	6		20
7	792		20				
8	495	5		10		3	15
9	220		10		4		
10	66	1		5			6
11	12		2				
12	1			1	1	1	1
	4,096	**32**	**64**	**32**	**16**	**8**	**64**

1	**6**	**8**	**6**	**3**
1	6	8	6	3
12	12	0	0	0
66	36	0	0	18
220	60	32	0	0
495	90	0	18	45
792	120	0	0	0
924	120	48	0	60
792	120	0	0	0
495	90	0	18	45
220	60	32	0	0
66	36	0	0	18
12	12	0	0	0
1	6	8	6	3
4,096	**768**	**128**	**48**	**192**

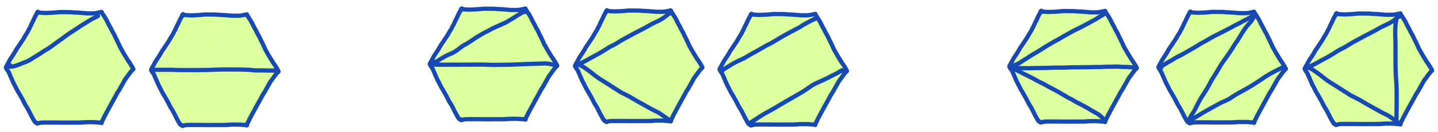
Total	**Count**
24	1
24	1
120	5
312	13
648	27
912	38
1,152	48
912	38
648	27
312	13
120	5
24	1
24	1
5,232	**218**



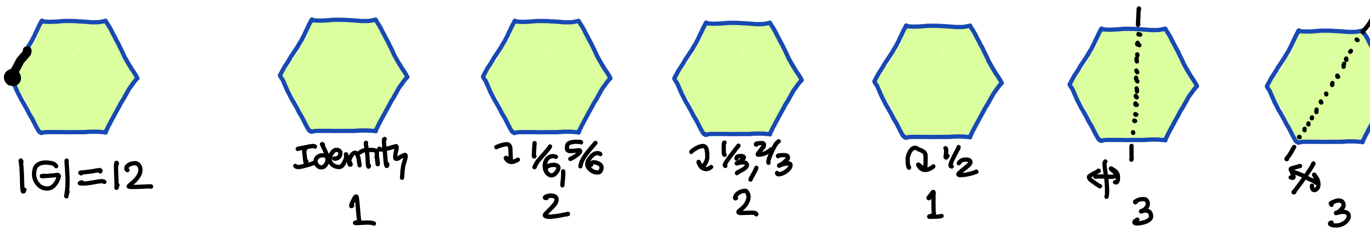
[4] Let $f(n)$ be the number of ways of dissecting an n-gon by at least one cut, up to the dihedral group of rotations and flips. As shown, $f(4) = 1$ and $f(5) = 2$. Find $f(6)$ two ways, by drawing the cases by hand and by using Burnside's lemma.

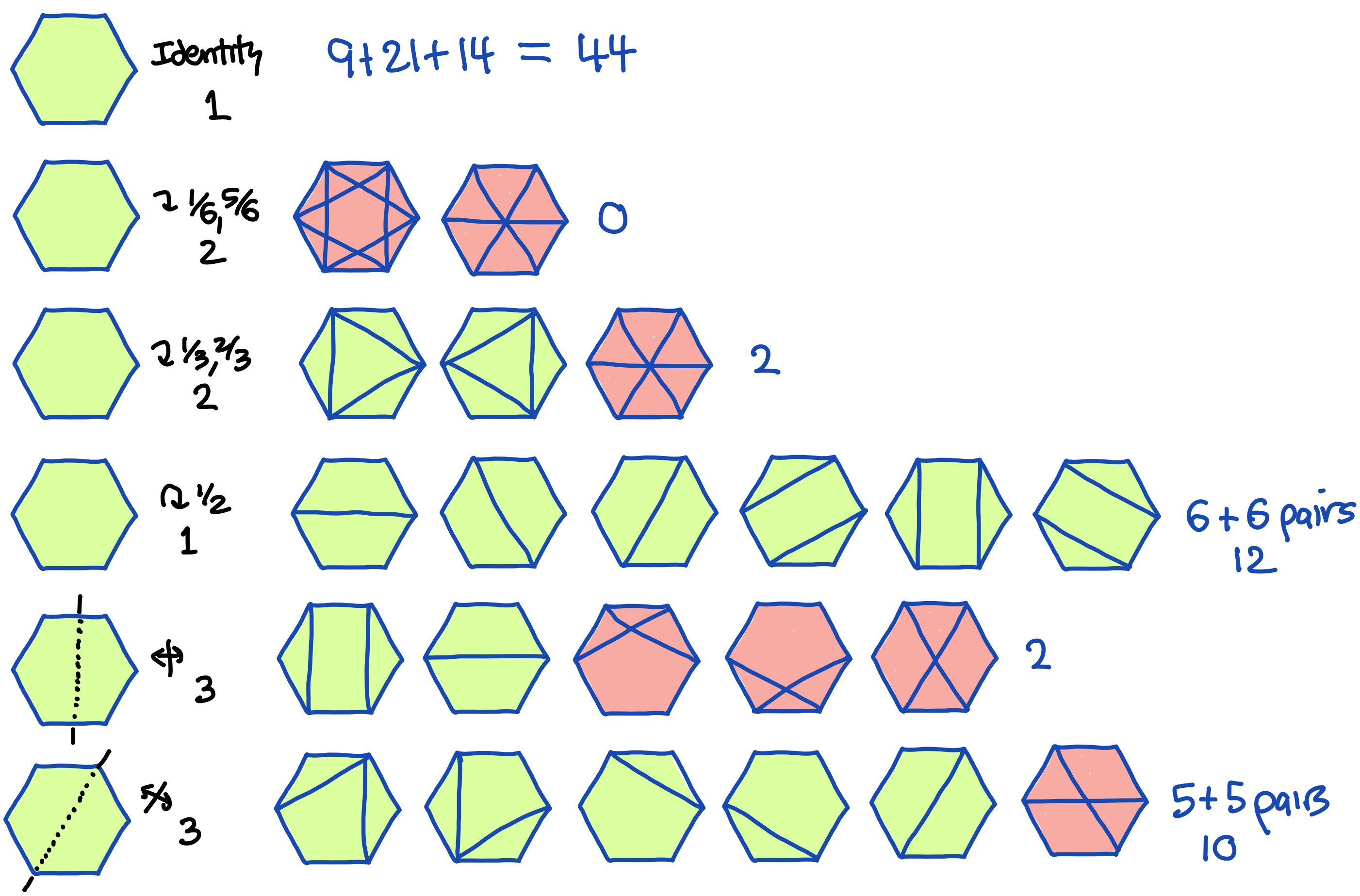


$f(6) = 8$



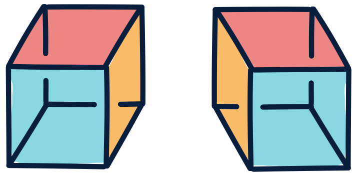
$6 + 3 = 9$ ☑ $12 + 6 + 3 = 21$ ☑ $6 + 6 + 2 = 14$ ☑



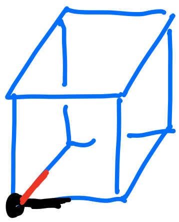


$(44 + 2\cdot 2 + 12 + 3\cdot 2 + 3\cdot 10)/12 = 96/12 = 8$ ☑

[5] How many ways can we color the faces of a cube using at most k colors, up to the group of symmetries generated by rotations and reflections ("look in the mirror")?



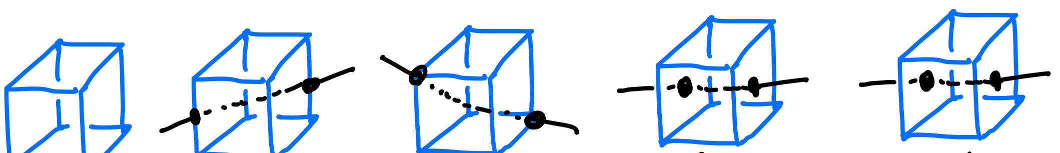
There are 48 symmetries of the cube, including reflections.



① pick a corner (8)
② pick an edge meeting that corner (3)
③ pick a rotational direction (orientation) (2)

$8 \cdot 3 \cdot 2 = \boxed{48}$ we have studied the 24 rotations that preserve orientation.

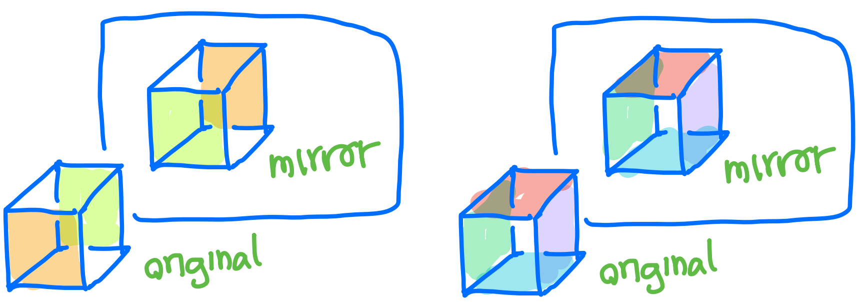
It is harder to classify the 24 symmetries that reverse orientation: Some involve not one but 3 reflections!



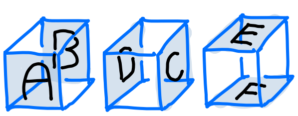
identity 1 | half turn 6 | third turn 8 | quarter turn 6 | half turn 3

$1+6+8+6+3=24$ ✓

For each rotation, we will also group faces by what happens in the mirror.



mirror swaps front and back faces, leaves sides alone

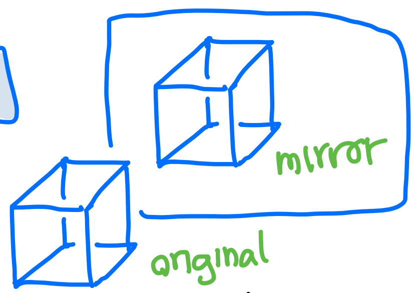


mirror swaps A and B

rotations

k^6

(A)(B)(C)(D)(E)(F)



identity
1

reflections

k^5

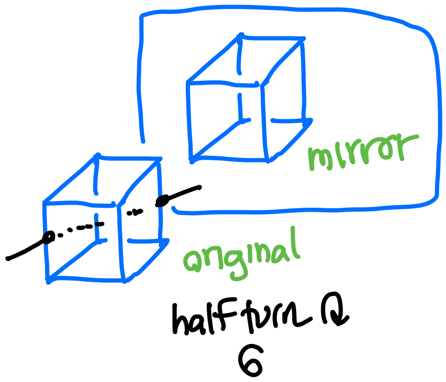
(AB)(C)(D)(E)(F)

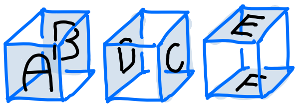
rotations

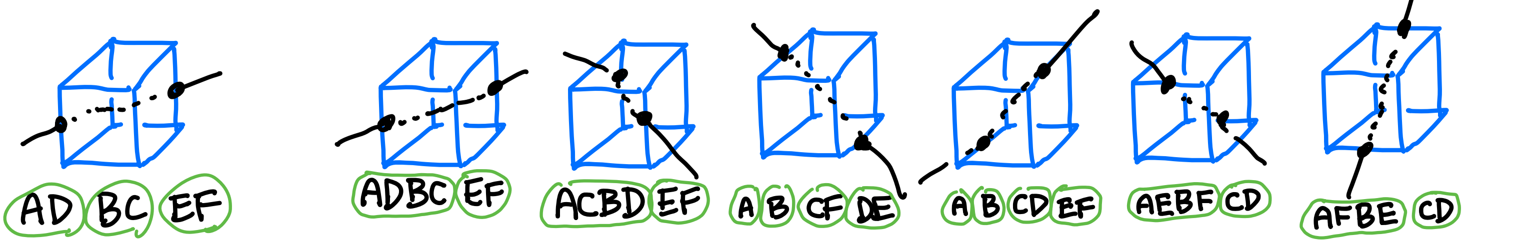
$6k^3$

reflections

$2k^4+4k^2$







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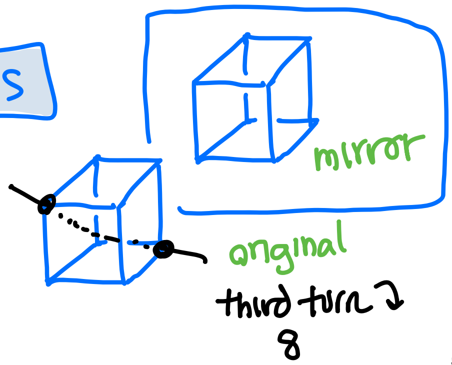
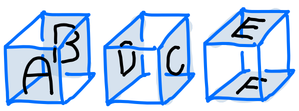
rotations

$8k^2$

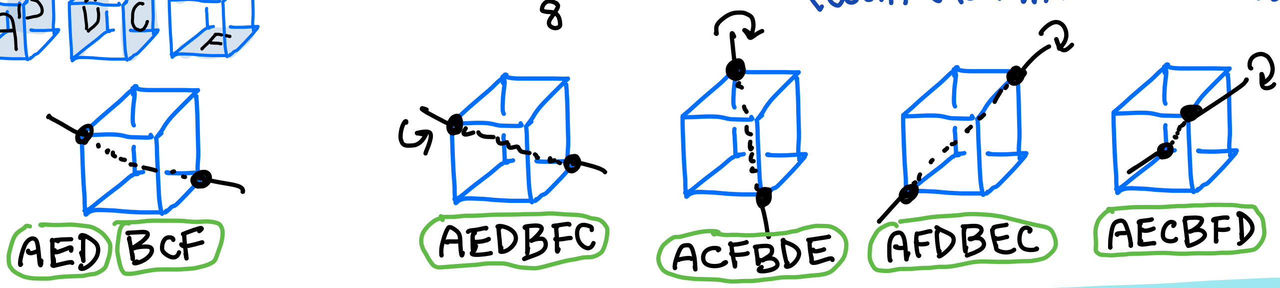
reflections

$8k$





(count each axis either way)



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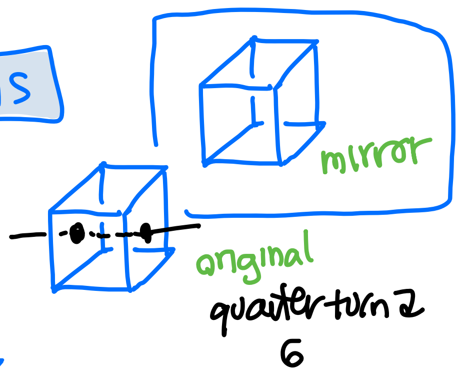
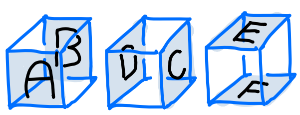
rotations

$6k^3$

reflections

$4k^4+2k^2$





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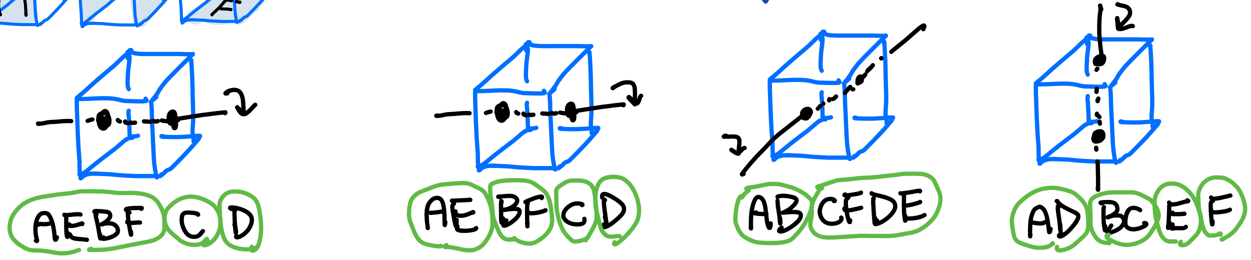
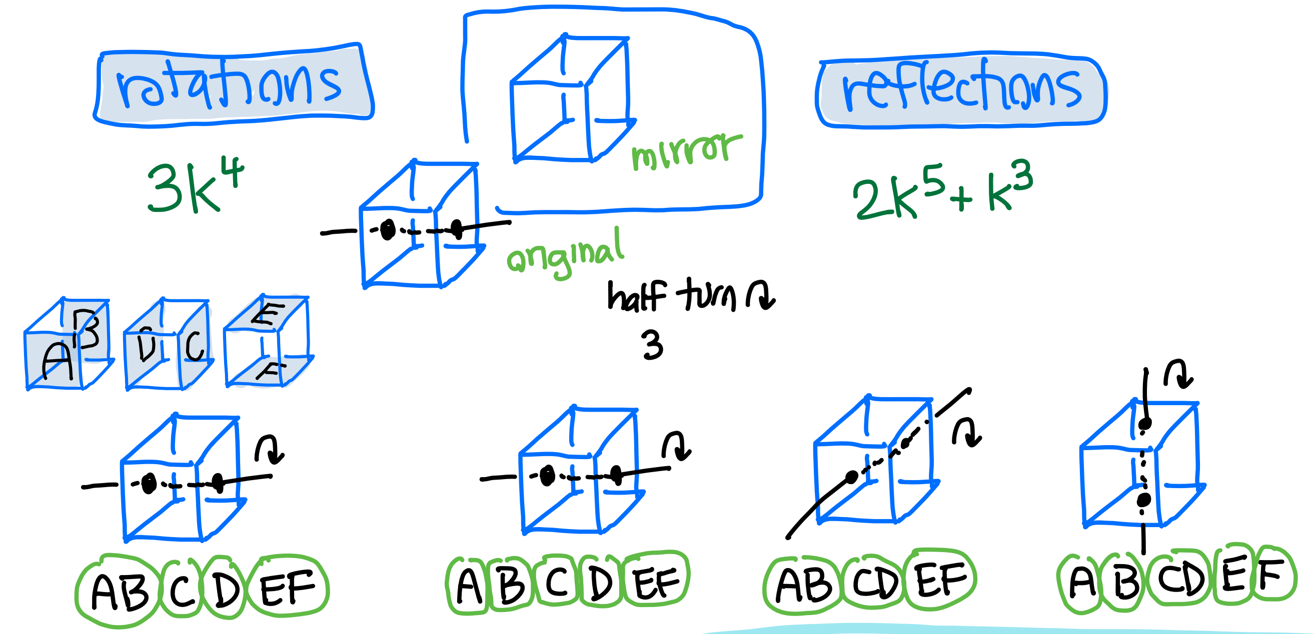




rotations $k^6 + 6k^3 + 8k^2 + 6k^3 + 3k^4$

reflections $k^5 + 2k^4 + 4k^2 + 8k + 4k^4 + 2k^2 + 2k^5 + k^3$

$$(8k + 14k^2 + 13k^3 + 9k^4 + 3k^5 + k^6)/48$$

	1	2	3	4	5	6		48
	8	14	13	9	3	1	total	count
1	8	14	13	9	3	1	48	1
2	16	56	104	144	96	64	480	10
3	24	126	351	729	729	729	2,688	56
4	32	224	832	2,304	3,072	4,096	10,560	220
5	40	350	1,625	5,625	9,375	15,625	32,640	680

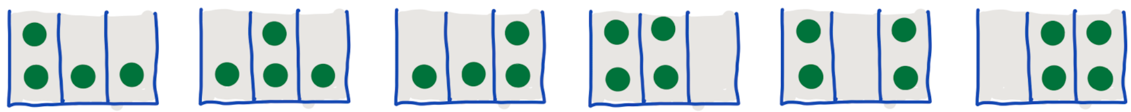
https://oeis.org/A198833

Final Exam

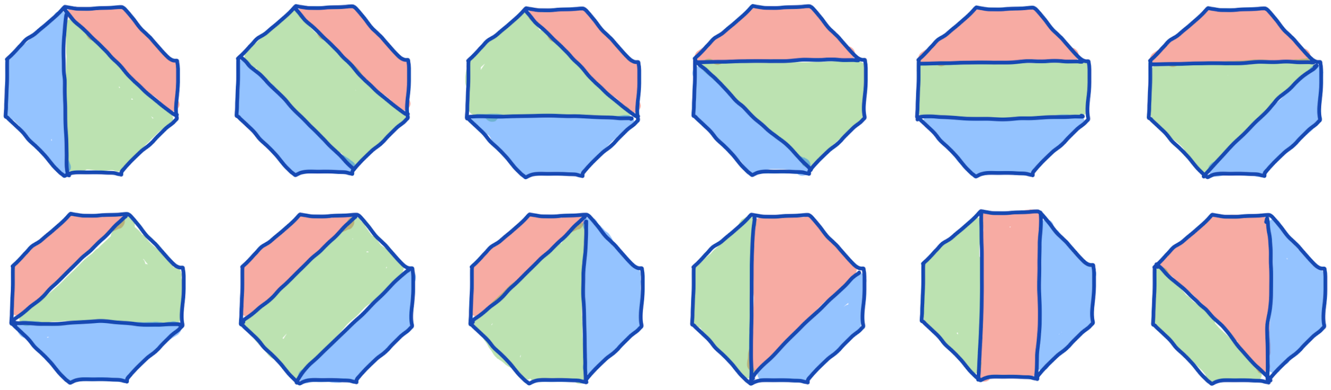
Combinatorics, Dave Bayer, May 3-13, 2022

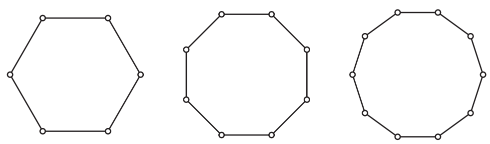
Please show all of your work. You will be graded for both your answers and your explanations. You need not complete the entire exam; the questions vary in difficulty.

[1] How many ways can we place four balls in n bins, if each bin has a capacity of two balls?

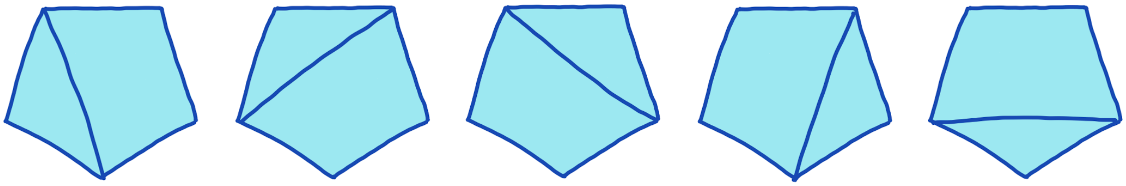


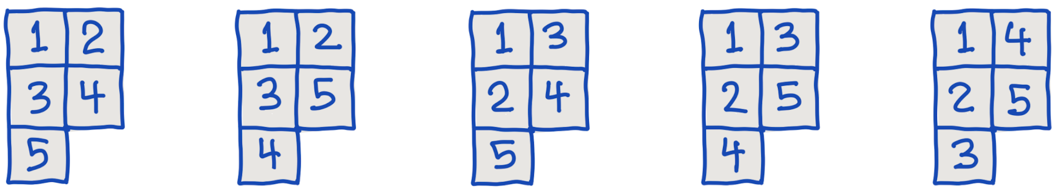
[2] There are twelve ways to dissect an octagon (8-gon) into quadrilaterals (4-gons), using noncrossing diagonals. How many ways can we dissect a decagon (10-gon) into quadrilaterals?



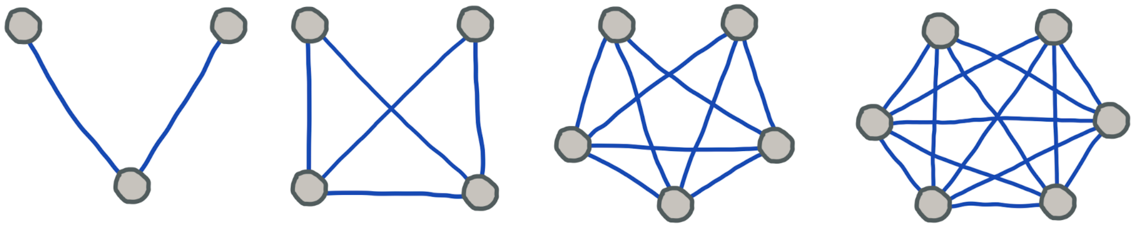


[3] There are five ways to dissect a pentagon (5-gon) making one cut. There are five Young tableaux with the corresponding shape under Stanley's correspondence. Which Young tableau goes with which dissection?

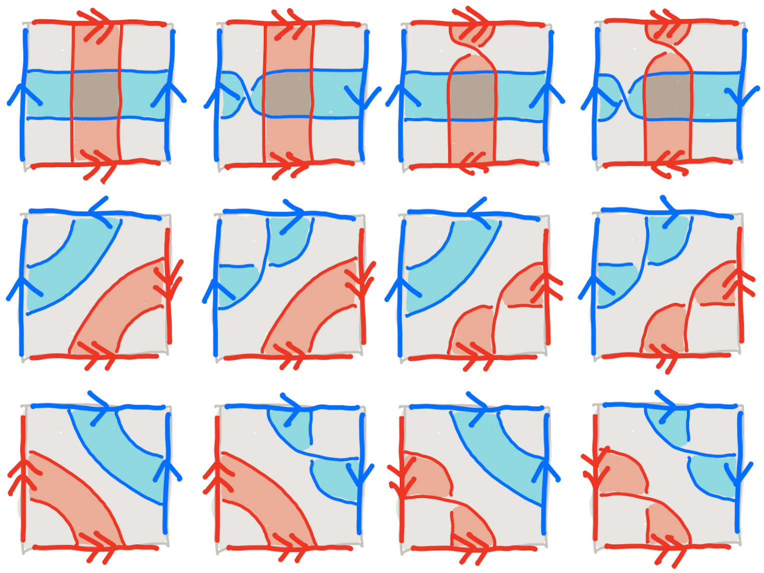




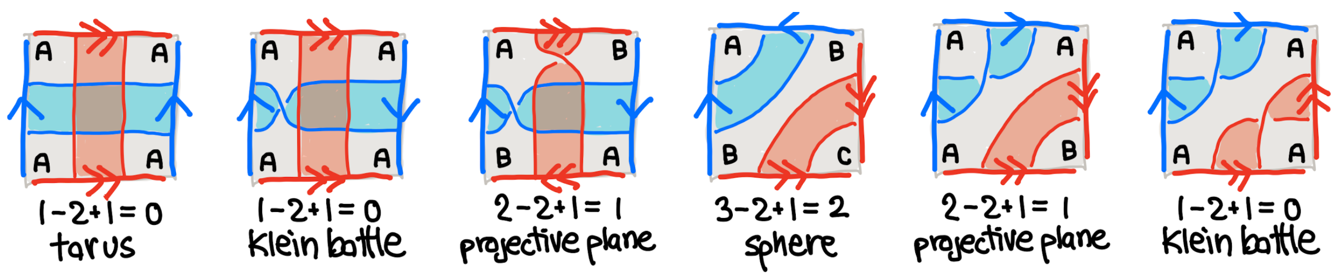
[4] Let G_k be the complete graph on k vertices, with one edge deleted. How many ways can we properly color the vertices of G_k using at most n colors? (For a proper coloring, adjacent vertices have distinct colors.)



[5] There are twelve ways to glue together pairs of sides of a square, while choosing which gluings reverse orientation.



There are six combinatorially distinct cases, which yield four distinct topological surfaces.



Understanding these gluings in general is a famous problem: The Harer-Zagier formula counts gluings that yield a genus g surface, and was applied to solve a deep problem in algebraic geometry.

What can you say about gluing a hexagon?

Final Exam

Combinatorics, Dave Bayer, May 3-13, 2022

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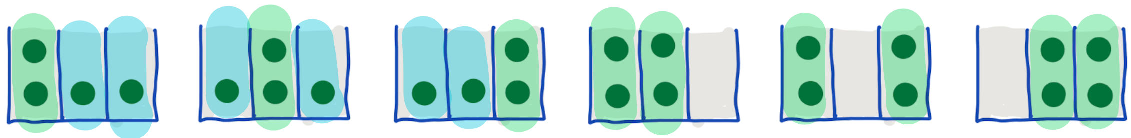
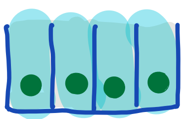
[1] How many ways can we place four balls in n bins, if each bin has a capacity of two balls?



First approach: $4 = 1+1+1+1 = 1+1+2 = 2+2$

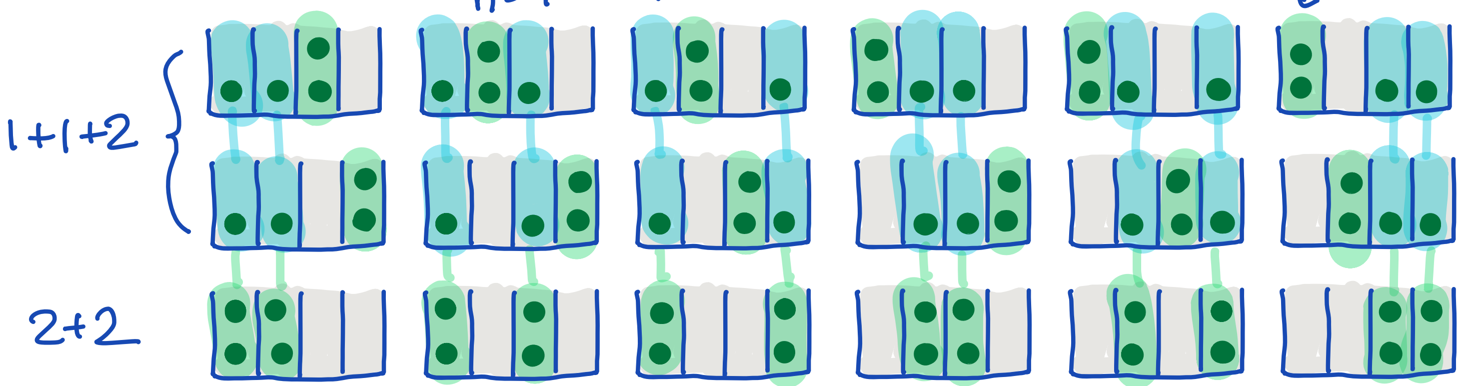
$$\binom{n}{4} + \binom{n}{2}\binom{n-2}{1}^{*} + \binom{n}{2}$$

$^{*}\binom{n}{2}\binom{n-2}{1} = n\binom{n-1}{2} = 3\binom{n}{3}$

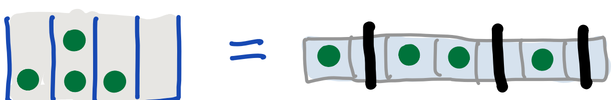
1+1+1+1 

$n=3$: $0 + 3\cdot 1 + 3 = 6$ ✓

$n=4$: $1 + 6\cdot 2 + 6 = 19$ ✓

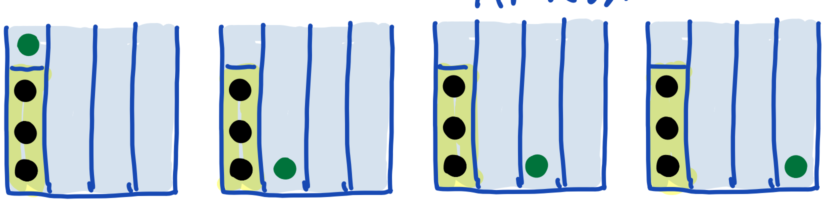


Second approach: 4 balls in n bins, no limits:



$\binom{n+3}{4}$

At least 3 balls in one bin:



n^2

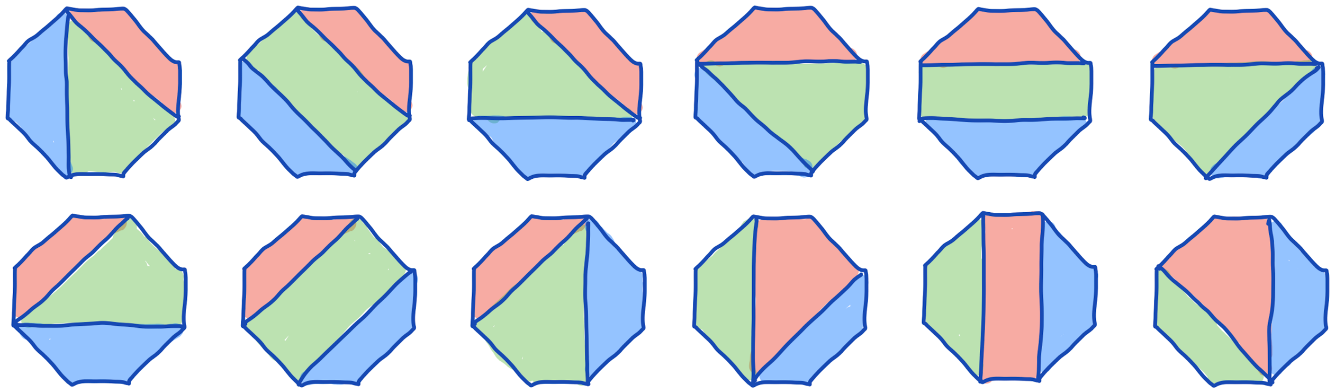
degree 4 in n
enough to check
5 values of n

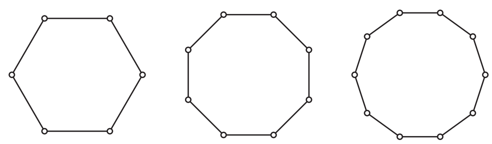
$$\binom{n+3}{4} - n^2 \overset{?}{=} \binom{n}{4} + \binom{n}{2}\binom{n-2}{1} + \binom{n}{2}$$

$n=0$:	0	−	0	=	0	+	0 · 0	+	0	=	0
$n=1$:	1	−	1	=	0	+	0 · 0	+	0	=	0
$n=2$:	5	−	4	=	0	+	1 · 0	+	1	=	1
$n=3$:	15	−	9	=	0	+	3 · 1	+	3	=	6
$n=4$:	35	−	16	=	1	+	6 · 2	+	6	=	19

✓

[2] There are twelve ways to dissect an octagon (8-gon) into quadrilaterals (4-gons), using noncrossing diagonals. How many ways can we dissect a decagon (10-gon) into quadrilaterals?

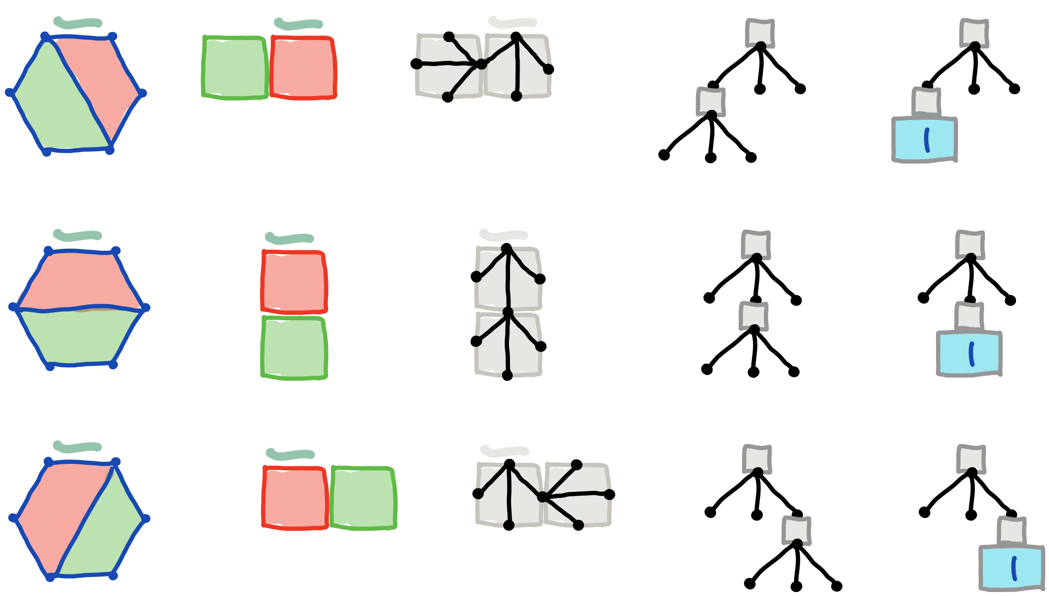




Let's work out these counts from the beginning.

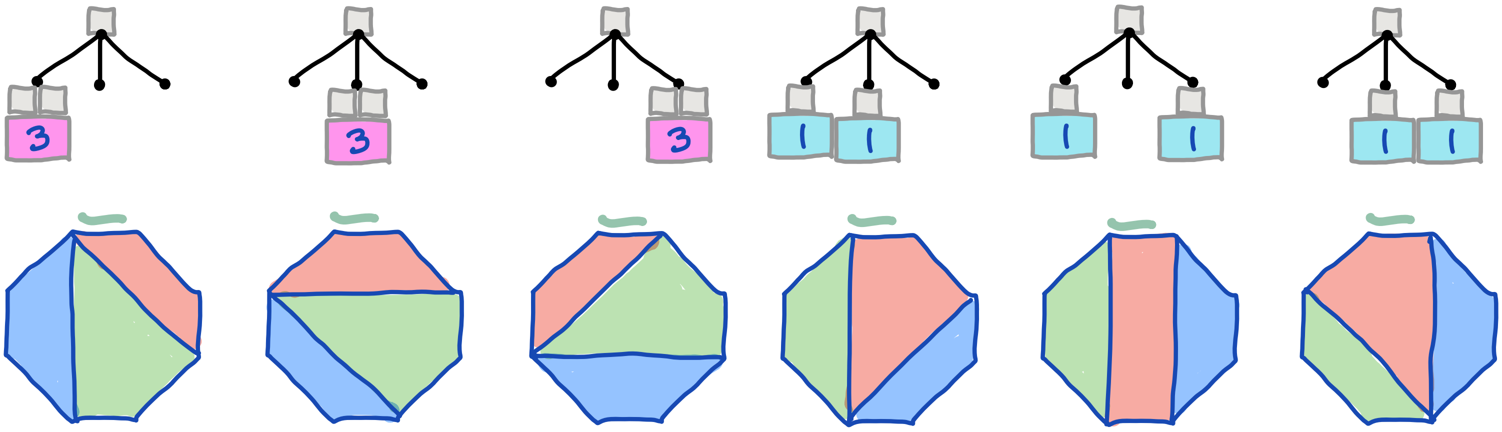
□'s	0	1	2	3	4	5
n-gon		4	6	8	10	12
count	1	1	3	12		

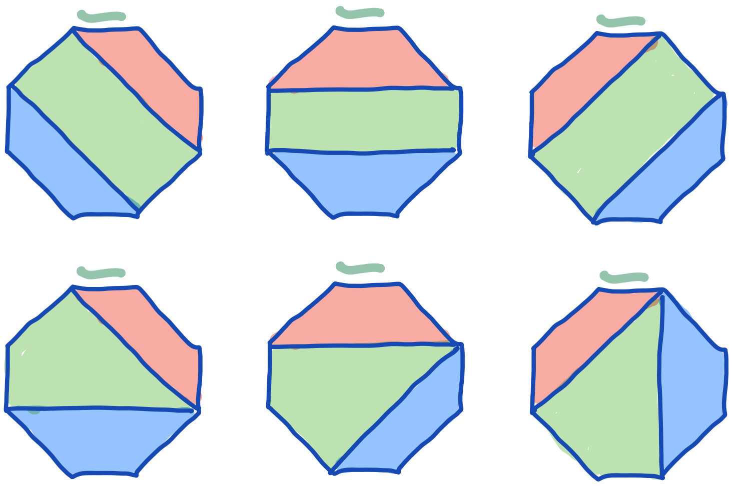
We can build hexagons by attaching two squares all possible ways:

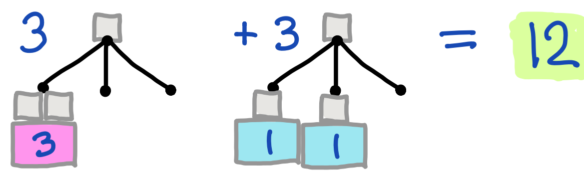


There is 1 way to hang a square off each side of the root square.

We can classify octagons the same way:







□'s	0	1	2	3
n-gon		4	6	8
count	1	1	3	12

We systematize this calculation from the table so far to recover 12

Continuing, we find that the count for a 10-gon is 55

0	1	2	3	4	5
	4	6	8	10	12
1	1	3	12	55	273

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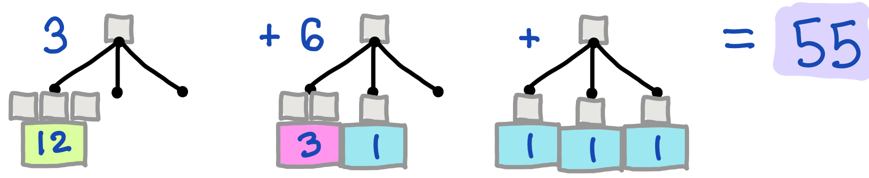
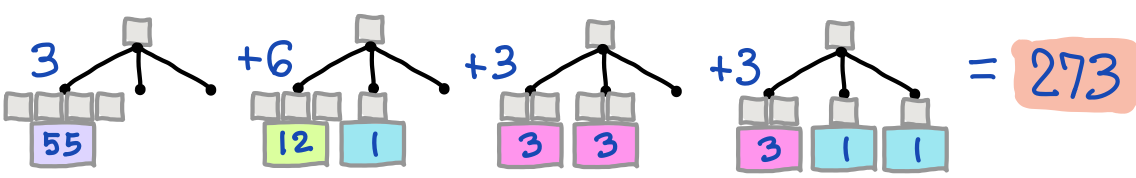




"Casting out nines" is a quick arithmetic check one can do in one's head.

$1 \equiv 10 = 100 = 1000 \ldots \mod 9$

so keep adding digits together till one left:

$1+6+5 \equiv 12 \equiv 3$

$7+2 \equiv 9 \equiv 0$ } cast out 9's

$2+7 \equiv 9 \equiv 0$ }

$9 \equiv 0$

$2+7+3 \equiv 12 \equiv 3$ ☑

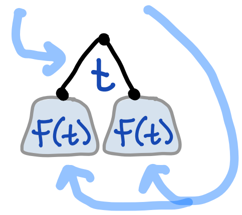
$3 \cdot 12 = 36$
$6 \cdot 3 = 18$
$1 \cdot 1 = 1$
total: 55

$3 \cdot 55 = 165$
$6 \cdot 12 = 72$
$3 \cdot 9 = 27$
$3 \cdot 3 = 9$
total: 273

This count is similar to the Catalan numbers.
The only difference there is binary (not 3-way) trees:

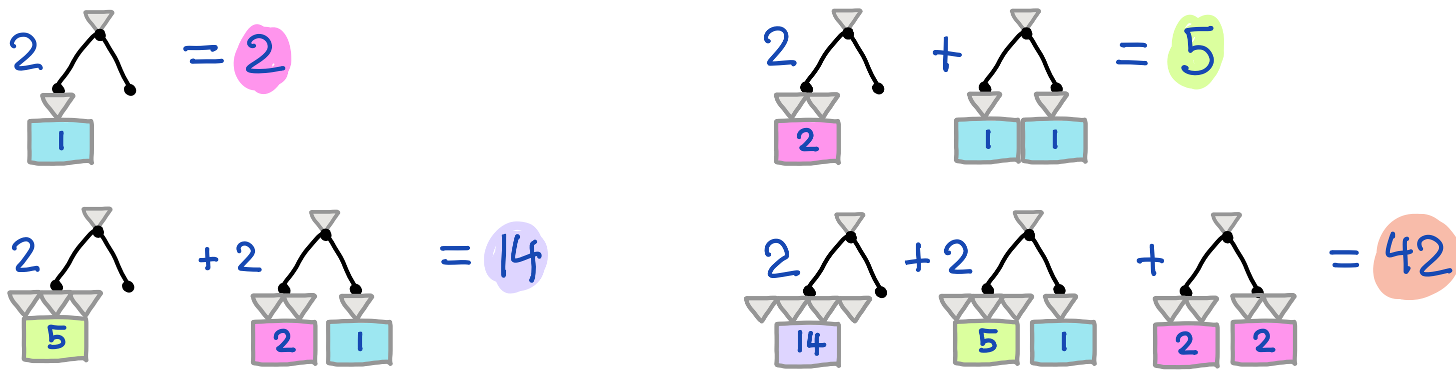
$$F(t) = 1 + t\,F(t)^2$$

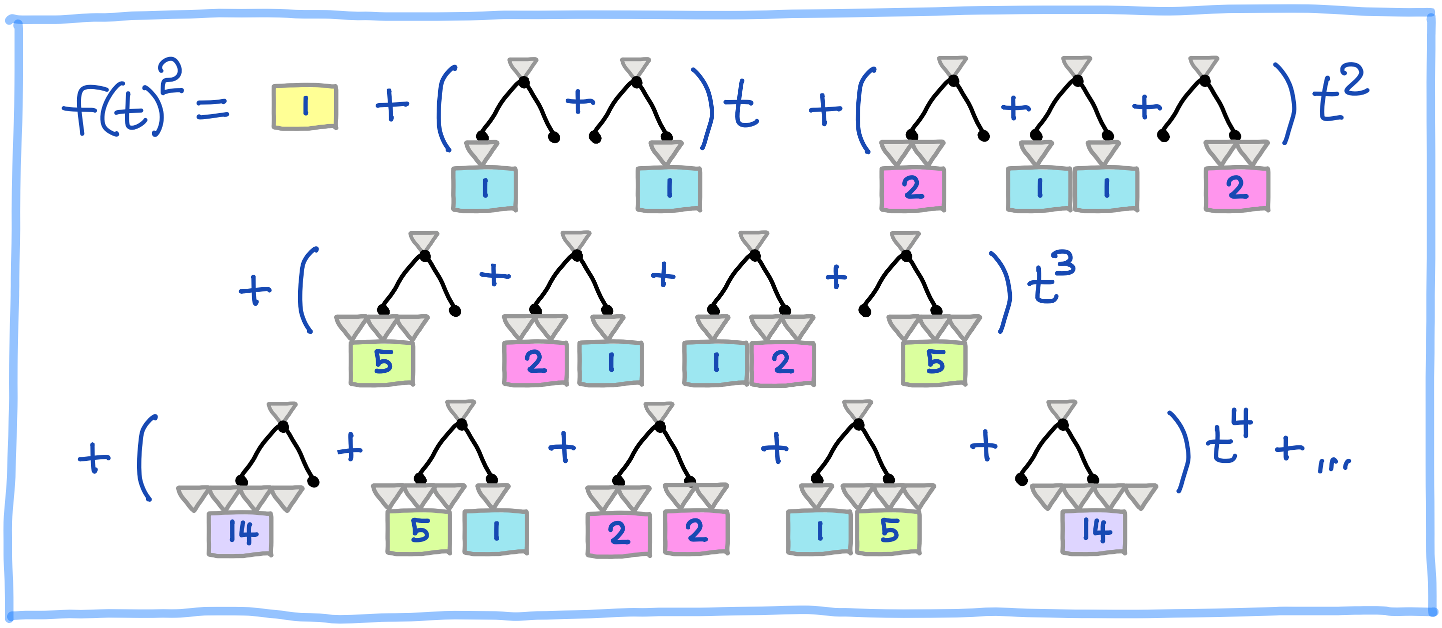
empty case



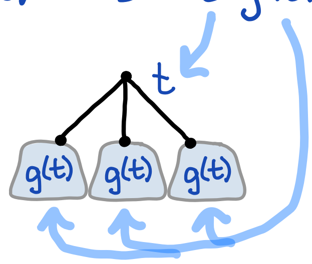
Δ's	0	1	2	3	4	5
n-gon		3	4	5	6	7
count	1	1	2	5	14	42

$$f(t) = 1 + 1\,t + 2\,t^2 + 5\,t^3 + 14\,t^4 + 42\,t^5 + \ldots$$





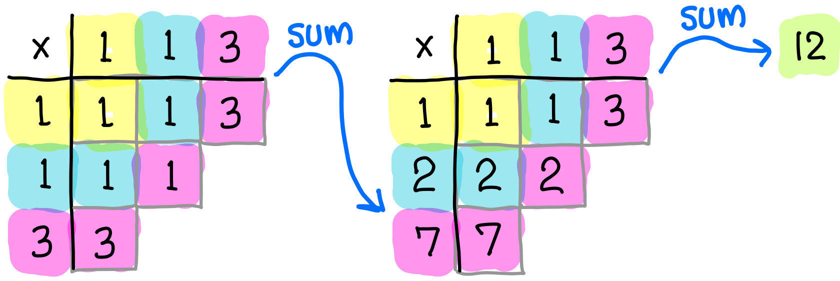
Our problem satisfies a similar equation: $g(t) = 1 + t\,g(t)^3$

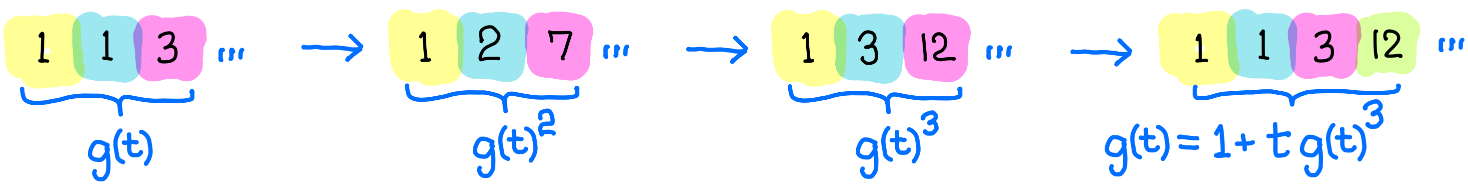


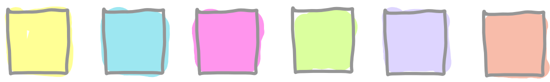
□'s	0	1	2	3	4	5
n-gon		4	6	8	10	12
count	1	1	3	12	55	273

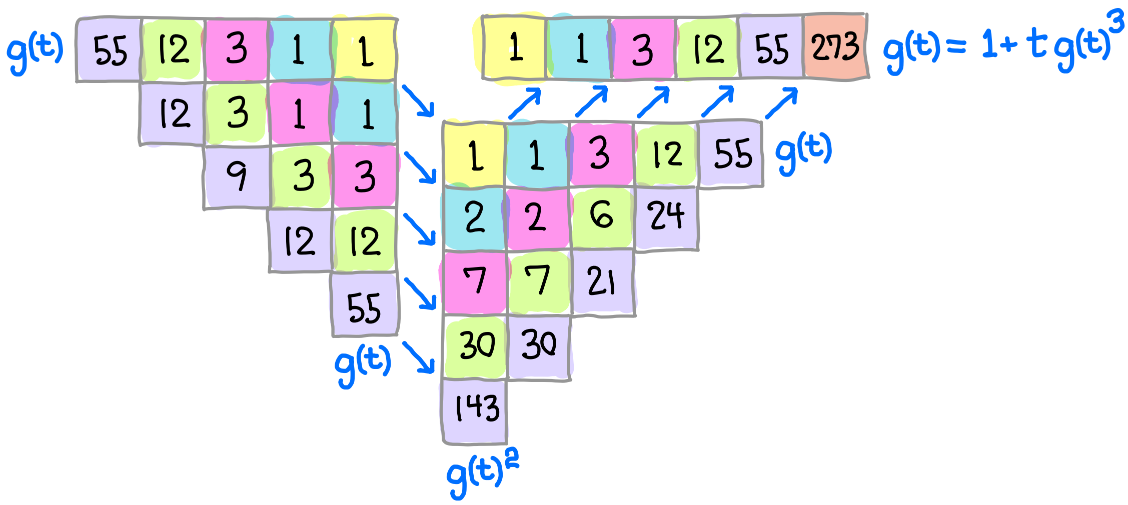
$$g(t) = 1 + 1\,t + 3\,t^2 + 12\,t^3 + 55\,t^4 + 273\,t^5 + \ldots$$

This equation gives an iterative algorithm for computing g(t):





Organized in one chart, fill in  in sequence.



The Haskell programming language uses call-by-need ("lazy") evaluation, allowing it to support infinite lists. Haskell easily expresses this generating function:

```haskell
convolve :: [Int] → [Int] → [Int]
convolve xs = map (sum . zipWith (*) xs) . tail . scanl (flip (:)) []

g :: [Int]
g = 1 : convolve g (convolve g g)
```

If you develop an interest in Haskell, I'm happy to offer support.

n	0	1	2	3	4
$\binom{2n}{n}$	1	2	6	20	70
$\div(n+1)$	1	1	2	5	14
$\binom{3n}{n}$	1	3	15	84	495
$\div(2n+1)$	1	1	3	12	55

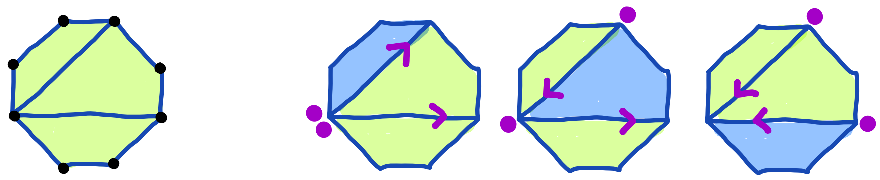
Just as there is a formula for the Catalan numbers

$$C(n) = \binom{2n}{n}/(n+1)$$

there is a formula for these numbers

$$C_3(n) = \binom{3n}{n}/(2n+1)$$

It is more satisfying to specialize the 2000 Przytycki, Sikora proof of Cayley's formula for arbitrary polygon dissections, than to generalize an ad hoc proof for Catalan numbers:



n squares form a (2n+2)-gon, dissected by (n-1) cuts.

By marking one of these n regions, we can orient each cut so the marked region is on the cut's left.

Each cut is then determined by its starting vertex, because the region to its right must be a quadrilateral.*

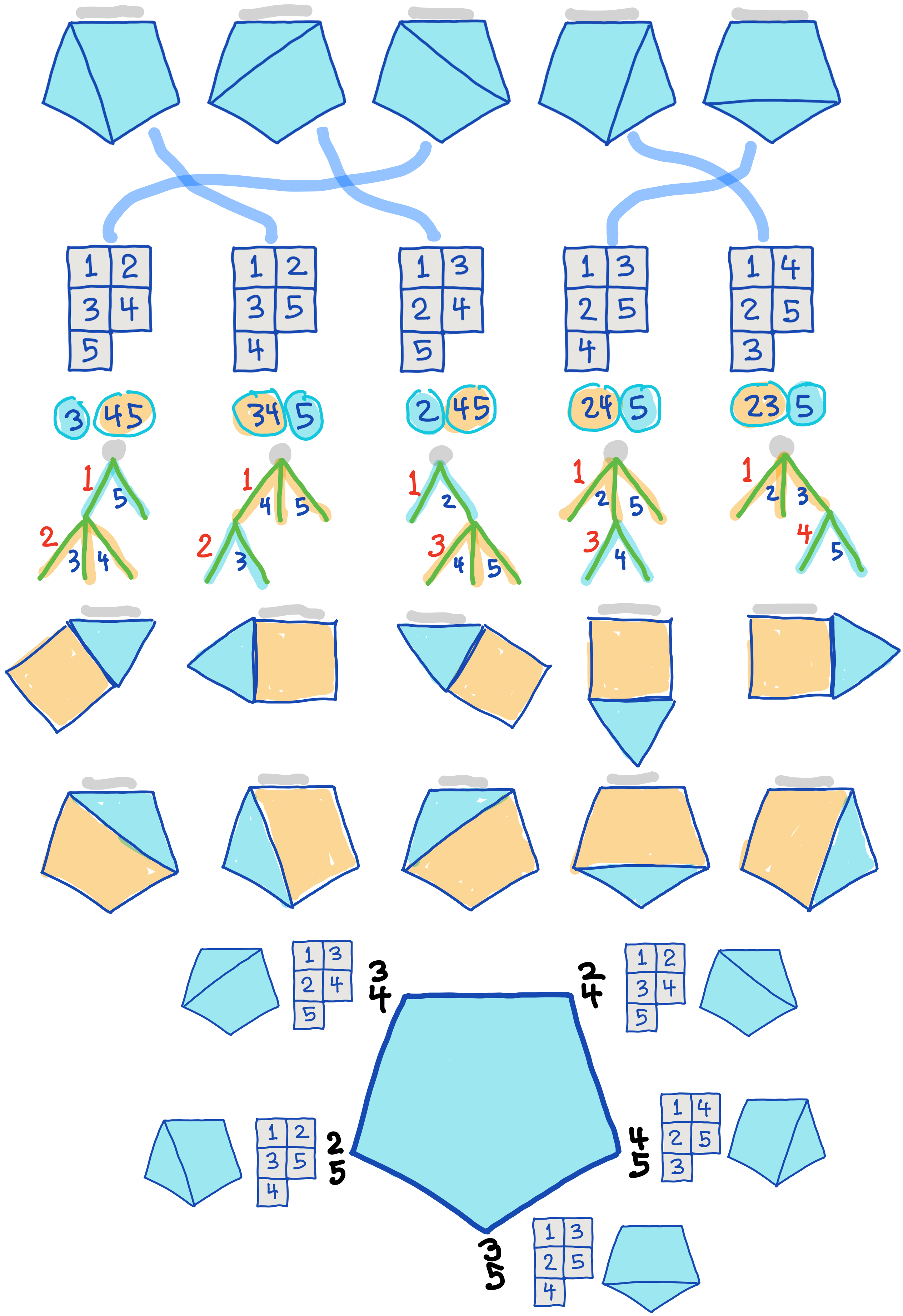
We are starting each of (n-1) cuts in one of (2n+2) bins. Dividing by our choice of one of n regions to mark, we have

$$\binom{(2n+2)+(n-1)-1}{(n-1)}/n = \binom{3n}{n-1}/n = \binom{3n}{n}/(2n+1)$$

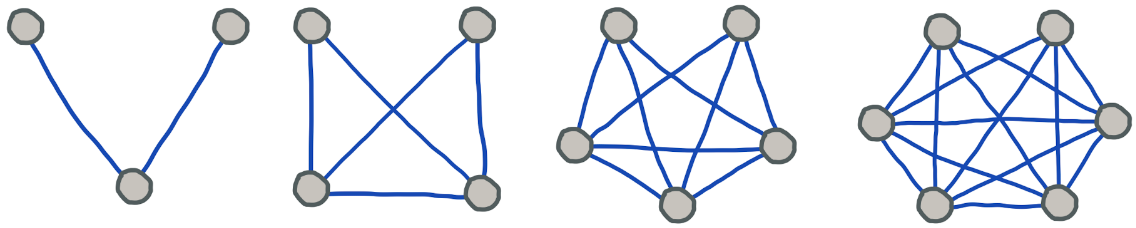
A similar argument recovers the formula for Catalan numbers.

* Yes, there are details to sort out. First make cuts that don't have other cuts in their way. This same argument works in general. Enumerate the possible region combinations, for the case where all cuts start at the same vertex. The same region combinations are possible, no matter where the cuts start.

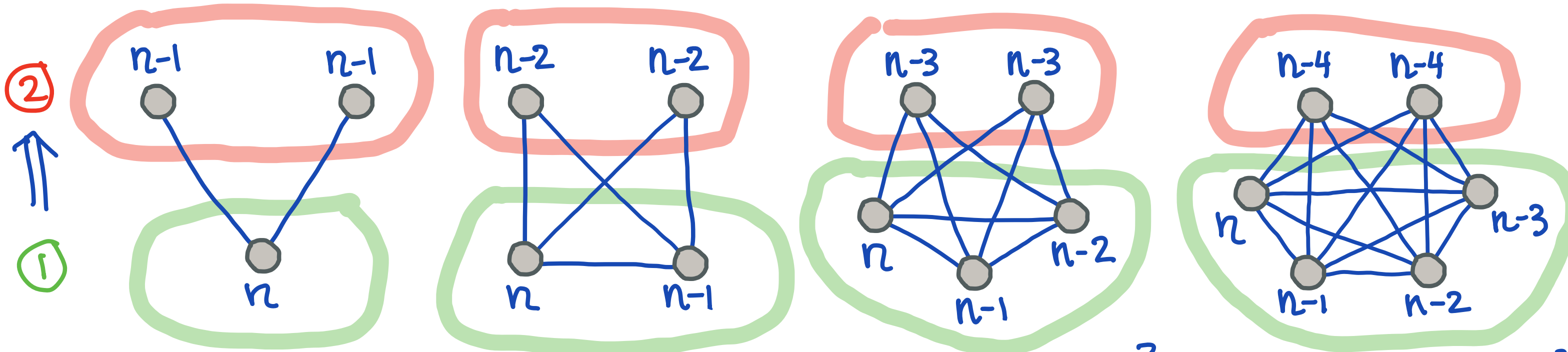
[3] There are five ways to dissect a pentagon (5-gon) making one cut. There are five Young tableaux with the corresponding shape under Stanley's correspondence. Which Young tableau goes with which dissection?



[4] Let G_k be the complete graph on k vertices, with one edge deleted. How many ways can we properly color the vertices of G_k using at most n colors? (For a proper coloring, adjacent vertices have distinct colors.)



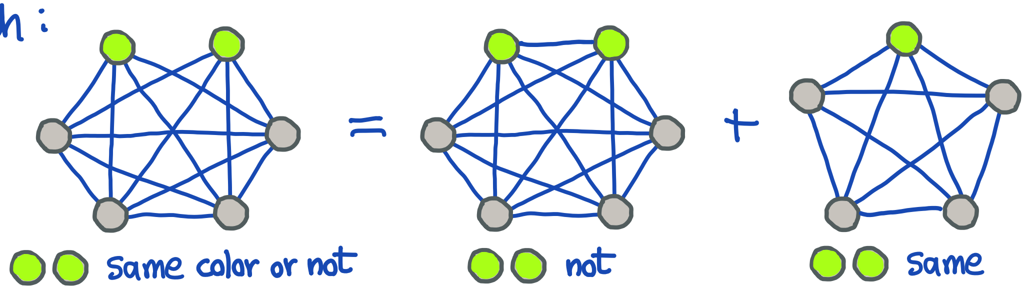
First approach:



$n(n-1)^2$ $k=3$ $\quad$ $n(n-1)(n-2)^2$ $k=4$ $\quad$ $n(n-1)(n-2)(n-3)^2$ $k=5$ $\quad$ $n(n-1)\cdots(n-3)(n-4)^2$ $k=6$

$$\boxed{n(n-1)\cdots(n-k+3)(n-k+2)^2}$$

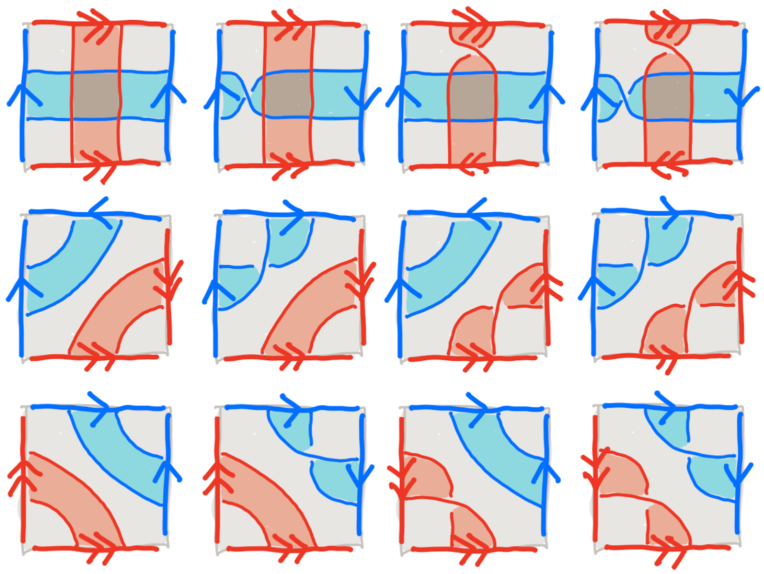
Second approach:



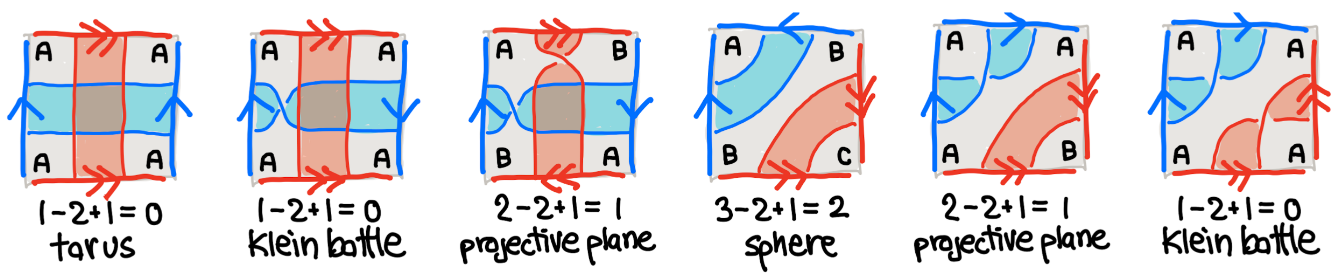
$$
\begin{array}{r}
n(n-1)\cdots(n-k+3)(n-k+2)(n-k+1) \\
+\ n(n-1)\cdots(n-k+3)(n-k+2)\,1 \\
\hline
n(n-1)\cdots(n-k+3)\underbrace{(n-k+2)(n-k+2)}_{(n-k+2)^2}
\end{array}
$$

☑

[5] There are twelve ways to glue together pairs of sides of a square, while choosing which gluings reverse orientation.



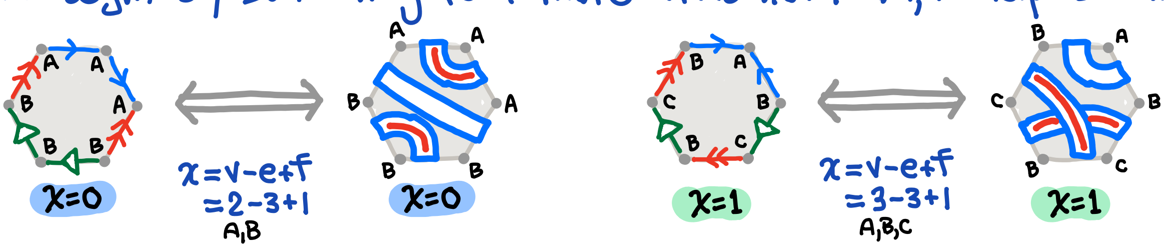
There are six combinatorially distinct cases, which yield four distinct topological surfaces.



Understanding these gluings in general is a famous problem: The Harer-Zagier formula counts gluings that yield a genus g surface, and was applied to solve a deep problem in algebraic geometry.

What can you say about gluing a hexagon?

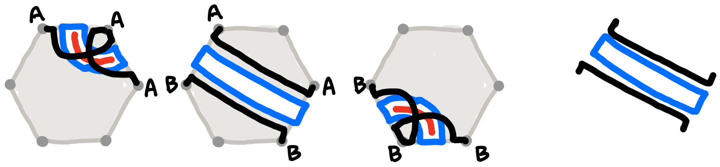
Let's begin by switching to a more vivid notation, to help us think quicker:



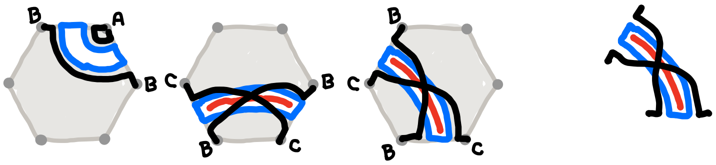
In standard notation, edge colors and arrows are arbitrary, serving only to identify glued pairs, and indicate orientation.

In this strip notation, the over/under pattern is irrelevant. Strips identify glued pairs, and a red line indicates orientation reversal.

To compute each Euler characteristic, we need to relearn how to count vertices:



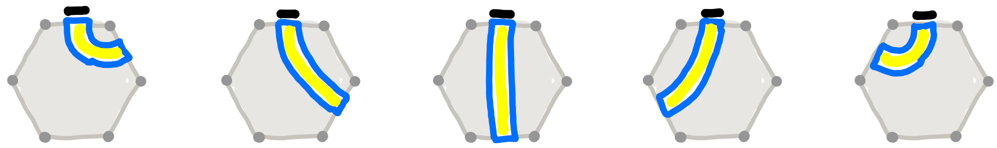
Preserve orientation
Follow strip edge to find neighboring corner (same vertex)



Reverse orientation
Cross over strip to find neighboring corner (same vertex)

How many pairing diagrams are there, before considering orientation?

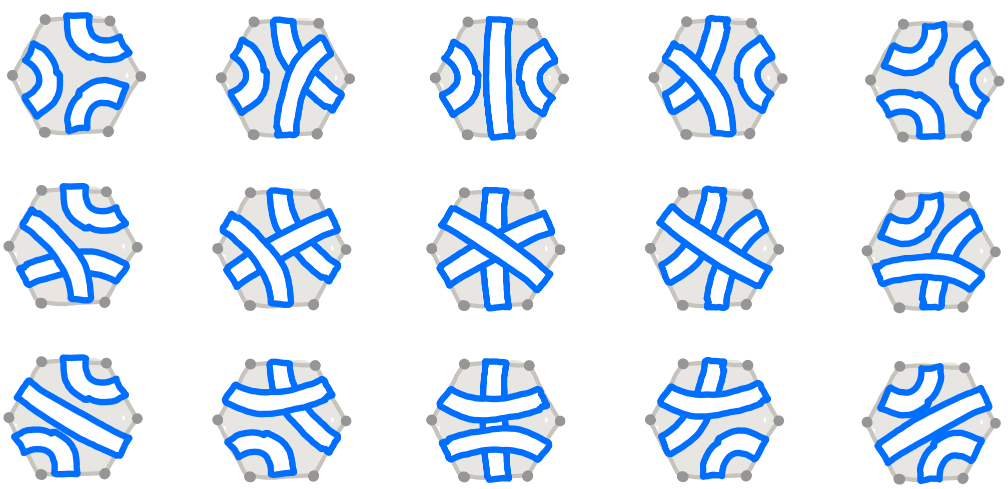
5 choices for other end of first strip:



3 choices for other end of next strip:

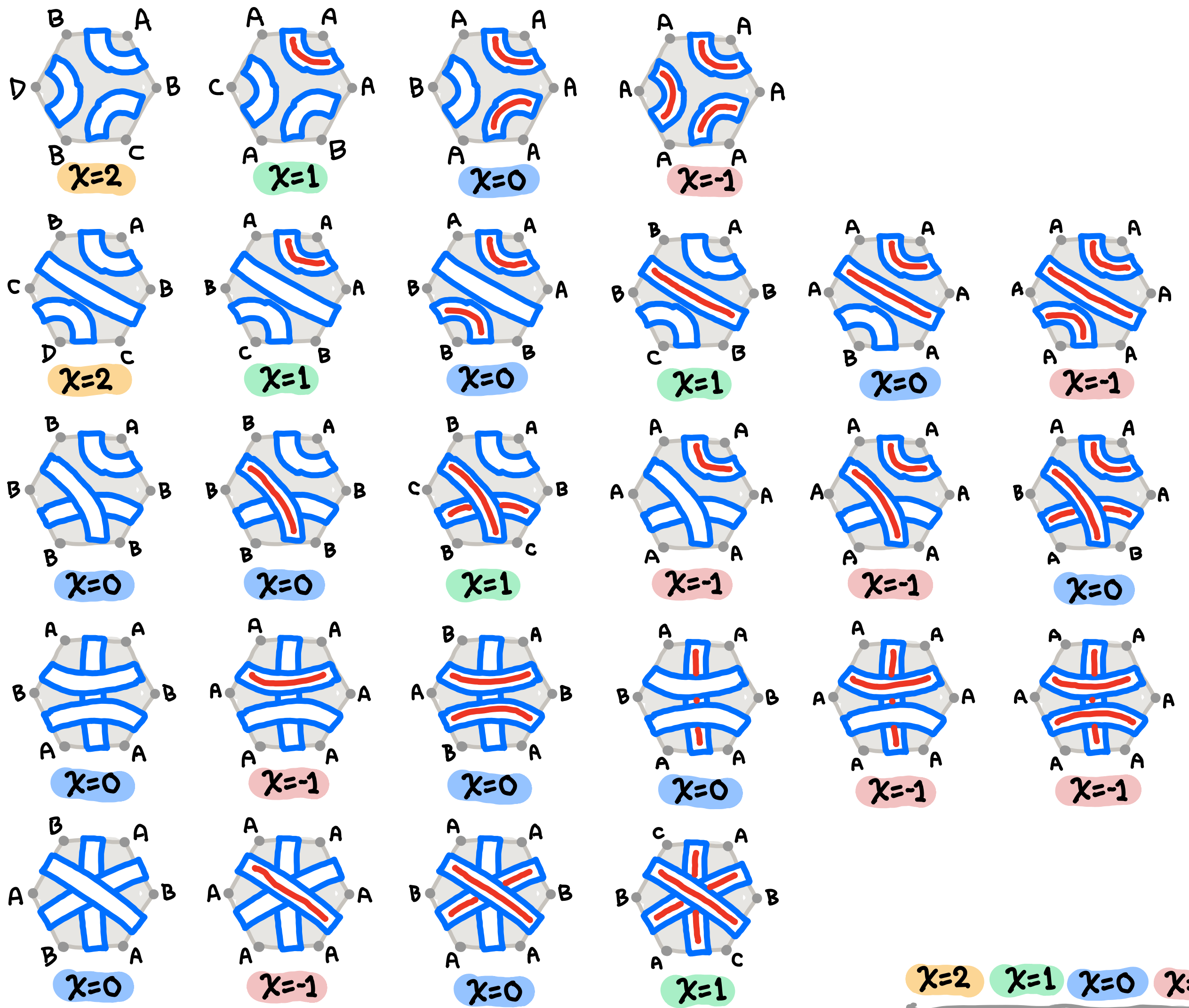
1 choice for other end of last strip:

So we have 15 pairing diagrams:



Up to symmetry there are 5 different diagrams:

For each of these 5 diagrams, we can now work out the possible orientations, and count vertices to compute each Euler characteristic:

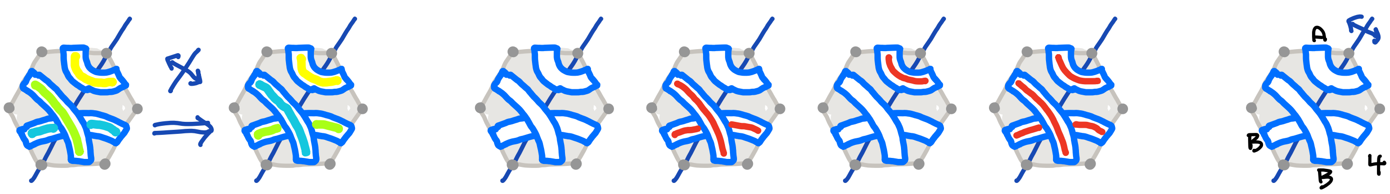


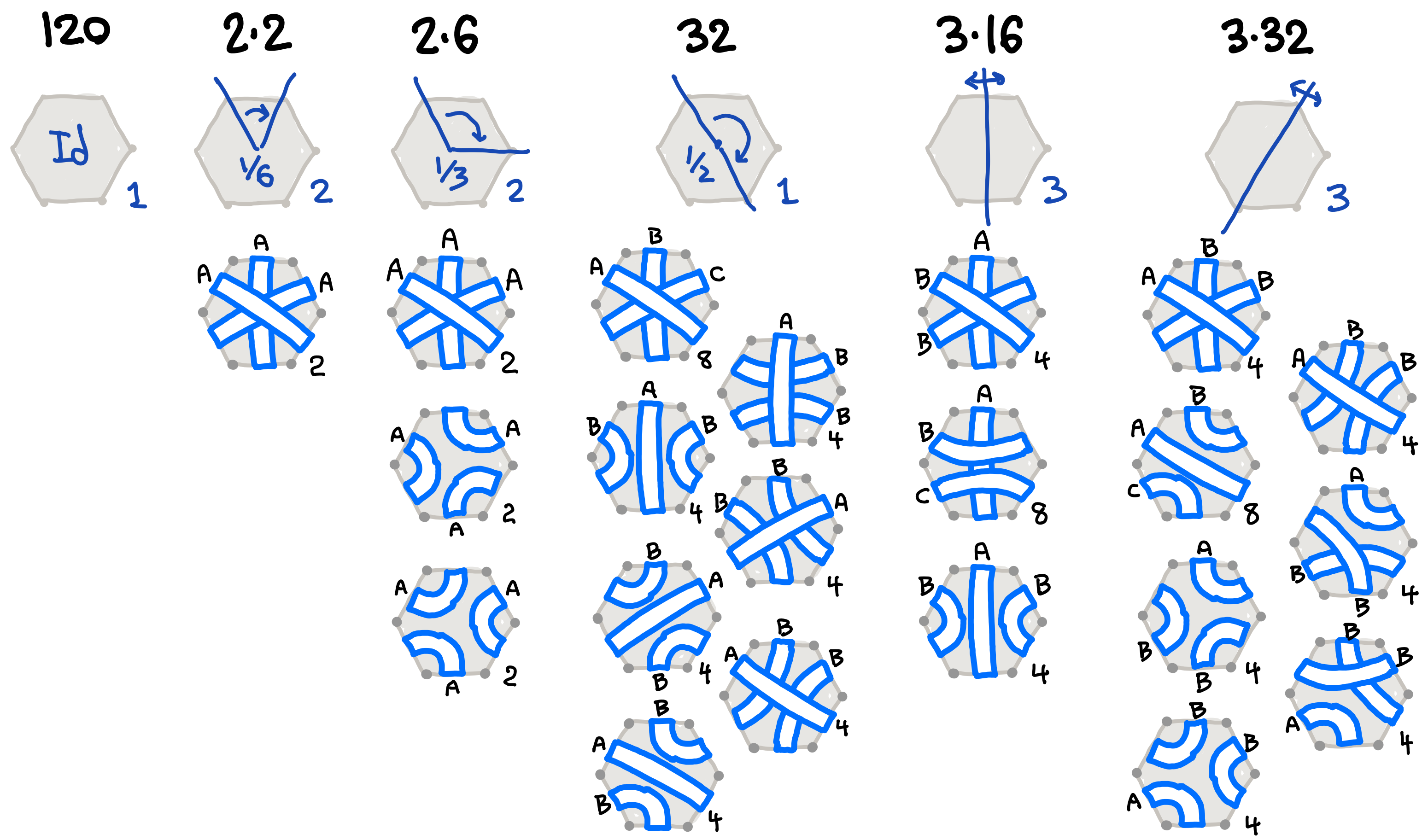
There are 26 different gluing diagrams, yielding 5 different surfaces:

	$\chi=2$	$\chi=1$	$\chi=0$	$\chi=-1$
	2		3	
		5	8	8

We can confirm this count using Burnside's lemma, letting the dihedral group act on the $2^3 \cdot 15 = 120$ gluing diagrams, taking all possible orientation choices for the 15 pairing diagrams.

The work here is learning to see which pairings are fixed by an action, and when two orientation choices must agree:





$|G| = 12$

$$\underbrace{(120}_{10} + \underbrace{2\cdot 2 + 2\cdot 6 + 32}_{4} + \underbrace{3\cdot 16 + 3\cdot 32)}_{12}/12 = 26 \quad ☑$$

Using just rotations we get the wrong answer:

$|G| = 6$

$$\underbrace{(120}_{20} + \underbrace{2\cdot 2 + 2\cdot 6 + 32}_{8})/6 = 28$$

Sure enough, there are two chiral pairs.
The 28 counts each of them; we count each pair once.