Combinatorics
January 12, 2021

Feller $\sum$ $\int$

Aigner

How many words length 2

#1 letter: A, B, C

#2 letter: x, y, z

	x	y	z
A	Ax	Ay	Az
B	Bx	By	Bz
C	Cx	Cy	Cz

$3 * 3 = 9$

Binomial Coefficients

$\binom{n}{k}$ "n choose k" = # number of subsets of size k of n things.

$6 = \binom{4}{2}$

A	B	C	D
x	x		
x		x	
x			x
	x	x	
	x		x
		x	x

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2+ & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} + & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

divide and conquer

$\binom{4}{2}=6$

1 2 3 4

pick 2

use 1?

yes $\rightarrow$ 1 … $\binom{3}{1}$ 12 13 14

no $\rightarrow$ $\binom{3}{2}$ 23 24 34

$$\binom{n+1}{k} = \binom{n}{k}+\binom{n}{k-1}$$

$$\boxed{\binom{n+1}{k}-\binom{n}{k}} = \binom{n}{k-1}$$

$f(x): \mathbb{R}\to\mathbb{R}$

$$\boxed{f'(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)-f(x)}{\varepsilon}}$$

$g(n): \mathbb{Z}\to\mathbb{Z}$

$$\boxed{\Delta g(n) = g(n+1)-g(n)}$$

$$g(n)=\binom{n}{k}$$

$$\Delta g(n) = \boxed{\binom{n+1}{k}-\binom{n}{k}} = \binom{n}{k-1}$$

$\frac{d}{dx}e^x=e^x$ $\qquad \Delta 2^n = 2^{n+1}-2^n=2^n$

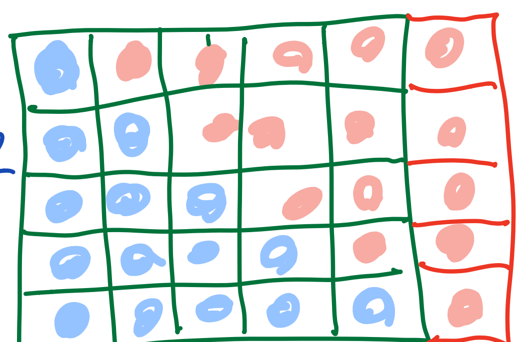
$\frac{d}{dx}x^n = nx^{n-1}$ $\qquad \binom{n}{0},\binom{n}{1},\binom{n}{2},\binom{n}{3}$

$\times 0$ $\leftarrow$ $1, x, x^2, x^3, \dots$

$\times 1$ $\times 2$ $\times 3$

$1+2+3+4+5 = 15$

$1+2+3+4 = 10$



$1+2+3+4+5$

$5+4+3+2+1$

$5\times 6/2 = 15$

$n(n+1)/2$

$$\sum_{i=1}^{n} i^m \quad \text{for any power} \qquad m=1$$

$$g(n) = 1+2+\dots+n$$

$$\Delta g(n) = g(n+1)-g(n) = n+1 = \binom{n}{1}+\binom{n}{0}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{0} = 1$$
$$\binom{n}{1} = n$$
$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\Delta g(n) = \binom{n}{1}+\binom{n}{0}$$

$$\Rightarrow \quad g(n) = \binom{n}{2}+\binom{n}{1}+\cancel{C}$$

$$= \frac{n(n-1)}{2} + \frac{2n}{2} = \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}$$

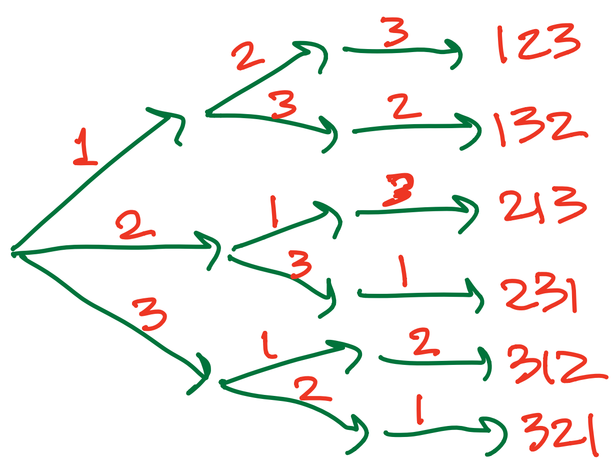
$\binom{n}{k}$ overcounting

$n!$ = all permutations of $1 \dots n$ (or any n things)

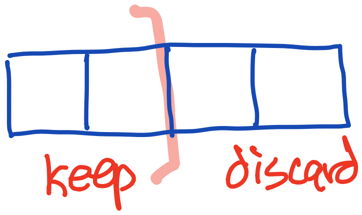
$$= n\cdot(n-1)\cdots 3\cdot 2\cdot 1$$

$n!$

$3! =$
$3\cdot 2\cdot 1 = 6$



$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2\cdot 2} = 6$$



ignore order

12	13	14
12 34, 21 34, 12 43, 21 43	13 24, 31 24, 13 42, 31 42	14 23, 41 23, 14 32, 41 32
23	**24**	**34**
23 14, 32 14, 23 41, 32 41	24 13, 42 13, 24 31, 42 31	34 12, 43 12, 34 21, 43 21

$4!/2!2!$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \overbrace{\underbrace{\boxed{1\ 2\ 3\ 4}}_{k \text{ keep}}\underbrace{\boxed{5\ 6\ 7\ 8\ 9}}_{n-k \text{ discard}}}^{n}$$

$$\binom{8}{3} = \frac{8\cdot 7\cdot 6\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{3\cdot 2\cdot 1\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$$

$$= \frac{\overset{4}{\cancel{8}}\cdot 7\cdot \overset{2}{\cancel{6}}}{\cancel{3}\cdot \cancel{2}\cdot 1} = 56$$

"bars & stars" argument

how many monomials of deg d in n variables?
terms

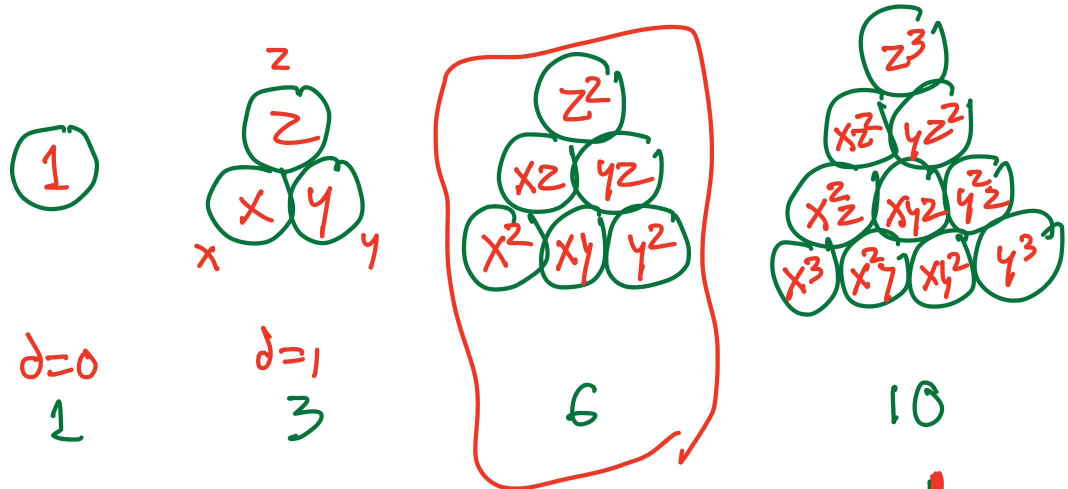
$n=3 \quad x, y, z$

$d=0 \quad x^0y^0z^0 = 1$

$d=1 \quad x, y, z$

$d=2 \quad x^2, y^2, z^2, xy, xz, yz$

$d=3$ $\quad x^3, y^3, z^3, x^2y, x^2z, xy^2, xz^2, yz^2, y^2z, xyz$



Pascal's Triangle

1

1 1

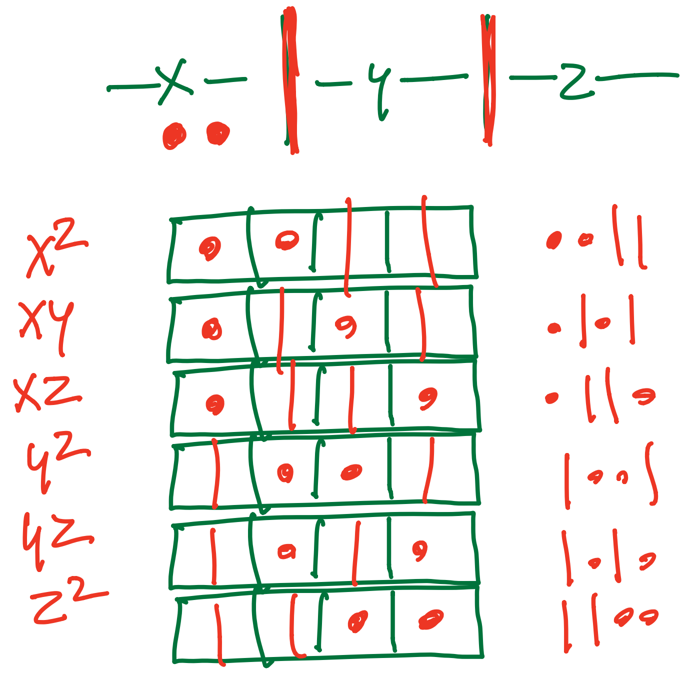
1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$\binom{4}{2}=6$



deg 3 in x, y, z

2 dividers $\quad x|y|z$

3 balls $\quad \bullet\bullet\bullet$

$\bullet\bullet|\bullet| = x^2y$

$x \quad y \quad z$

$\binom{5}{2} = \frac{5\cdot \cancel{4}^{2}}{\cancel{2}\cdot 1} = 10$

$xyz \quad \bullet|\bullet|\bullet$

Jan 14 Thurs

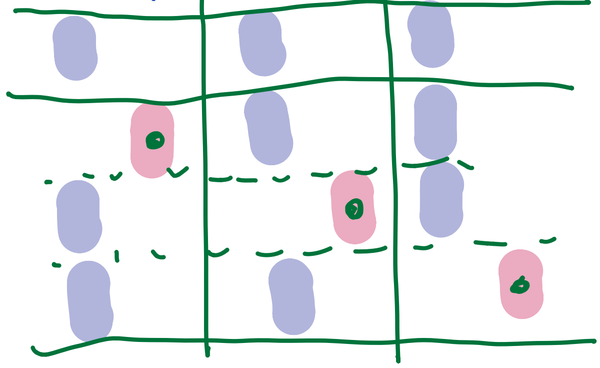
1 $\quad (x+y)^0 = 1$

1 1 $\quad (x+y)^1 = x+y$

1 2 1 $\quad (x+y)^2 = x^2+2xy+y^2$

1 3 3 1 $\quad (x+y)^3 = x^3+3x^2y+3xy^2+y^3$

$(x+y)^3 = (x+y)(x+y)(x+y)$



x^3

x^2y

$\binom{3}{1}x^2y$

$$\boxed{(x+y)^n = \underbrace{\binom{n}{0}}_{=1}x^n + \underbrace{\binom{n}{1}}_{=n}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots \binom{n}{n}y^n}$$

binomial theorem

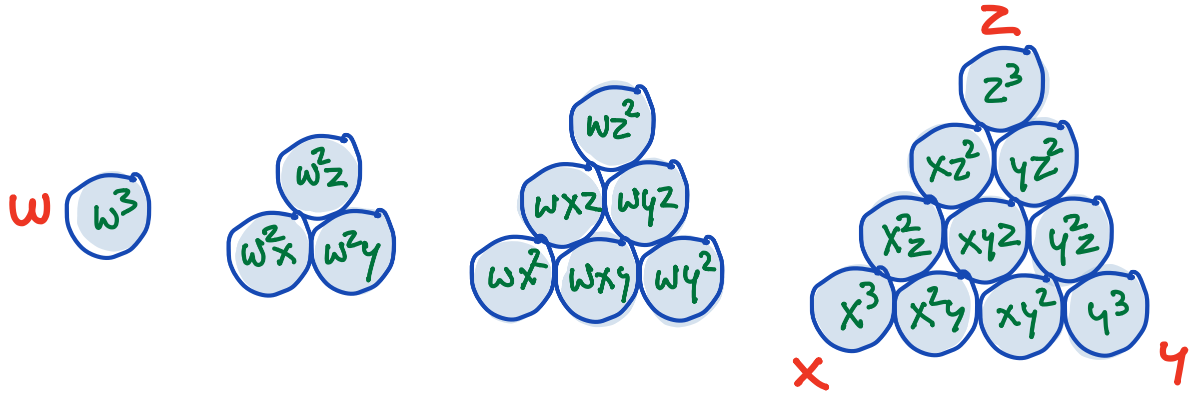
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\begin{array}{c} 1 \\ 1-1=0 \\ 1-2+1=0 \\ 1-3+3-1=0 \\ \boxed{1-4+6-4+1} \end{array}$$

$(x+y)^n$

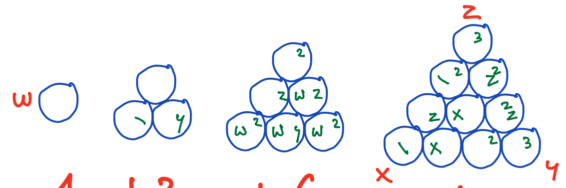
$x=y=1$: $\quad 2^n = \boxed{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}}$

$x=1,\ y=-1$: $\quad 0 = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\dots\pm\binom{n}{n}$



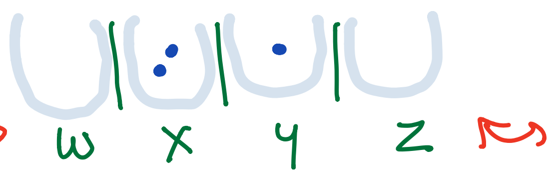
monomials of deg 3 in 4 variables w, x, y, z

d $\quad n \quad \binom{n-1+d}{d} \quad \binom{6}{3}$ ✓



$1 + 3 + 6 + 10 = 20$ ✓ monomials

x^2y ↔



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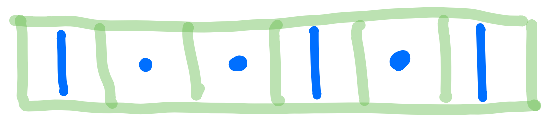


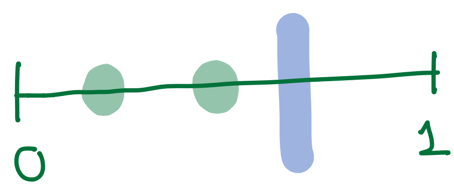
$\binom{6}{3} = \frac{\cancel{6}\cdot 5\cdot 4}{\cancel{3}\cdot\cancel{2}\cdot 1} = 20$ ✓

Jerry Tersoff

AB|	1/4
A|B B|A	1/2
|AB	1/4

Bose-Einstein

••|	1/3
•|•	1/3
|••	1/3



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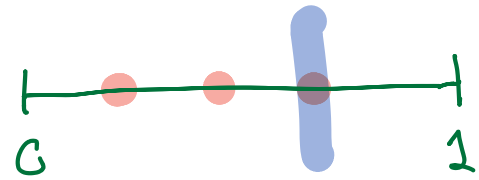
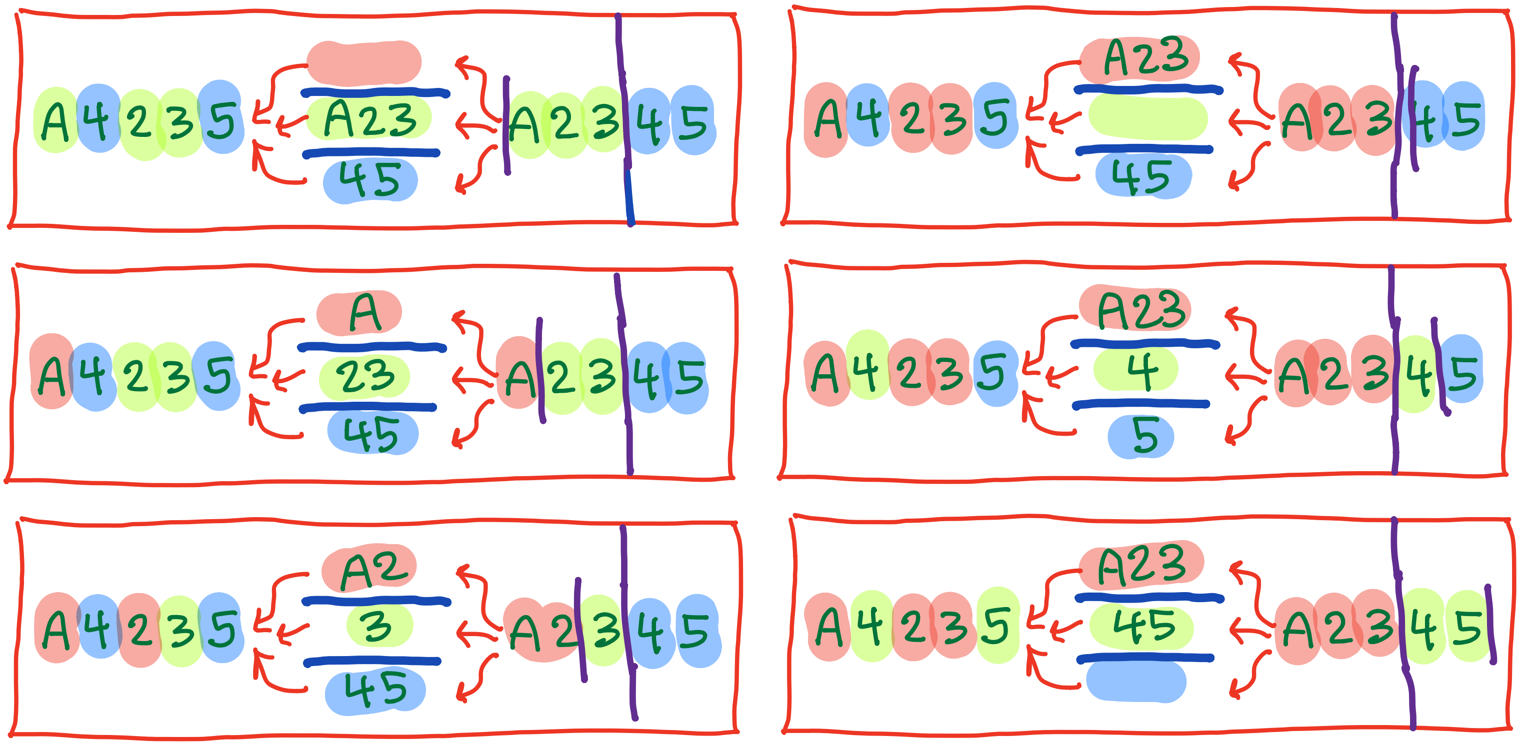


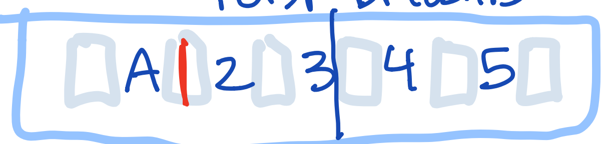


A4235

|A|4|2|3|5|

Persi Diaconis

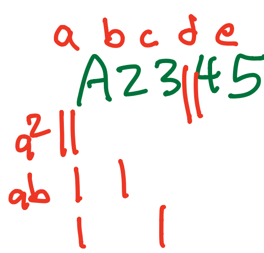
$\binom{6}{1}$



d^2

|A|2|3|4|5|

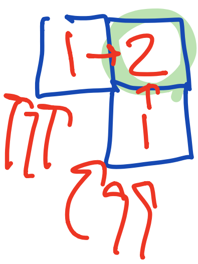
$\frac{A2}{\frac{3}{45}}$

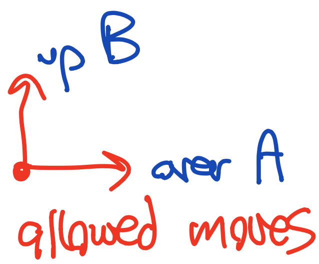


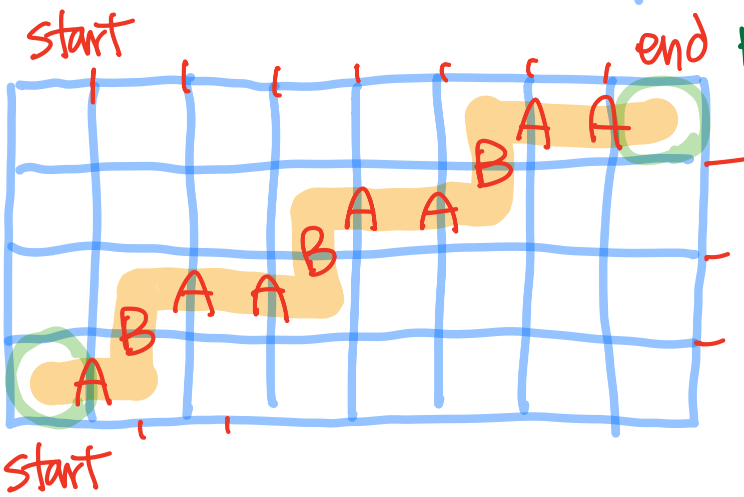
end

1	4	10	20	35	56	84	120
1	3	6	10	15	21	28	36
1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1

start







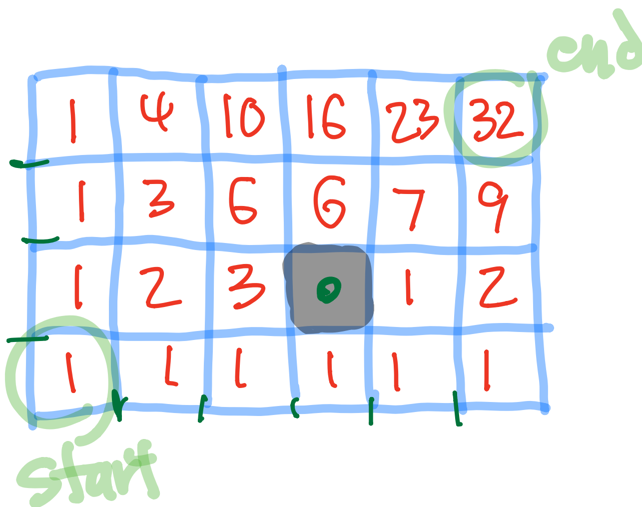
How many paths start to end

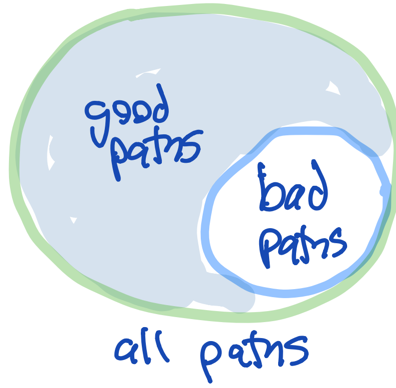
7 A's 3 B's 10 all

ABAABAABAA

$\binom{10}{3} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 120$ ✓

Inclusion-Exclusion





all − bad = good

$$\binom{8}{3} - \binom{4}{1}\binom{4}{2}$$

$$\frac{8\cdot 7\cdot \cancel{6}}{\cancel{3}\cdot\cancel{2}\cdot 1} - \frac{4}{1}\cdot\frac{\cancel{4}^{2}\cdot 3}{\cancel{2}}$$

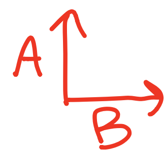
$$56 - 24 = 32$$

Tue Jan 19

end

1	6	18	41	41	67	132	254
1	5	12	23		26	65	122
1	4	7	11	17	26	39	57
1	3	3	4	6	9	13	18
1	2		1	2	3	4	5
1	1	1	1	1	1	1	1

start



Casting out nines?

$\mathbb{Z}/n\mathbb{Z}$ integers mod n $\quad n \equiv 0$

$\mathbb{Z}/3\mathbb{Z}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

mod 9 $\quad 10 = 1+9 = 1$

$100 = 1+99 = 1+9\cdot 11 = 1$

$1356 \equiv \cancel{1+3+5}+6 = 6$

end

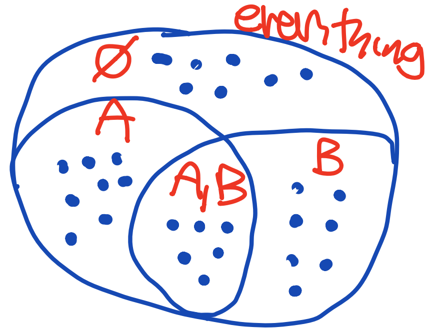
1	6	0	5	5	4	6	2
1	5	3	5		8	2	5
1	4	7	2	8	8	3	3
1	3	3	4	6	0	4	0
1	2		1	2	3	4	5
1	1	1	1	1	1	1	1

start

25 $\boxed{6 \mid 6 \mid 4 \mid 3 \mid 6}$

7 $\quad 0$

1	6	18	41	41	67	132	254 (end)
1	5	12	23	B	26	65	122
1	4	7	11	17	26	39	57
1	3	3	4	6	9	13	18
1	2	A	1	2	3	4	5
1 (start)	1	1	1	1	1	1	1



can compute easily

$\geq\emptyset$
$\geq A$
$\geq B$
$\geq AB$

want

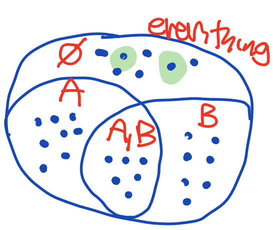
$\emptyset$
A
B
AB

$$\geq\emptyset = \emptyset + A + B + AB$$
$$\geq A = A + AB$$
$$\geq B = B + AB$$
$$\geq AB = AB$$

$$\begin{bmatrix} \geq\emptyset \\ \geq A \\ \geq B \\ \geq AB \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \emptyset \\ A \\ B \\ AB \end{bmatrix}$$

$$\emptyset = \geq\emptyset - \geq A - \geq B + \geq AB$$
$$A = \geq A - \geq AB$$
$$B = \geq B - \geq AB$$
$$AB = \geq AB$$

$$\begin{bmatrix} \emptyset \\ A \\ B \\ AB \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \geq\emptyset \\ \geq A \\ \geq B \\ \geq AB \end{bmatrix}$$

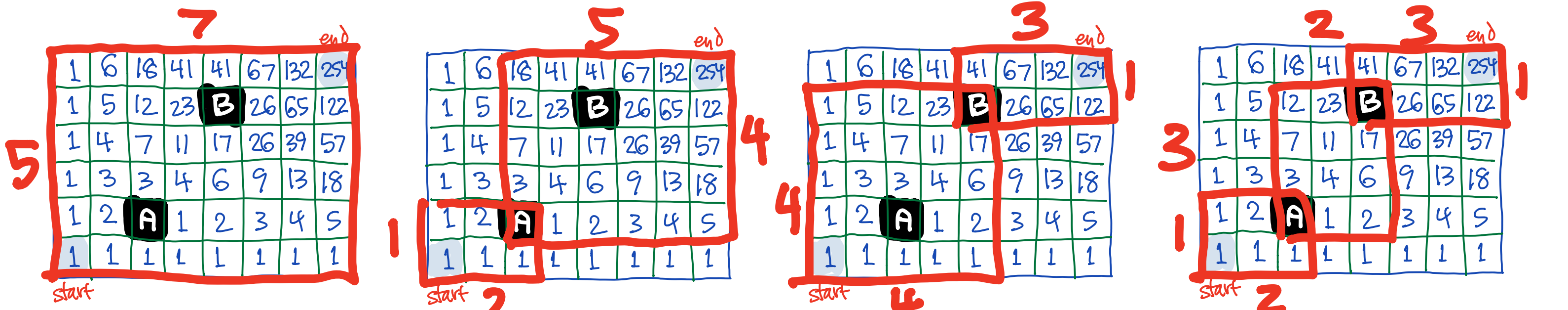


	$\geq\emptyset$	$-\geq A$	$-\geq B$	$+\geq AB$	
$\emptyset$	1				1
A	1	−1			0
B	1		−1		0
AB	1	−1	−1	1	0

ABC $1 \mid \underbrace{-1 \;\; -1 \;\; -1} \mid \underbrace{1 \;\; 1 \;\; 1} \mid -1$

1
1 1
1 2 1
$1-3+3-1=0$
1 4 6 4 1

$(1-1)^n = 0$



$\geq \emptyset$ $\qquad \geq A$ $\qquad \geq B$ $\qquad \geq AB$

$$\binom{12}{5} - \binom{3}{1}\binom{9}{4} - \binom{8}{4}\binom{4}{1} + \binom{3}{1}\binom{5}{3}\binom{4}{1}$$

$$\frac{12\cdot 11\cdot \cancel{10}^{\,3}\cdot \cancel{9}^{\,2}\cdot \cancel{8}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} - \frac{\cancel{3}\cdot 9\cdot \cancel{8}\cdot 7\cdot 6}{1 \quad \cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} - \frac{\cancel{8}\cdot 7\cdot \cancel{6}^{\,2}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}\,\frac{4}{1} + \frac{3}{1}\,\frac{5\cdot 4\cdot \cancel{3}}{\cancel{3}\cdot\cancel{2}\cdot 1}^{\,2}\,\frac{\cancel{4}}{1}$$

12·66 $\qquad$ 9·42

660 $\qquad$ 420−42

132

$$\boxed{254} = \boxed{792} - \boxed{378} - \boxed{280} + \boxed{120}$$

casting out nines $\quad 0 \qquad 0 \qquad -1 \qquad +3 \qquad = 2$

254
= 2+5+4
= 2 ✓

Exercise: How many integers in 1..60
are not divisible by 2, 3, or 5 ?
A B C

Hat check problem

How many permutations are fixed point free

n=1

~~1~~

n=2

~~12~~
21

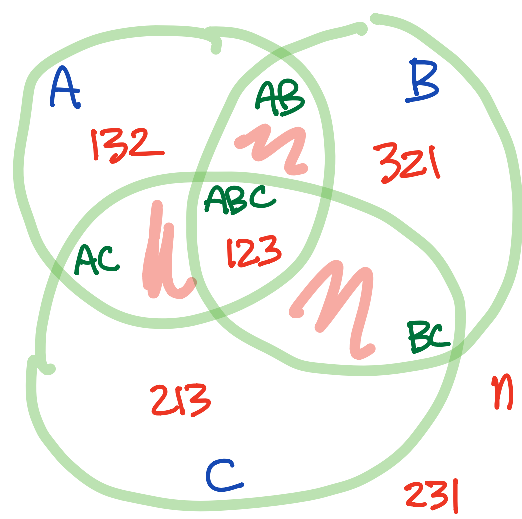
n=3

~~123~~ ABC
~~132~~ A
~~213~~ C
231 ∅
312 ∅
~~321~~ B

A B C

2134

AB C D



n=3

231
312

work with arbitrary n

$$\Sigma\emptyset \;-\; \Sigma A - \Sigma B - \Sigma C \;+\; \Sigma AB + \Sigma AC + \Sigma BC \;-\; \Sigma ABC$$

$$n! \;-\; \binom{n}{1}(n-1)! \;+\; \binom{n}{2}(n-2)! \;-\; \binom{n}{3}(n-3)!$$

$$\binom{n}{2} = \frac{n(n-1)}{2\cdot 1}$$

$$\binom{n}{3}(n-3)! = \frac{n!}{3!}$$

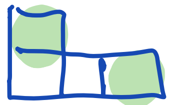
$$n! \;-\; \frac{n!}{1} \;+\; \frac{n!}{2!} \;-\; \frac{n!}{3!} \;+ \ldots$$

$$n!\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} \ldots\right) = \left[n!/e\right] \qquad 6/27$$

$$e^x = 1 + x + x^2/2! + x^3/3!$$

$$e^{-1} = 1 - 1 + 1/2 - 1/6 + 1/24 \cdots$$

Thurs Jan 21



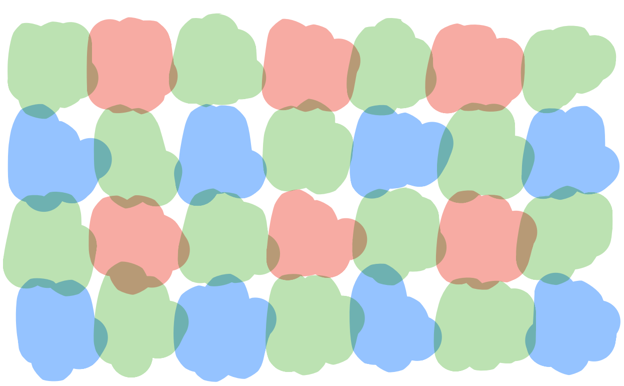
Prime sieve

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48

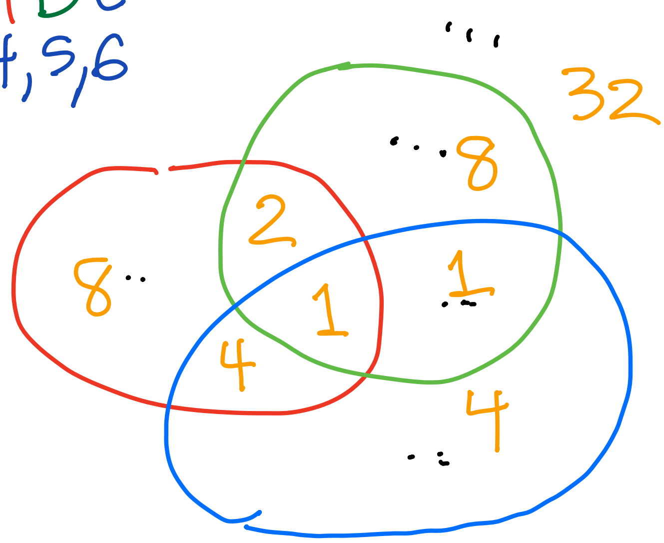
Relatively prime

divisibility by A B C
4, 5, 6

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60



A B C
4, 5, 6



$\geq A \quad 15 = 60/4$

$\geq B \quad 12 = 60/5$

$\geq C \quad 10 = 60/6$

$\Sigma AB \quad 3 = 60/20 \qquad 20 = 4 \cdot 5 \qquad \text{lcm}(4,6)$

$\Sigma AC \quad 5 = 60/12 \qquad 12 = \cancel{\ldots}$

$\Sigma BC \quad 2 = 60/30 \qquad 30 = 5 \cdot 6$

$\Sigma ABC \quad 1 = 60/60 \qquad 60 = \text{lcm}(4,5,6)$

$$\phi = 60 - (15+12+10) + (3+5+2) - 1$$

$$60 - 37 + 10 - 1 = \boxed{32}$$

$$60\left(1 - \tfrac{1}{4} - \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{20} + \tfrac{1}{12} + \tfrac{1}{30} - \tfrac{1}{60}\right)$$

1..30 not divisible by 2 or 3

$$30 - 15 - 10 + 5 = 10$$

$0! = 1$

$$= 30\left(1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{6}\right)$$

$$30\left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right) = 30 \cdot \tfrac{1}{\cancel{2}} \cdot \tfrac{\cancel{2}}{3} = 10$$

Euler's totient function $\varphi(n)$ "phi"

integers < n, relatively prime to n

(include 1)

$20 = 2 \cdot 2 \cdot 5$

$2, 5$

$A \; B$

$\varphi(20) = 1, 3, 7, 9, 11, 13, 17, 19$

$(1-A)(1-B)$

$= 1 - A - B + AB$

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

$$20\left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{5}\right)$$

$$2 \cdot 20 \cdot \tfrac{1}{\cancel{2}} \cdot \tfrac{4}{5} = 40/5 = 8 \quad ✓$$

primes for m,n

if m,n relatively prime

p,q prime factors of m

r,s prime factors of n

$$\varphi(m) \quad * \quad \varphi(n) \quad = \quad \varphi(mn)$$

$$m\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) * n\left(1-\frac{1}{r}\right)\left(1-\frac{1}{s}\right) = mn\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)\left(1-\frac{1}{s}\right)$$

(after class...)

$12 = 2\cdot2\cdot3 \qquad 5=5 \qquad 60 = 2\cdot2\cdot3\cdot5$

$\varphi(12)$

$\varphi(5)=4$

2 3 5

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60

$\varphi(5) = 5(1-1/5) = 5\cdot 4/5 = 4$ ✓

1 2 3 4 5

$\varphi(12) = 12(1-1/2)(1-1/3) = 12\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3} = 4$

$\varphi(60) = 4\cdot4 = 16$

1	2	3	4	5	6
7	8	9	10	11	12

$p_1, \dots, p_j$ prime factors of m

$q_1, \dots, q_k$ prime factors of n

$$\varphi(m) = m \prod_{i=1}^{j} (1-1/p_i)$$

$$\varphi(n) = n \prod_{i=1}^{k} (1-1/q_i)$$

$$\varphi(mn) = mn \prod_{i=1}^{j} (1-1/p_i) \prod_{i=1}^{k} (1-1/q_i)$$

Jan 26

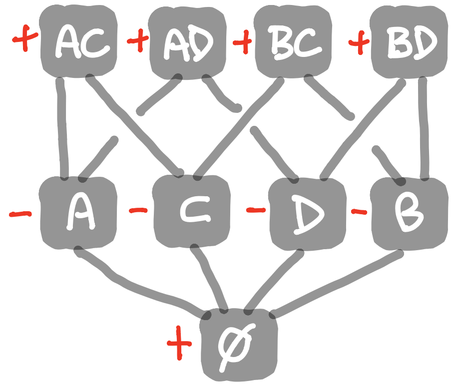
Möbius inversion

poset

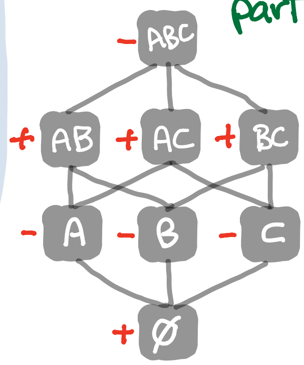
end

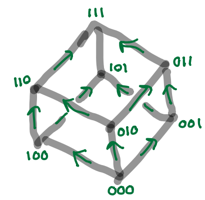
1	2	2	2	4
1	1	C	0	2
1	A	0	D	2
1	2	B	1	2
1	1	1	1	1

start



partially ordered set





(Special case:
Inclusion-Exclusion
looks like an n-cube)

+ ∅ $\binom{8}{4} = \frac{\cancel{8}^{2}\cdot 7\cdot \cancel{6}\cdot 5}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 70$

− A $\binom{3}{1}\binom{5}{3} = 3\cdot 10 = 30$

− B $\binom{3}{2}\binom{5}{2} = 3\cdot 10 = 30$

− C $\binom{5}{2}\binom{3}{2} = 30$

− D $\binom{5}{3}\binom{3}{1} = 30$

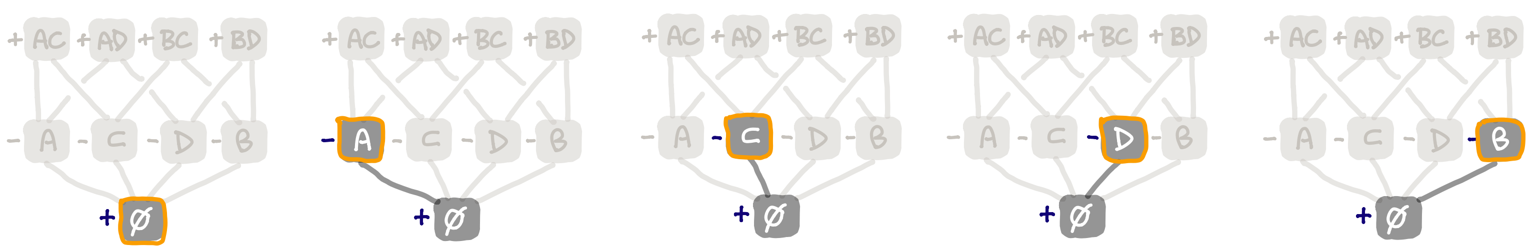
+ AC $\binom{3}{1}\binom{2}{1}\binom{3}{2} = 18$

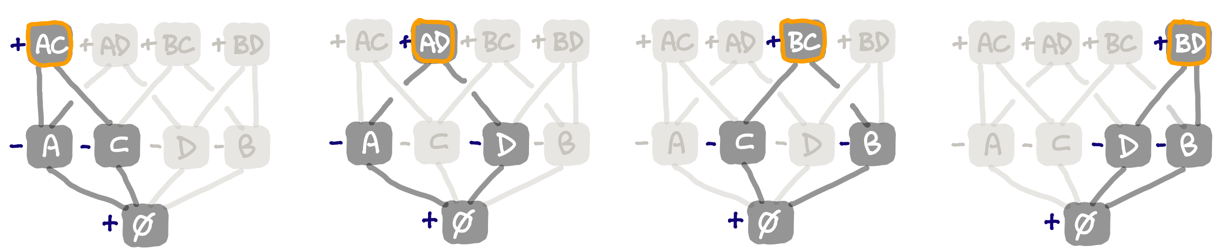
+ AD $\binom{3}{1}\binom{2}{2}\binom{3}{1} = 9$

+ BC $\binom{3}{2}\binom{2}{0}\binom{3}{2} = 9$

+ BD $\binom{3}{2}\binom{2}{1}\binom{3}{1} = 18$

$\boxed{70 - 4\cdot 30 + 6\cdot 9 = 70 - 120 + 54 = 4}$

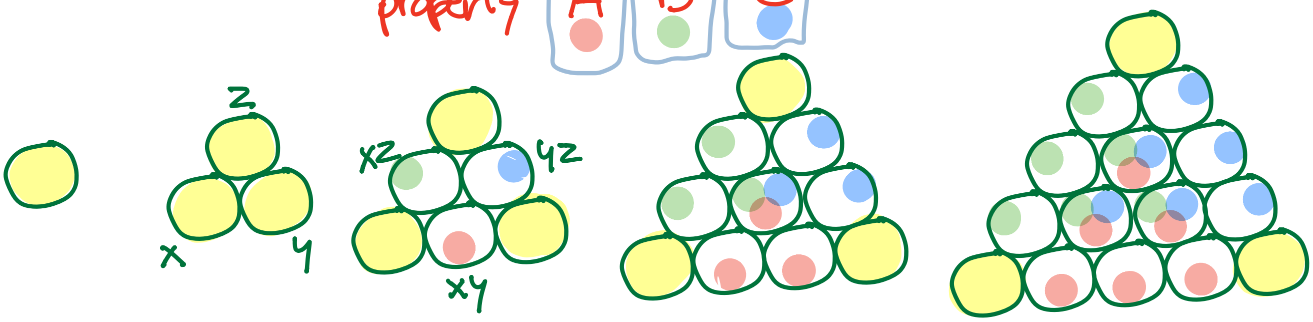




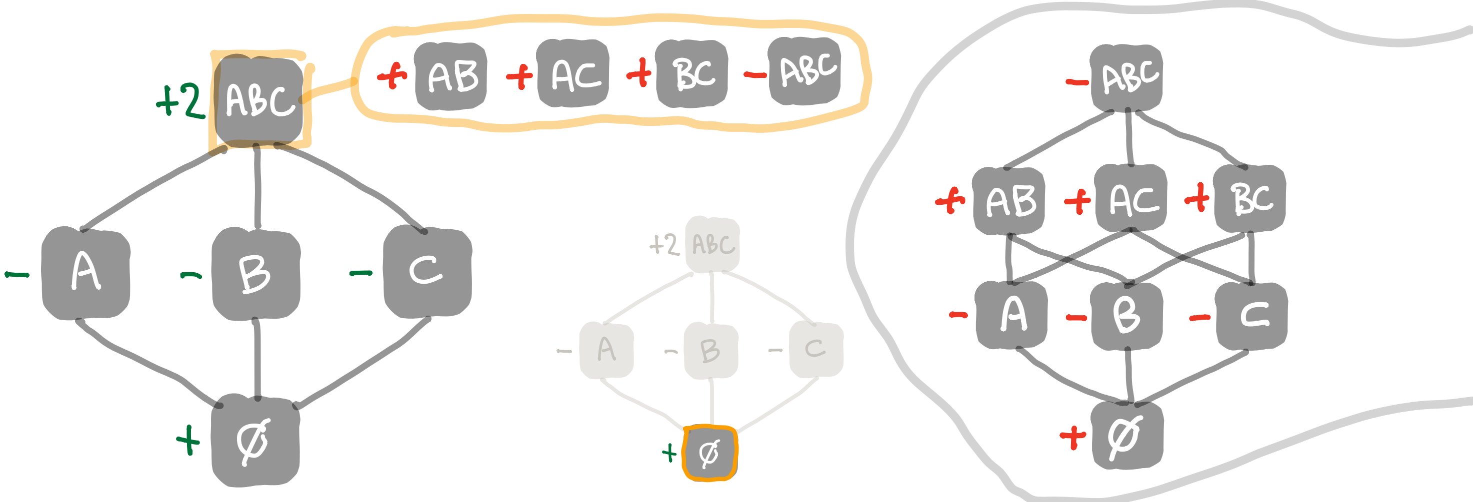
How many monomials of degree 4 in x, y, z are not divisible by any of xy, xz, yz?

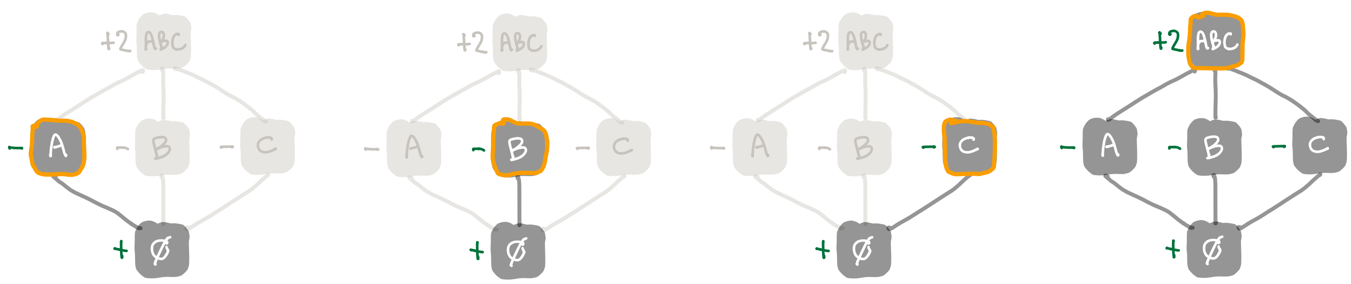
Soma cubes

property A B C



$\emptyset$	$-A$	$-B$	$-C$	$+AB$	$+AC$	$+BC$	$-ABC$		
1	xy	xz	yz	xyz	xyz	xyz	xyz		
15	-6	-6	-6	$+3$	$+3$	$+3$	-3	$=$	$\boxed{3}$



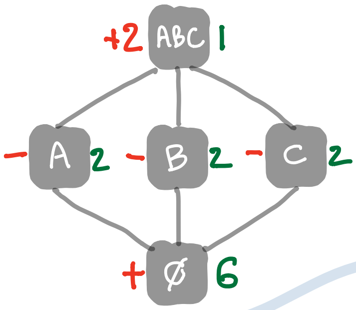


Can't have two of A, B, C without all three.
Where have we seen this before?

Permutations of 1, 2, 3

A = 1 in 1st position
B = 2 in 2nd position
C = 3 in 3rd position

1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1



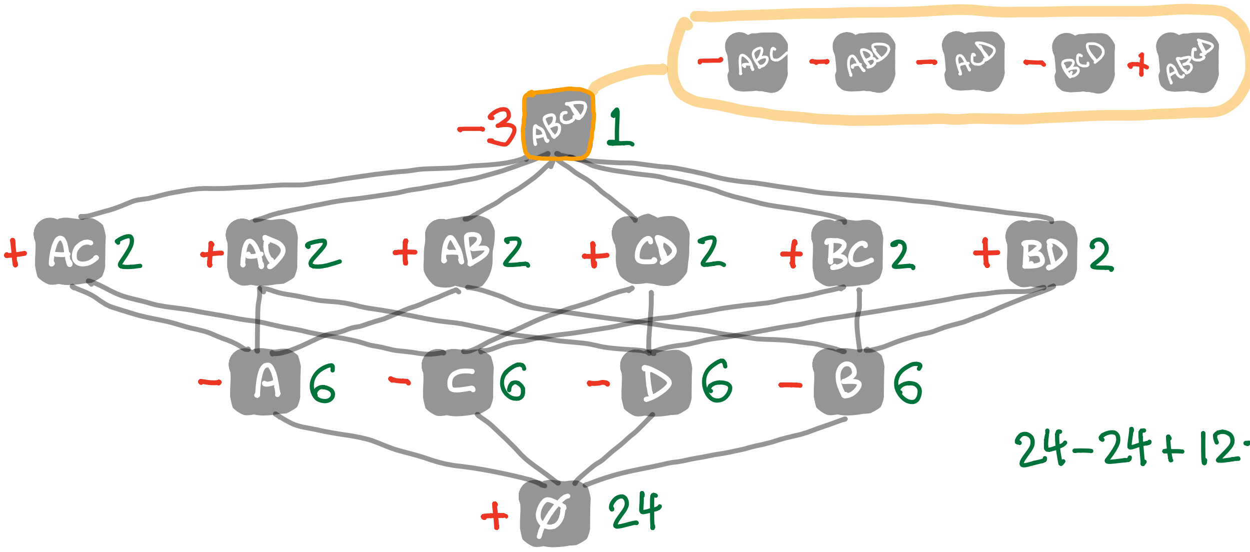
$$\varnothing - A - B - C + 2ABC$$

$$6 - 2 - 2 - 2 + 2\cdot 1 = \boxed{2}$$

A B C D

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 3 1 4	3 2 1 4	4 2 1 3
1 3 4 2	2 3 4 1	3 2 4 1	4 2 3 1
1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

$\boxed{9}$



$$24 - 24 + 12 - 3 = \boxed{9}$$

How many permutations have no adjacent ascending pairs?

A B C
1 2 3 4

1	1 2	1 2 3	1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
	2 1	1 3 2	1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
		2 1 3	1 3 2 4	2 3 1 4	3 2 1 4	4 2 1 3
		2 3 1	1 3 4 2	2 3 4 1	3 2 4 1	4 2 3 1
		3 1 2	1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
		3 2 1	1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

$$\varnothing - A - B - C + AB + AC + BC - ABC$$

$$24 - (6+6+6) + (2+2+2) - 1 = \boxed{11}$$

✓

$$n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \ldots$$

$$(n-1)!\left[n - (n-1) + \frac{n-2}{2} - \frac{n-3}{6} \ldots\right]$$

$n=1 \qquad 0!\,[1] = 1$

$n=2 \qquad 1!\,[2-1] = 1$

$n=3 \qquad 2!\,[3-2+\frac{1}{2}] = 3$

$n=4 \qquad 3!\,[4-3+1-\frac{1}{6}] = 11$

How many permutations have no adjacent pairs, ascending or descending?

A B C

1 2 3 4

1	12	123	1234	2134	3124	4123
	21	132	1243	2143	3142	4132
		213	1324	2314	3214	4213
		231	1342	2341	3241	4231
		312	1423	2413	3412	4312
		321	1432	2431	3421	4321

$$(n-1)!\left[n-2\left((n-1)+\frac{n-2}{2}-\frac{n-3}{6}\cdots\right)\right]$$? a quick guess...

$n=1 \qquad 0!\,[1] = 1$

$n=2 \qquad 1!\,[2-2\cdot 1] = 0$

$n=3 \qquad 2!\,[3-2(2-\frac{1}{2})] = 0$

$n=4 \qquad 3!\,[4-2(3-1+\frac{1}{6})] = -2$?

$\emptyset - A - B - C + AB + AC + BC - ABC$

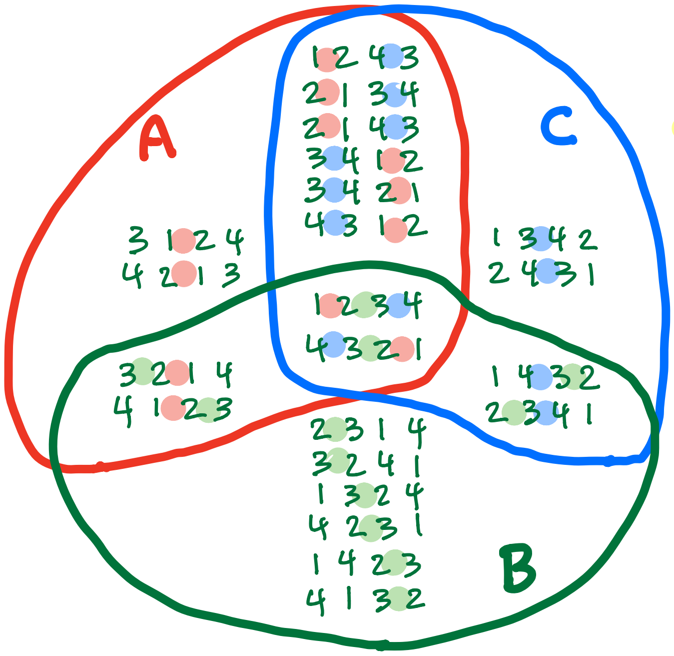
$24-(12+12+12)+(4+4+4) \quad -2$

$24-36+12-2 = \boxed{-2}$?

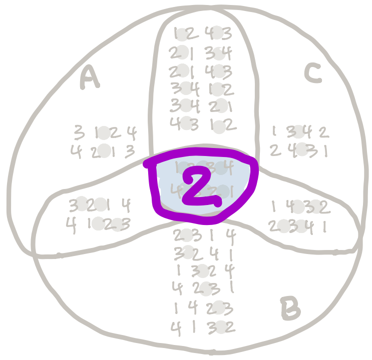
No. Too fast...

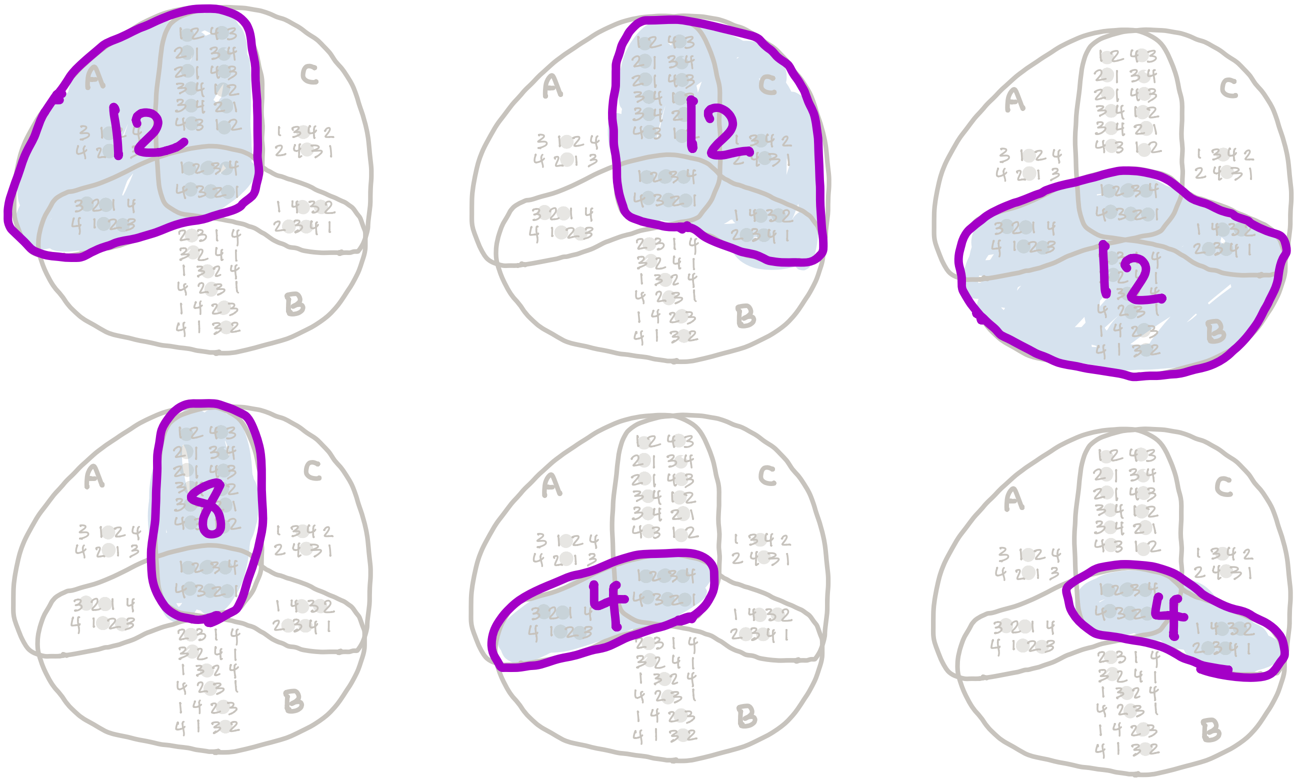
Check work.

A B C
1 ● 2 ● 3 ● 4



2413
3142





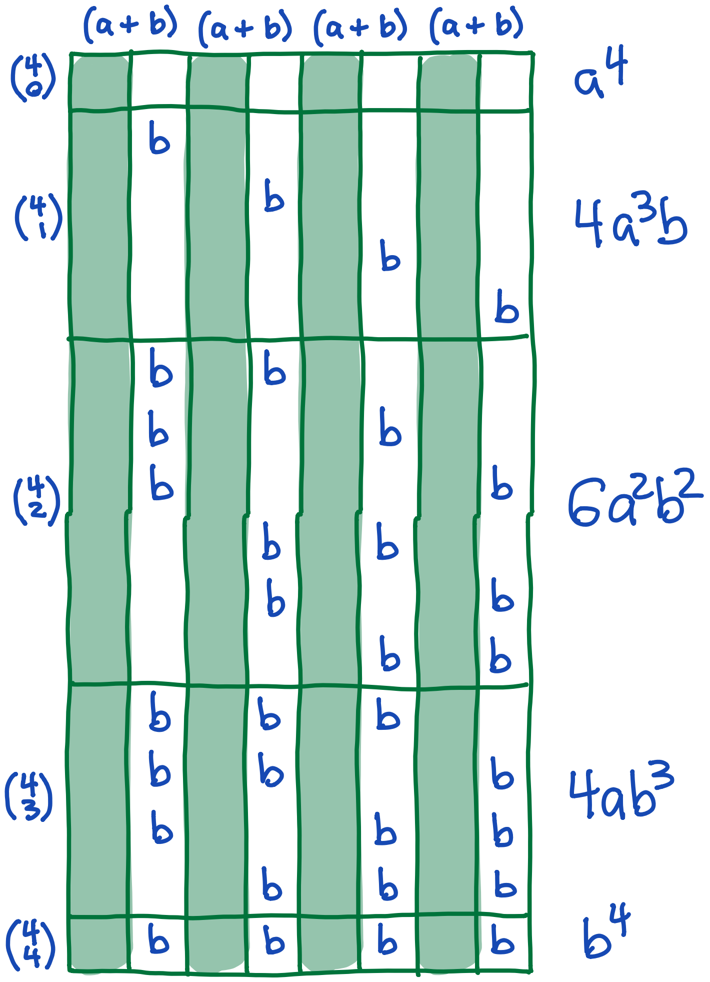
$\emptyset - A - B - C + AB + AC + BC - ABC$

$24 - (12+12+12) + (4+8+4) \quad - 2$

$24 - 36 + 16 - 2 \; = \; \boxed{2}$ ✓

Generating functions

Prototype: Binomial theorem



$$(a+b)^4 =$$
$$\binom{4}{0}a^4+\binom{4}{1}a^3b+\binom{4}{2}a^2b^2+\binom{4}{3}ab^3+\binom{4}{4}b^3$$

4 terms in product
2 choose b
rest choose a

General pattern:
Algebra does a combinatorial dance we want to harness.

Monomials in two variables

$$(1+x+x^2+x^3+x^4+\dots)\cdot$$
$$(1+y+y^2+y^3+y^4+\dots)$$
$$= 1 + (x+y)$$
$$+ (x^2+xy+y^2)$$
$$+ (x^3+x^2y+xy^2+y^3)$$
$$+ \dots$$

convolution FFT

+	1	y	y^2	y^3	y^4	...
	x	xy	xy^2	xy^3	xy^4	
	x^2	x^2y	x^2y^2	x^2y^3	x^2y^4	
	x^3	x^3y	x^3y^2	x^3y^3	x^3y^4	
	x^4	x^4y	x^4y^2	x^4y^3	x^4y^4	
	⋮					

Geometric series

$$1+x+x^2+x^3+x^4+\dots = \frac{1}{1-x}$$
$$1+y+y^2+y^3+y^4+\dots = \frac{1}{1-y}$$

So product is

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)$$

Recall proof:

$$(1+x+x^2+x^3+x^4+\dots)(1-x)$$
$$\begin{aligned} = \; & 1+x+x^2+x^3+x^4+\dots \\ & \;\;\; -x-x^2-x^3-x^4 \;\; \dots \\ \hline & 1 \end{aligned}$$

setting $x=y=t$, product is

$$\left(\frac{1}{1-t}\right)\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2}$$

$$\boxed{\frac{1}{(1-t)^2}} = 1+2t+3t^2+4t^3+\dots = \boxed{\sum_{n=0}^{\infty} f(n)t^n}$$

n	0	1	2	3	4	...
$f(n)$	1	2	3	4	5	...

These are same thing!

These are same thing!

$f(n)$ = # monomials of degree n in x, y

Monomials in three variables

$g(n)$ = # monomials of degree n in x, y, z

$f(n) = 1 = \binom{n}{0}$ $\quad x$

$f(n) = n+1 = \binom{n+1}{1}$ $\quad x, y$

$f(n) = \dots = \binom{n+2}{2}$ $\quad x, y, z$

$$(1+x+x^2+x^3+\dots)(1+y+y^2+y^3+\dots)(1+z+z^2+z^3+\dots)$$
$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)\left(\frac{1}{1-z}\right)\Bigg|_{x=y=z=t} = \boxed{\frac{1}{(1-t)^3}} = \boxed{\sum_{n=0}^{\infty} g(n)t^n}$$

n	0	1	2	3	4	...
$g(n)$	1	3	6	10	15	...

check: $\quad 1/(1-t)^2$

×	1	2	3	4	...
1	1	2	3	4	
1	1	2	3	4	
1	1	2	3	4	
1	1	2	3	4	
⋮					

$\frac{1}{1-t}$

$g(n) = \binom{n+2}{2}$

n balls
2 dividers

x^3 $\quad$ ●●●|| $\quad$ (x y z)

x^2z $\quad$ ●●||● $\quad$ (x y z)

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
```

← row n

choose 2

$$\boxed{\text{we prefer } \frac{1}{(1-t)^3} \text{ to } \binom{n+2}{2}}$$

Generating function:

For any function $f: \mathbb{N} \to \mathbb{Z}$ (or $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}, \ldots$)

consider instead the series $\boxed{\sum_{n=0}^{\infty} f(n)\, t^n}$

Compare Laplace transform from ODE's

$$f(t) \implies F(s) = \boxed{\int_0^{\infty} f(t)\, e^{-st}\, dt}$$

$\sum_{n=0}^{\infty}$	$\int_0^{\infty} dt$

sum

$f(n)$	$f(t)$

use function

t^n	$(e^{-s})^t$

take power

William Feller

An Introduction to Probability Theory and its Applications

Volumes 1,2

Straddles these worlds

Cult status book

Example: Making change for 20¢ using 1¢ 2¢ 5¢ (a, b, c)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1¢	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2¢	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11
5¢	1	1	2	2	3	4	5	6	7	8	10	11	13	14	16	18	20	22	24	26	29
						1	1	2	2	3	4	5	6	7	8	10	11	13	14	16	

1111112
111122
11222
2222
1111111111

	0	2	4	6	8	10	12	14	16	18	20
0	0	2	4	6	8	10	12	14	16	18	20
5	5	7	9	11	13	15	17	19			
10	10	12	14	16	18	20					
15	15	17	19								
20	20										

$$\left.\begin{array}{c} 11 \\ 8 \\ 6 \\ 3 \\ 1 \end{array}\right\} \boxed{29}$$

all ways of getting within 20 using 2,5

finish with pennies

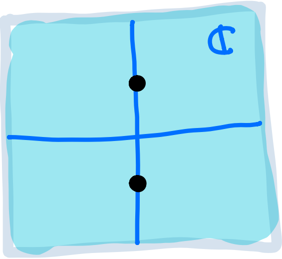
$$\left(\frac{1}{1-a}\right)\left(\frac{1}{1-b}\right)\left(\frac{1}{1-c}\right)\Bigg|_{\substack{a=t \\ b=t^2 \\ c=t^5}} = \boxed{\frac{1}{(1-t)(1-t^2)(1-t^5)}} = \ldots + \boxed{29t^{20}} + \ldots$$

Algebraic Geometry

Study geometry of zeros of polynomial systems of equations.

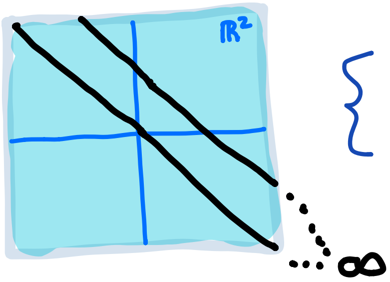
This is "bonus" material. It won't be on exams

Need zeros!



$x^2+1=0$ Fix: work with $\mathbb{C}$ not $\mathbb{R}$

$(x+i)(x-i)=0$ zeros $i,-i$



$\left\{\begin{array}{l} x+y=0 \\ x+y=1 \end{array}\right\}$ Fix: work with projective space of ratios

$1:-1:0$

$\mathbb{R}^1 = \{x\}$ $\quad \mathbb{P}^1 = \{x:y\}$ ratio of x to y

$x \longmapsto x:1$

$\infty \longmapsto 1:0$

$\mathbb{R}^2 = \{(x,y)\}$ $\quad \mathbb{P}^2 = \{x:y:z\}$

$(x,y) \longmapsto x:y:1$ All possible ratios

$\mathbb{P}^1$ at $\infty \longmapsto x:y:0$ $x:y$ are points at ∞

$\left\{\begin{array}{l} x+y=0 \\ x+y=1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} x+y=0 \\ x+y=z \end{array}\right\}$ "homogenize" using z

$1:-1:0$ is common solution at ∞

Ratios need homogeneous polynomials $\quad 1:-1:0 \approx 2:-2:0$ same ratio

all terms same degree $d \iff f(\lambda x,\lambda y,\lambda z) = \lambda^d f(x,y,z)$ (both vanish or neither does)

Integers mod p : $m \approx n$ if they differ by a multiple of p

$\{0,1,\ldots,p-1\}$

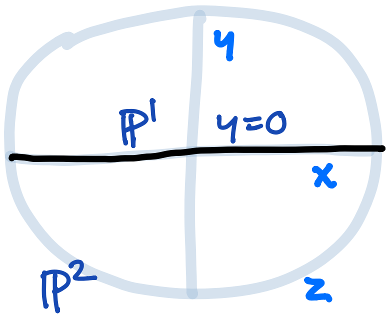
Polynomials mod a "variety" X (solution set) : $f \approx g$ if $f-g$ vanishes on X

Combinatorial examples

$\mathbb{P}^1$ ratios $x:y$
homogeneous polynomials in x, y

n	0	1	2	3	4	...
$f(n)$	1	2	3	4	5	...

$$\boxed{\frac{1}{(1-t)^2}}$$

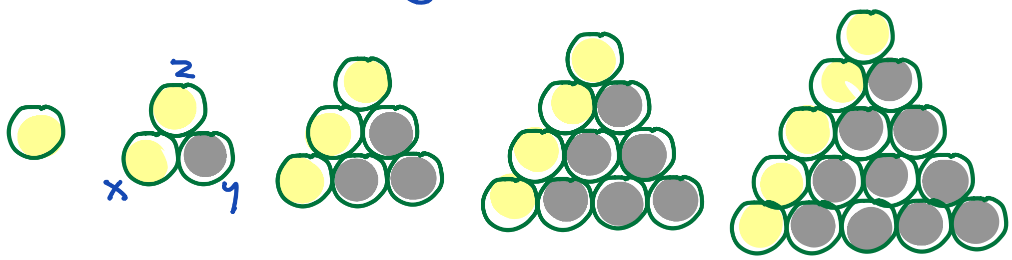


$\mathbb{P}^2$ ratios $x:y:z$
homogeneous polynomials in x, y, z

n	0	1	2	3	4	...
$g(n)$	1	3	6	10	15	...

$$\boxed{\frac{1}{(1-t)^3}}$$

What about $\mathbb{P}^1$ sitting inside $\mathbb{P}^2$ as $\{y=0\}$?



multiples of y, ≈ 0

n	0	1	2	3	4	...
$f(n)$	1	2	3	4	5	...

$$\boxed{\frac{1}{(1-t)^2}}$$

After modding by $X = \mathbb{P}^1$ defined by $y=0$, same answer.

Twisted cubic curve

$$f: \mathbb{R} \hookrightarrow \mathbb{R}^3$$
$$t \longmapsto (t, t^2, t^3)$$

use instead

$$g: \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$
$$s:t \longmapsto s^3 : s^2t : st^2 : t^3$$

$s=1$

$a \quad b \quad c \quad d$ variables

Equations

$$\left\{\begin{aligned} b^2 &= ac \\ bc &= ad \\ c^2 &= bd \end{aligned}\right\}$$

$(s^2t)^2 = (s^3)(st^2)$ $\qquad$ $21 + 21 = 30 + 12 = 42$ ✓

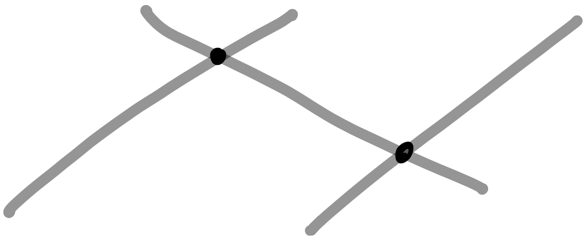
$(s^2t)(st^2) = (s^3)(t^3)$ $\qquad$ $21 + 12 = 30 + 03 = 33$ ✓

$(st^2)^2 = (s^2t)(t^3)$ $\qquad$ $12 + 12 = 21 + 03 = 24$ ✓

Monomials in a, b, c, d mod X (these equations)

1	a, b, c, d	$a^2, ab, ac, ad, bd, cd, d^2$	$a^3, \ldots$
		$b^2 \quad bc \quad c^2$ (under ac, ad, bd)	
1	4	7	10

$$1+4t+7t^2+10t^3+\dots = \frac{3}{(1-t)^2} - \frac{2}{1-t}$$



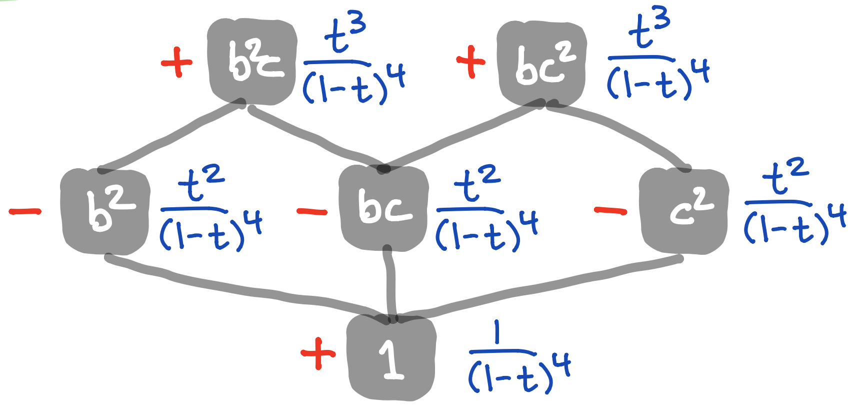
$3\mathbb{P}^1 - 2\mathbb{P}^0$

"size" of 3 lines − 2 points

A degenerate object like original curve

Leading terms give counting problem we've already studied:

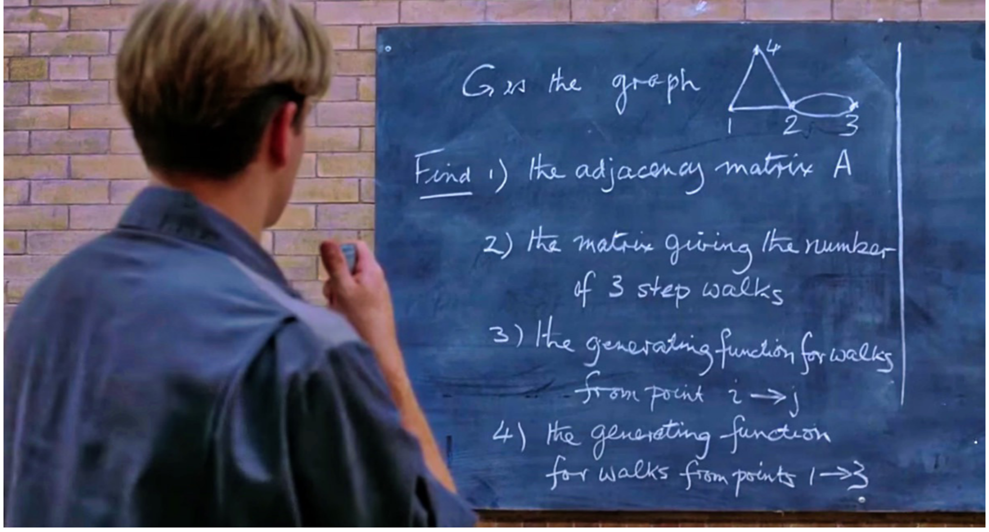
$\boxed{\text{How many monomials of degree } n \text{ in } a,b,c,d \text{ are not divisible by any of } b^2, bc, c^2\text{?}}$



$$\frac{1-3t^2+2t^2}{(1-t)^4} = \frac{3}{(1-t)^2} - \frac{2}{1-t} = \boxed{3\mathbb{P}^1 - 2\mathbb{P}^0}$$

n	0	1	2	3	4	...
$\frac{1}{(1-t)^4}$	1	4	10	20	35	...
$\frac{t^2}{(1-t)^4}$			1	4	10	...
$\frac{t^3}{(1-t)^4}$				1	4	...

$$\boxed{\sum_{n=0}^{\infty} g(n)t^n}$$

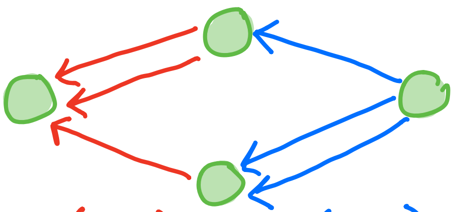


Feb 2

Good Will Hunting blackboard scene

(Easy if you know what to do...)

1 Matrix multiplication counts paths

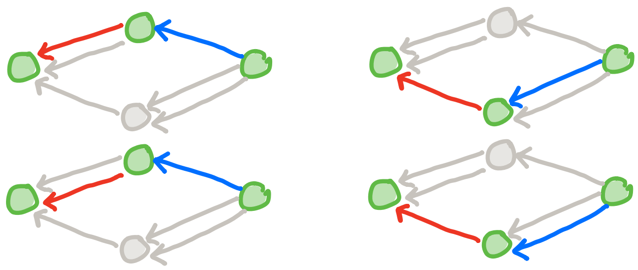


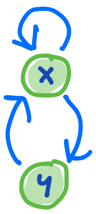
(2,1) (1,2)

dot product

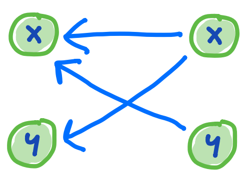
$(2,1)\cdot(1,2) = 2\cdot 1 + 1\cdot 2 = 4$

$2\cdot 1 \quad + \quad 1\cdot 2$



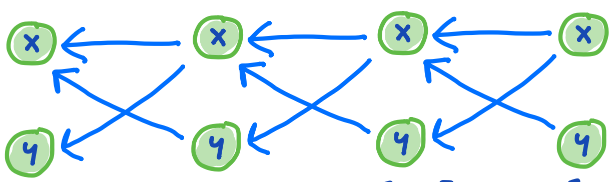


unfold as



$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \ \text{start} \\ \text{end} \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

one step



$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} = \begin{array}{cc} & \begin{array}{cc} x & y \end{array} \ \text{start} \\ \text{end} \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \end{array}$$

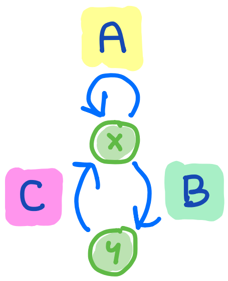
3 steps

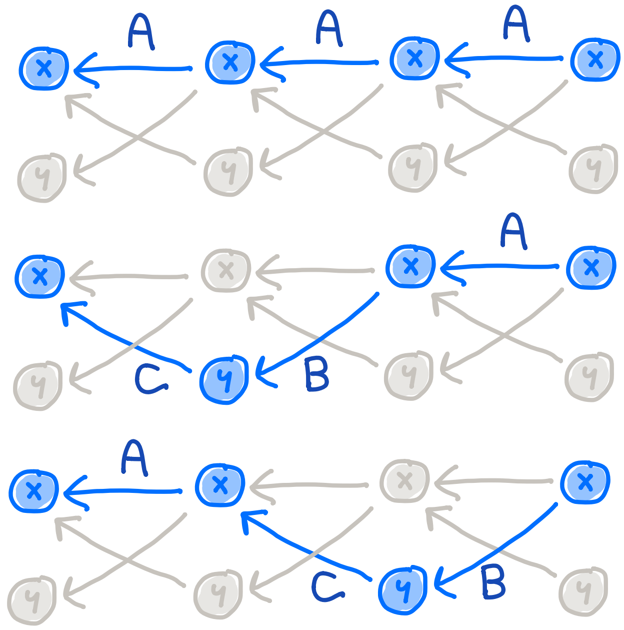
Right to left is function composition order:

$f(g(x)) \quad f \circ g$

$\Leftarrow \Leftarrow$

check:



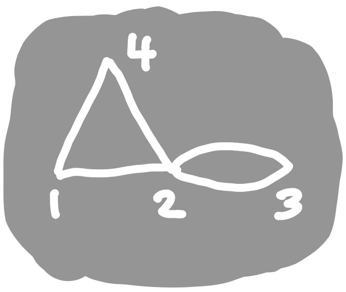


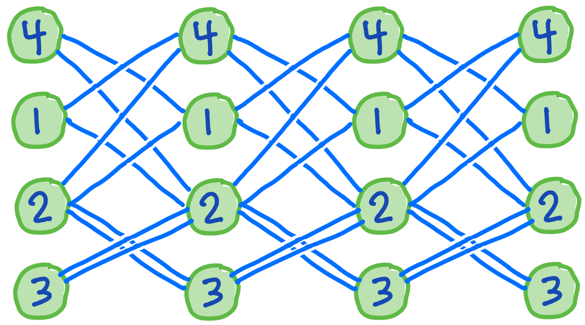
AAA

ABC

BCA

$$\begin{array}{c c} & \begin{array}{cc} x & y \end{array} \ \text{start} \\ \begin{array}{r} x \\ \text{end } y \end{array} & \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \end{array}$$





1)

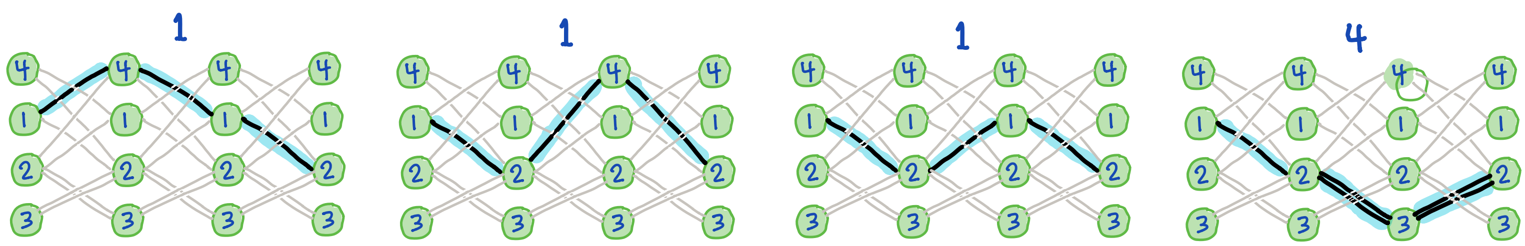
$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \ \text{right} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{left} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = M \end{array}$$

$$M^2 = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 6 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

2)

$$M^3 = \begin{bmatrix} 2 & 7 & 2 & 3 \\ 7 & 2 & 12 & 7 \\ 2 & 12 & 0 & 2 \\ 3 & 7 & 2 & 2 \end{bmatrix}$$

check: 7

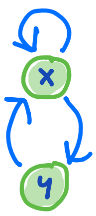


$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \ \text{right} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{left} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}
\qquad
\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \ \text{left} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{right} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \ \text{end} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{start} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}
\qquad
\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \ \text{start} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{end} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

row, column convention is arbitrary (and same if symmetric)

Generating functions



$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = M$$

M^n_{ij} = (i, j) entry of n^{th} power of M

= # paths of length n from i to j

$$\boxed{g_{ij}(t) = \sum_{n=0}^{\infty} M^n_{ij} t^n}$$

generating function, all n at once

Divide into cases:

$$g_{ij}(t) = \underbrace{\delta_{ij}}_{\substack{1 \text{ if } i=j \\ 0 \text{ if } i \neq j \\ (0 \text{ steps})}} + \underbrace{t\, g_{i1}(t) M_{1j}}_{\substack{\text{get to 1} \\ \text{step 1 to } j}} + \underbrace{t\, g_{i2}(t) M_{2j}}_{\substack{\text{get to 2} \\ \text{step 2 to } j}}$$

$f: \mathbb{N} \to \mathbb{Z}$

$f(n)$ counts something

$\sum_{n=0}^{\infty} f(n) t^n$

$$\underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{G} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + t \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{M}$$

$$G = I + tGM$$
$$GI - tGM = I$$
$$G(I - Mt) = I$$
$$G = (I - Mt)^{-1}$$

inverse matrix

Or sum geometric series

$$g = \sum_{n=0}^{\infty} a^n \Rightarrow g = \frac{1}{1-a}$$

$$G = \sum_{n=0}^{\infty} M^n t^n \Rightarrow G = (I - Mt)^{-1}$$

works for matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Big/ (ad-bc)$$

$$I - Mt = \begin{bmatrix} 1-t & -t \\ -t & 1 \end{bmatrix}$$

$$\boxed{(I - Mt)^{-1} = \begin{bmatrix} 1 & t \\ t & 1-t \end{bmatrix} \Big/ (1-t-t^2)}$$

How can we understand $\frac{1}{1-t-t^2}$?

$$(1 + \ \ t + \ \ t^2 + \ \ t^3 + \ \ t^4 + \ldots)(1 - t - t^2) = 1$$

figure out step by step as recurrence relation

$$
\begin{array}{r|l}
1 & 1 + 1t + 2t^2 + 3t^3 + 5t^4 + \dots \\
-t & \quad -1t - 1t^2 - 2t^3 - 3t^4 - 5t^5 + \dots \\
-t^2 & \qquad -1t^2 - 1t^3 - 2t^4 - 3t^5 - 5t^6 + \dots \\
\hline
 & 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{array}
$$

Each term is sum of previous two. $\boxed{\text{Fibonacci sequence}}$

n	0	1	2	3	4	5	6	...
m_{11}^n	1	1	2	3	5	8	13	...

$\boxed{1 \quad 1}$ ... $\boxed{1 \quad 1}$

$-t^2 \ -t + 1 = 0$

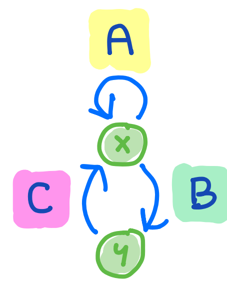
$1 - t - t^2 = 0$

now $\to 1 = t + t^2$

reach back 1 step   reach back 2 steps

How can we understand $\dfrac{1-t}{1-t-t^2}$?

n	0	1	2	3	4	5	6	...
$1/1-t-t^2$	1	1	2	3	5	8	13	...
$-t/1-t-t^2$		1	1	2	3	5	8	13 ...
	1	0	1	1	2	3	5	...



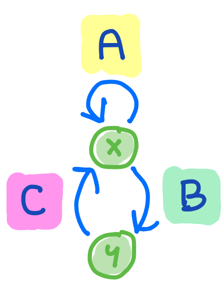
Huh? It sure looks like

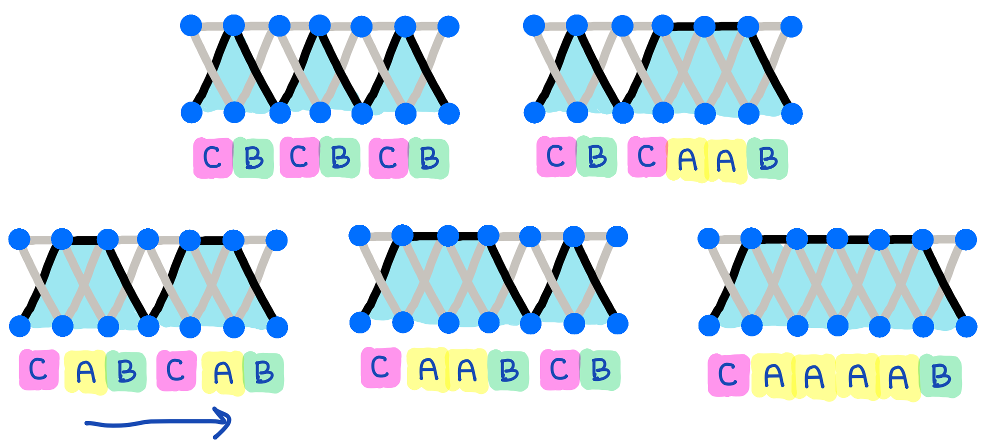
$$\boxed{\frac{1-t}{1-t-t^2} = 1 + \frac{t^2}{1-t-t^2}}$$

$$\frac{1-t-\cancel{t^2}+\cancel{t^2}}{1-t-t^2}$$ ☑ yes!

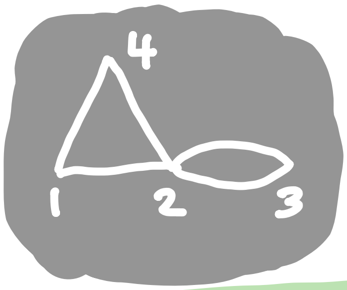
explain

Check: $n=6$, y to y: 5 paths





Adopt left to right convention from probability, CS



$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad I - Mt = \begin{bmatrix} 1 & -t & 0 & -t \\ -t & 1 & -2t & -t \\ 0 & -2t & 1 & 0 \\ -t & -t & 0 & 1 \end{bmatrix}$$

$$\boxed{3)\ (I - Mt)^{-1} = \frac{1}{1 - 7t^2 - 2t^3 + 4t^4} \begin{pmatrix} 1 - 5t^2 & t + t^2 & 2t^2 + 2t^3 & t + t^2 - 4t^3 \\ t + t^2 & 1 - t^2 & 2t - 2t^3 & t + t^2 \\ 2t^2 + 2t^3 & 2t - 2t^3 & 1 - 3t^2 - 2t^3 & 2t^2 + 2t^3 \\ t + t^2 - 4t^3 & t + t^2 & 2t^2 + 2t^3 & 1 - 5t^2 \end{pmatrix}}$$

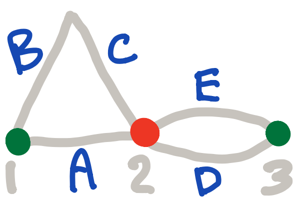
$$\boxed{4)\ (I - Mt)^{-1}_{13} = \frac{2t^2 + 2t^3}{1 - 7t^2 - 2t^3 + 4t^4}}$$

Using Mathematica (Can be done by hand)

Check: $1 - 7t^2 - 2t^3 + 4t^4 = 0$

$1 = 7t^2 + 2t^3 - 4t^4$

n	0	1	2	3	4	5	6		
$1/(1-7t^2-2t^3+4t^4)$	1	0	7	2	45	28	...		
	$\boxed{-4}$	$\boxed{2}$	$\boxed{7}$						
$2t^2$			2	0	14	4	90	56	...
$+\,2t^3$				2	0	14	4	90	56 ...
	0	0	2	2	14	18	94	146	...

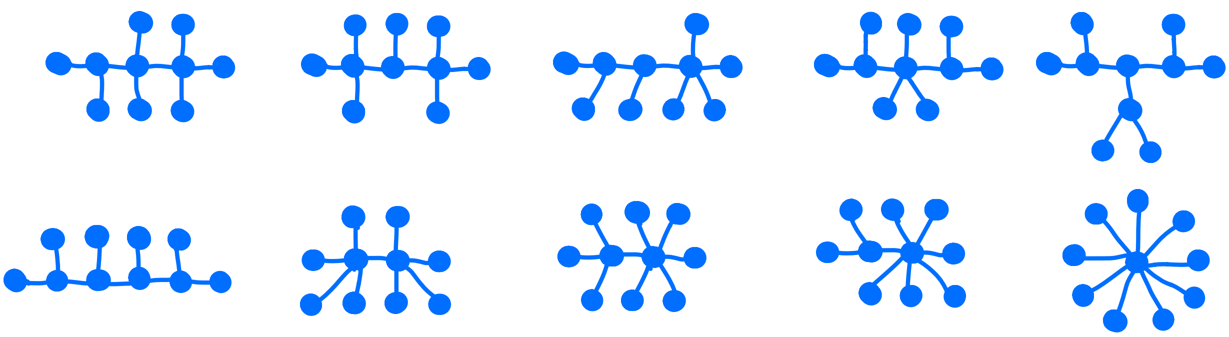


A	A	B	C	D
A	C	B	A	D
B	B	B	C	D
B	C	A	A	D
B	C	C	C	D
B	C	D	D	D
B	C	D	E	D
B	C	E	D	D
B	C	E	E	D

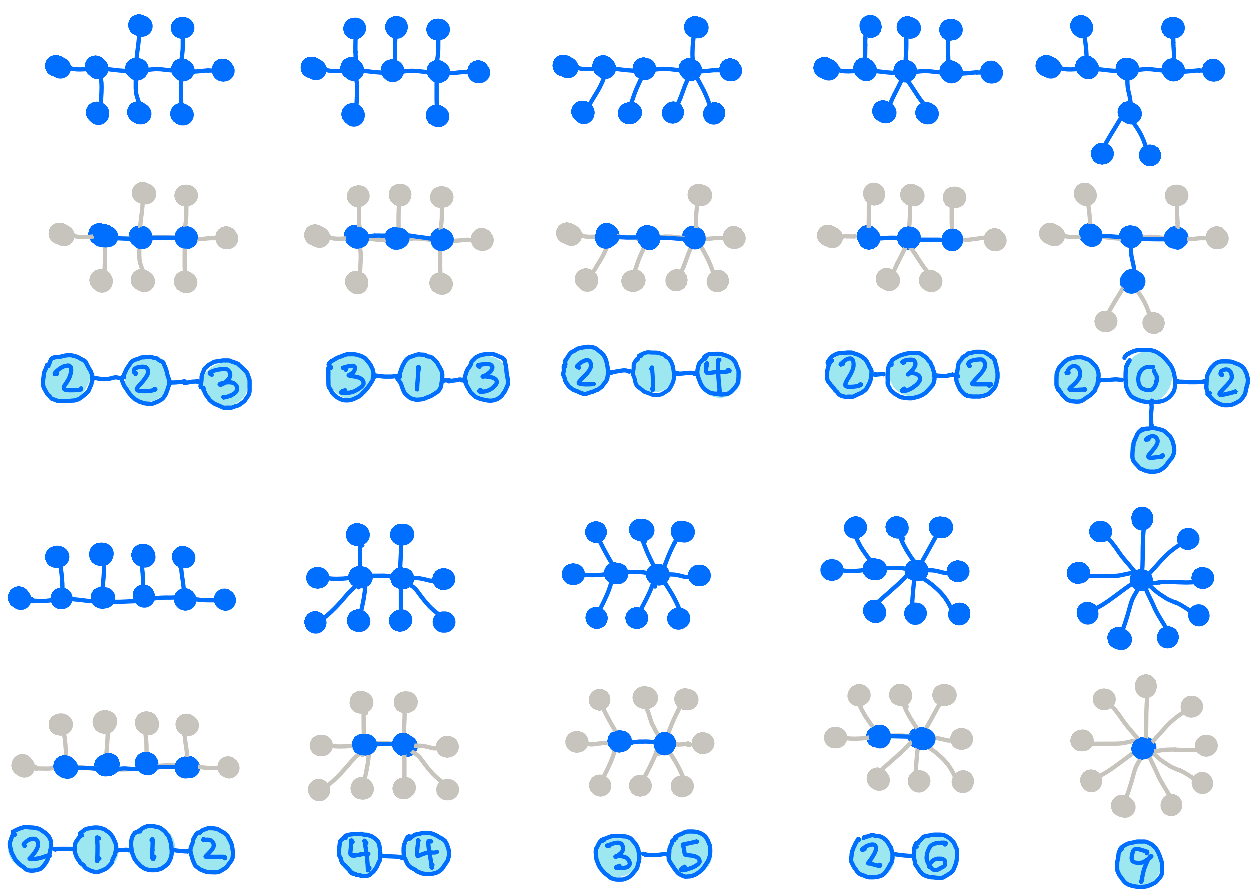
A	A	B	C	E
A	C	B	A	E
B	B	B	C	E
B	C	A	A	E
B	C	C	C	E
B	C	D	D	E
B	C	D	E	E
B	C	E	D	E
B	C	E	E	E

Bonus: 2nd problem in Film is actually easier

Draw all trees on 10 nodes up to symmetry (no nodes of degree 2)

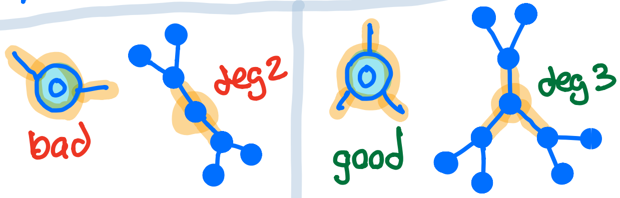


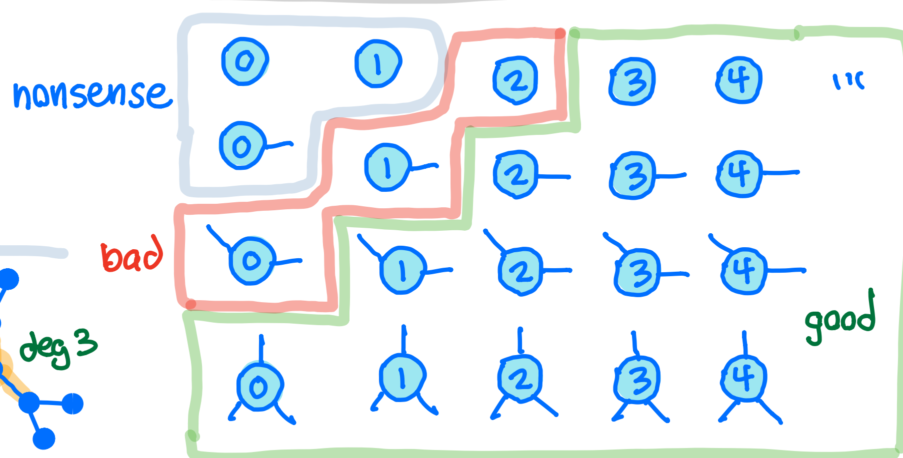
How can we make the drawings easier? Imply "leaves"



Leaves are implied.
Revised degree rule:

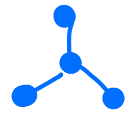
$n + \text{edges} \geq 3$





4 nodes : Non-leaves + numbers sum to 4

1 non-leaf : 3



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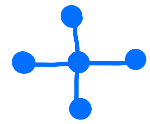
5 nodes : Non-leaves + numbers sum to 5

1 non-leaf : 4



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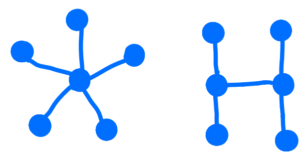
6 nodes : 1 non-leaf : 5

2 non-leaves : 2—2



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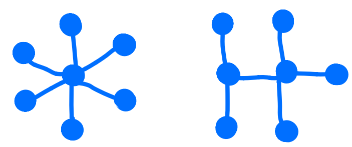
7 nodes : 1 non-leaf : 6

2 non-leaves : 2—3



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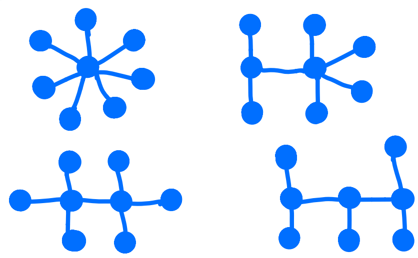
8 nodes : 1 non-leaf : 7

2 non-leaves : 2—4 3—3

3 non-leaves : 2—1—2



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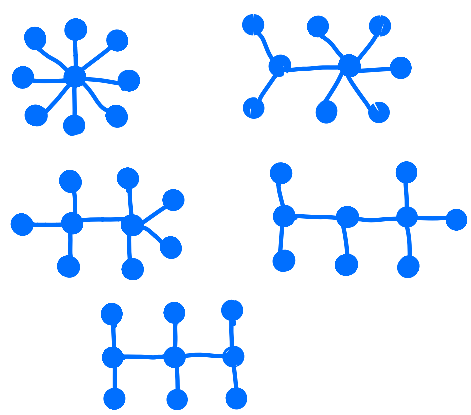
9 nodes : 1 non-leaf : 8

2 non-leaves : 2—5

3—4

3 non-leaves : 2—1—3

2—2—2



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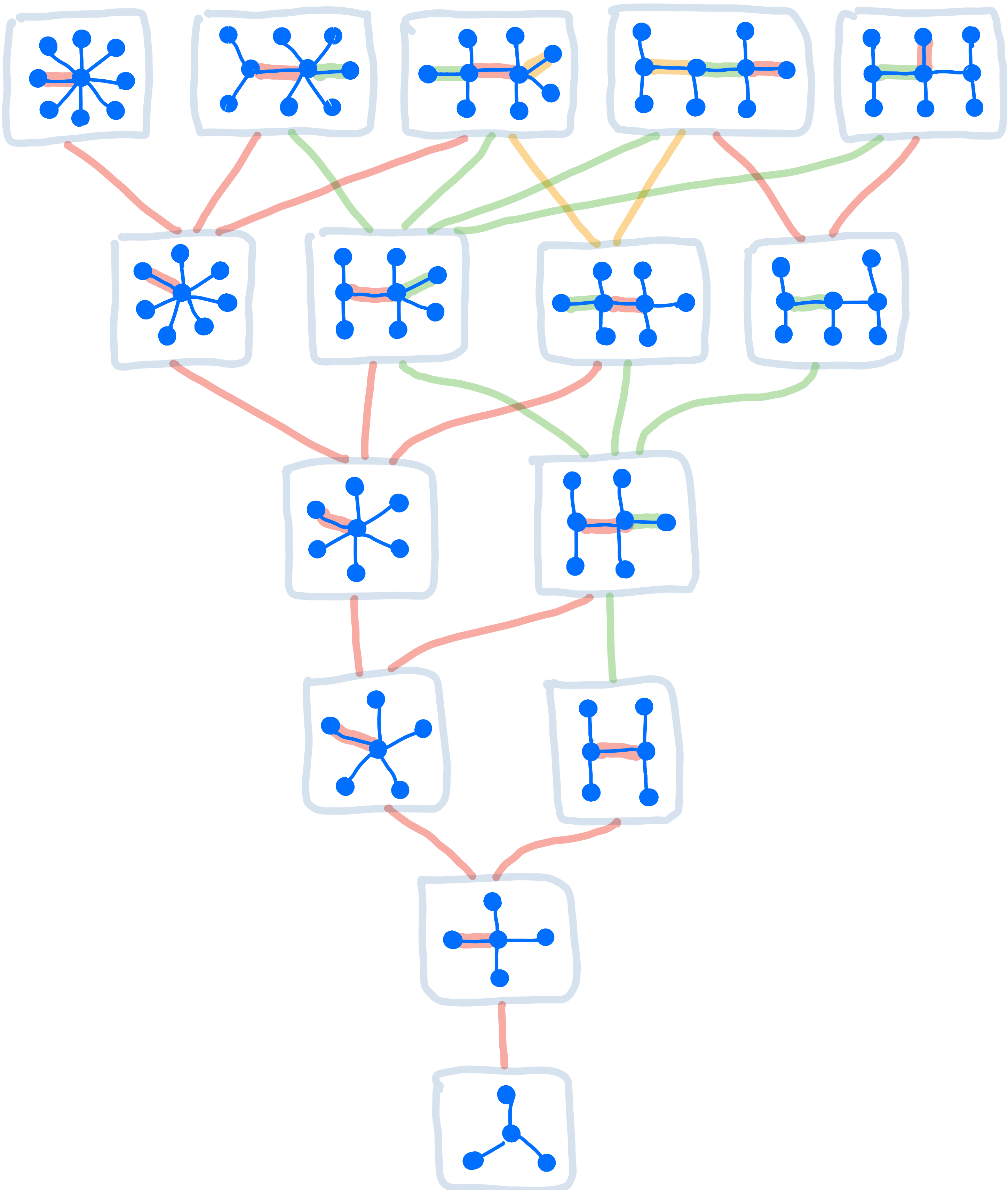
10 nodes : 1 non-leaf : 9

2 non-leaves : 2—6 3—5 4—4

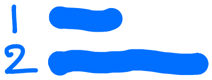
3 non-leaves : 2—1—4 2—2—3 2—3—2 3—1—3

4 non-leaves : 2—1—1—2 2—0—2

2

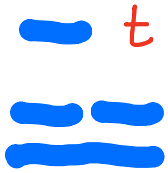


Partially ordered set induced by contracting edges
Some edge contractions are not allowed, create degree 2 nodes.



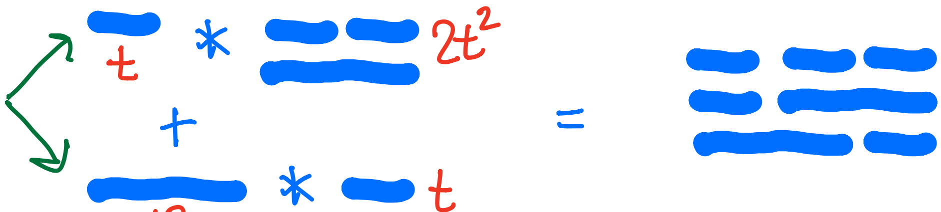
tube length

n	f(n)	list
0	1	1
1	1	t
2	2	$2t^2$



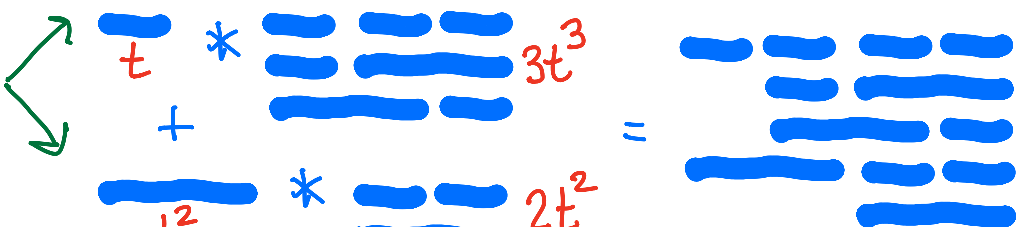
After class:
Fibonacci sequence as sticks length 1 or 2 filling tube of length n
recurrence,
and generating function

$f(3) = f(2) + f(1)$ 3 3



$t \ast 2t^2 + t^2 \ast t = 3t^3$

$f(4) = f(3) + f(2)$ 4 5



$t \ast 3t^3 + t^2 \ast 2t^2 = 5t^4$

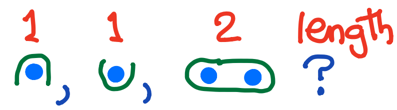
$$g(t) = \sum_{n=0}^{\infty} f(n)\, t^n = 1 + \ldots$$

$$\boxed{g(t) = 1 + t\,g(t) + t^2 g(t)} \;\Rightarrow\; (1 - t - t^2)\, g(t) = 1$$

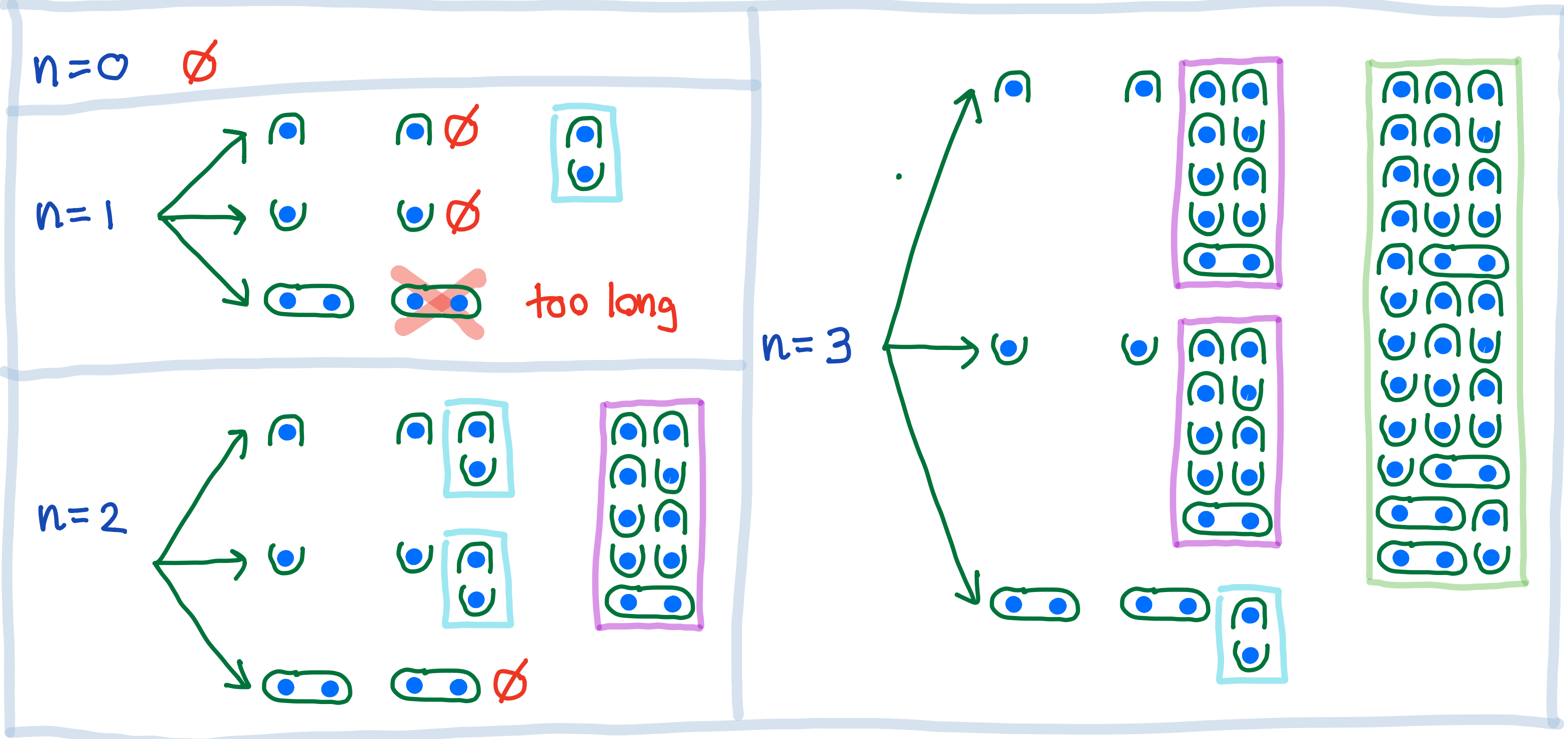
$$\boxed{g(t) = \frac{1}{1 - t - t^2}}$$

1	$=$	$1 +$	
$+t$	$=$	$1 \cdot t$	
$+2t^2$	$=$	$+t \cdot t$	$1 \cdot t^2$
$+3t^3$	$=$	$+2t^2 \cdot t$	$+t \cdot t^2$
$+5t^4$	$=$	$+3t^3 \cdot t$	$+2t^2 \cdot t^2$
$\vdots$		$\vdots$	

t shifts by 1
t^2 shifts by 2

How many words of length n can be formed from 

?



$$f(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 2f(n-1)+f(n-2), & n>0 \end{cases}$$

We want to write this without cases, and have it make sense.

$$\boxed{f(n) = 1_0 + 2f(n-1) + f(n-2)}$$

n	0	1	2	3	4	5
1_0	1	0	0	0	0	0
$2f(n-1)$		2 1	2 2	2 5	2 12	2 29
$f(n-2)$			1	2	5	12
$f(n)$	1	2	5	12	29	70

To convert to algebra, use powers of t to record table position

$$\boxed{g(t) = \sum_{n=0}^{\infty} f(n)t^n}$$

generating function

1	1_0	1
$2tg(t)$	$2f(n-1)$	$2t(1+2t+5t^2+12t^3+29t^4+\dots)$
$t^2g(t)$	$f(n-2)$	$t^2(1+2t+5t^2+12t^3+\dots)$
$g(t)$	$f(n)$	$1+2t+5t^2+12t^3+29t^4+70t^5+\dots$

$$f(n) = 1_0 + 2f(n-1) + f(n-2)$$
$$g(t) = 1 + 2tg(t) + t^2g(t)$$

into generating function

$$g(t) = \sum_{n=0}^{\infty} f(n)t^n$$

$$\Downarrow$$

$$g(t) - 2tg(t) - t^2g(t) = 1$$
$$g(t)(1-2t-t^2) = 1$$

learn to read same way

$$\boxed{g(t) = \frac{1}{1-2t-t^2}}$$

$*$	$1+2t+5t^2+12t^3+29t^4+70t^5+\dots$
1	$1+2t+5t^2+12t^3+29t^4+70t^5+\dots$
$-2t$	$-2t(1+2t+5t^2+12t^3+29t^4+\dots)$
$-t^2$	$-t^2(1+2t+5t^2+12t^3+\dots)$
	$1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

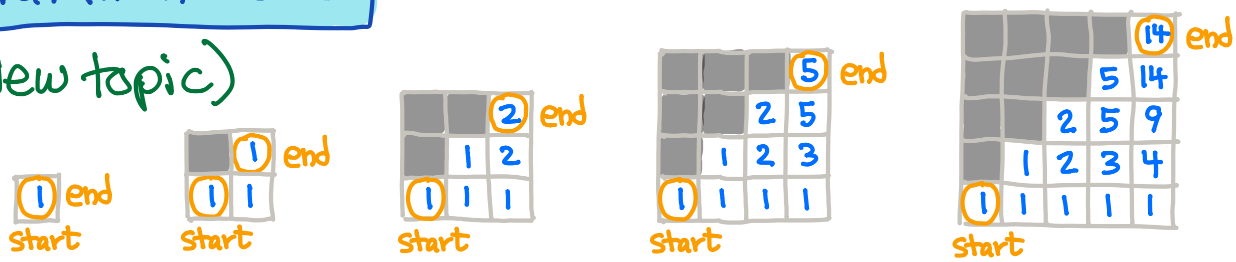
Same calculation more concisely

n	0	1	2	3	4	5
$f(n)$	1	2	5	12	29	70
		$-t^2$	$-2t$	1		

sliding rule for recurrence

Catalan numbers

(New topic)

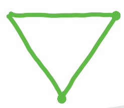


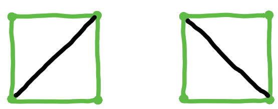
$\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{5}$ $\boxed{14}$

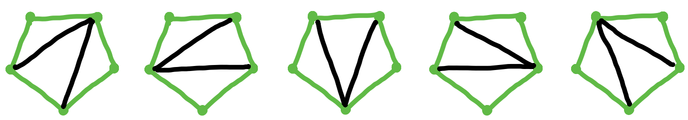
How many lattice paths stay on or below the diagonal?

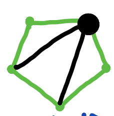
How many ways can we triangulate an n-gon?

$n=2$ $\boxed{1}$ (The empty case, we'll see)

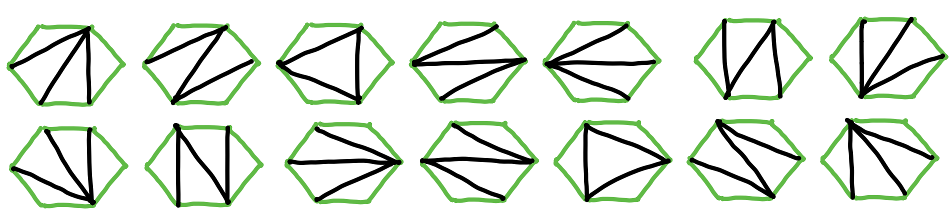
$n=3$ $\boxed{1}$ 

$n=4$ $\boxed{2}$ 

$n=5$ $\boxed{5}$ 

 Pick a corner, make a wishbone

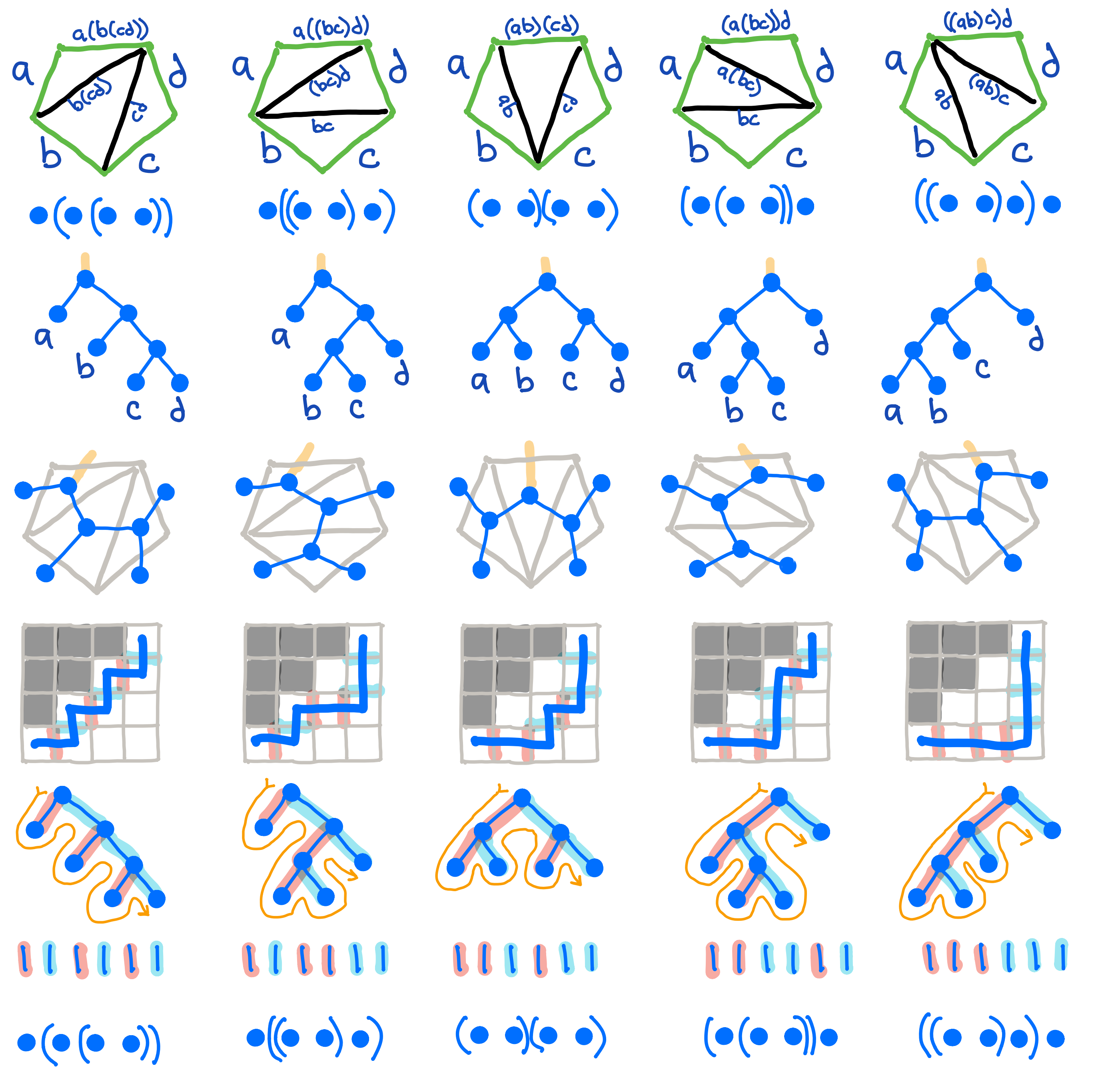
(5 rotations)

$n=6$ $\boxed{14}$ 

(6 rotations) (3 rotations) (3 rotations) (2 rotations)

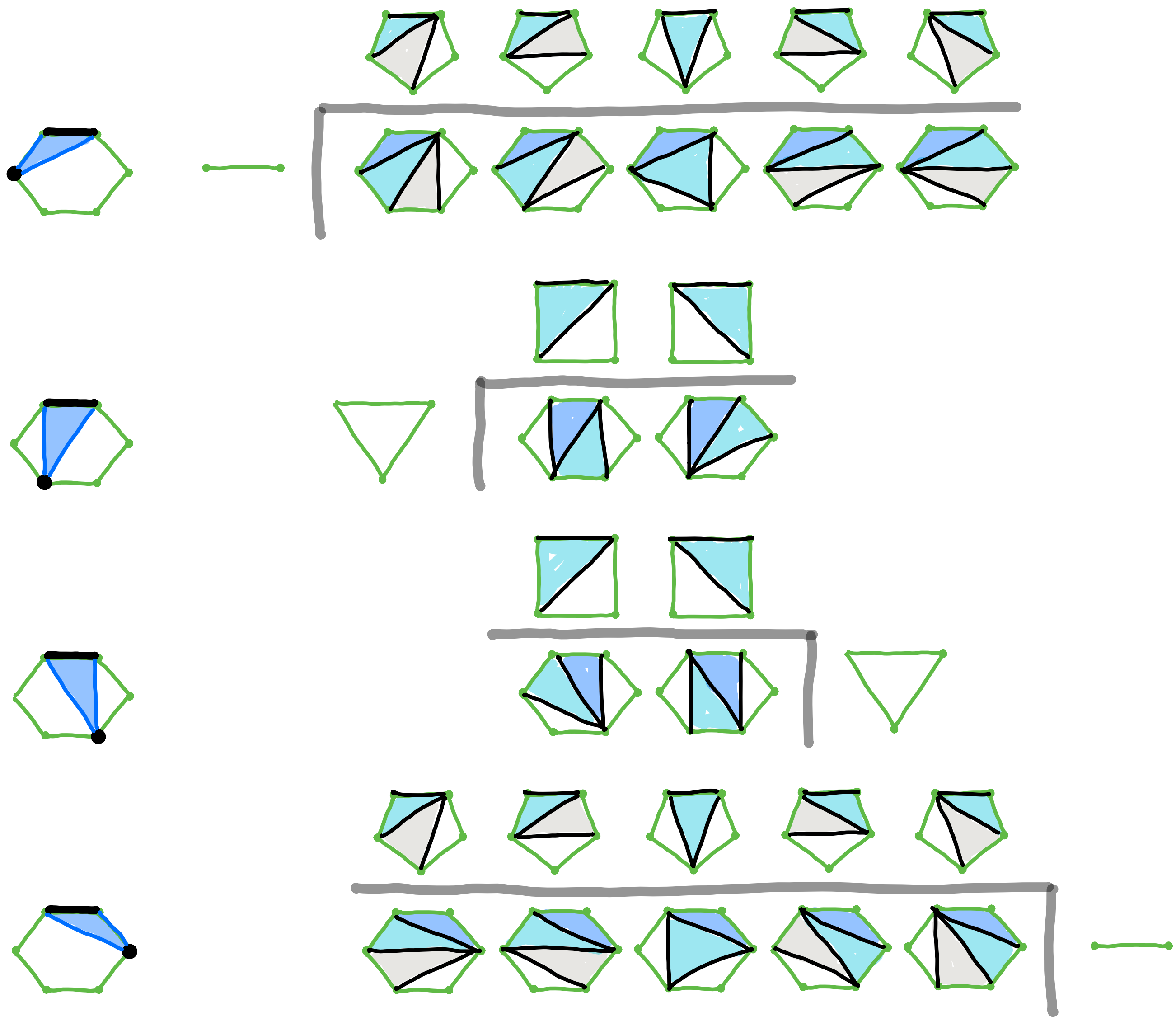
Associative law: How many ways can we parenthesize n terms?

n=1	1	•	No work to do
n=2	1	• •	Only one way to combine terms
n=3	2	(• •)• •(• •)	
n=4	5	•(•(• •)) •((• •)•) (• •)(• •) (•(• •))• ((• •)•)•	
n=5	14	•(•(•(• •))) •(•((• •)•)) •((• •)(• •)) •((•(• •))•) •(((• •)•)•) (• •)(•(• •)) (• •)((• •)•) (•(• •))(• •) ((• •)•)(• •) (•(•(• •)))• (•((• •)•))• ((• •)(• •))• ((•(• •))•)• (((• •)•)•)•	



What is the common pattern? The recurrence?

Each step depends on all previous steps.



1	1	2	5	14	42	...

1
1
1

1	1
1	1
1	1
	2

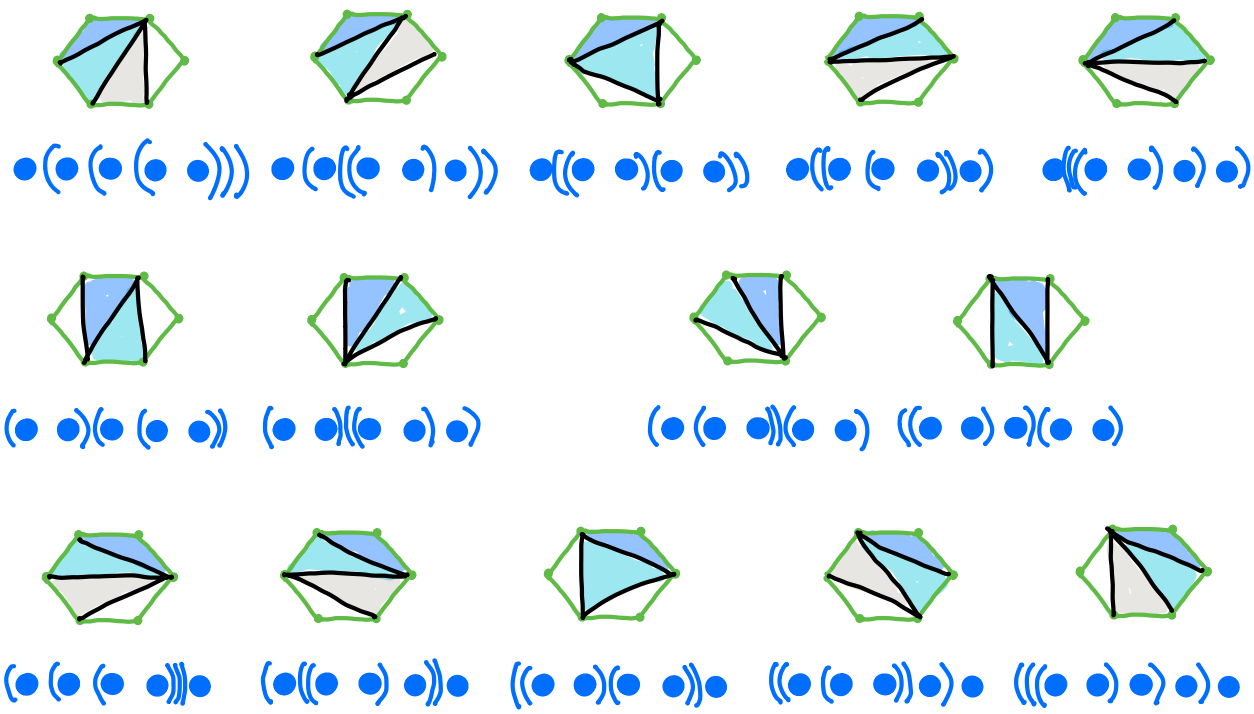
1	1	2
2	1	1
2	1	2
		5

1	1	2	5
5	2	1	1
5	2	2	5
			14

1	1	2	5	14
14	5	2	1	1
14	5	4	5	14
				42

Flip numbers so far, take dot product.

How did I actually figure out the parentheses for 5 terms?



$$f(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 2f(n-1)+f(n-2), & n>0 \end{cases}$$

We want to write this without cases, and have it make sense.

$$\boxed{f(n) = 1_0 + 2f(n-1) + f(n-2)}$$

n	0	1	2	3	4	5
1_0	1	0	0	0	0	0
2f(n-1)		2 1	2 2	2 5	2 12	2 29
f(n-2)			1	2	5	12
f(n)	1	2	5	12	29	70

last class

Haskell

```
Prelude> words = 1 : [ 2*a + b | (a, b) <- zip words (0 : words) ]
Prelude> take 6 words
[1,2,5,12,29,70]
```

lazy evaluation

1	2	5	12	29	words
0	1	2	5	12	0:words
1,0	2,1	5,2	12,5	29,12	zip
2	5	12	29	70	2a+b

words = 1 2 5 12 29 70 1: list

Catalan numbers

$$C_n = \underset{0}{\boxed{1}}, \underset{1}{\boxed{1}}, \underset{2}{\boxed{2}}, \underset{3}{\boxed{5}}, \underset{4}{\boxed{14}}, \ldots$$

They can be found in Pascal's triangle

1 $\boxed{1}$
1 1
1 2 $\boxed{1}$ 1
1 3 3 1
1 4 6 $\boxed{2}$ 4 1
1 5 10 10 5 1
1 6 15 20 $\boxed{5}$ 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 $\boxed{14}$ 56 28 8 1

$$\boxed{C_n = \binom{2n}{n} - \binom{2n}{n+1}}$$

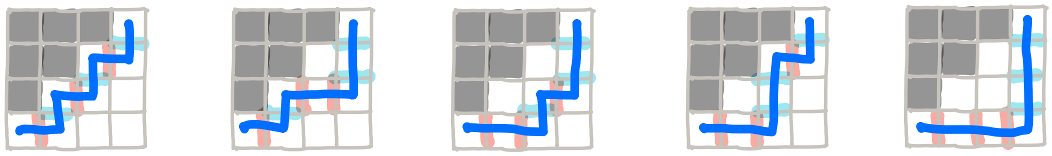
$$\frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} - \frac{2n(2n-1)\cdots(n+1)n}{(n+1)n(n-1)\cdots 1}$$

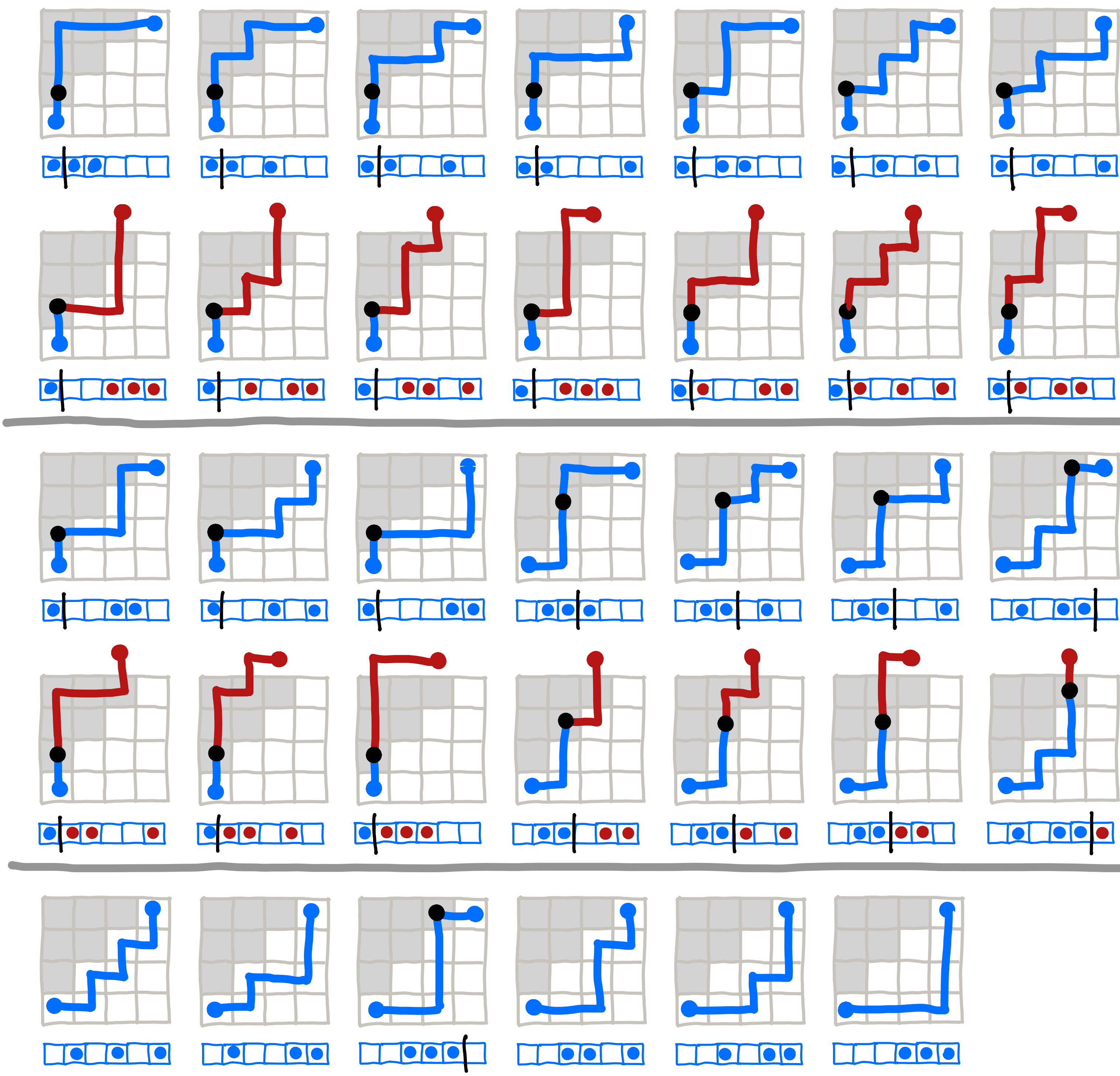
$$\frac{2n(2n-1)\cdots(n+1)(n+1)}{(n+1)n(n-1)\cdots 1} - \frac{2n(2n-1)\cdots(n+1)n}{(n+1)n(n-1)\cdots 1}$$

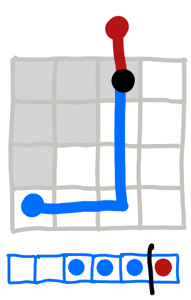
$$\frac{1}{(n+1)} \frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} = \boxed{\frac{1}{n+1}\binom{2n}{n}}$$

Why? André's reflection method.

$C_3 = 5$, valid paths on 4x4 grid $\quad \boxed{5} = 20 - 15$







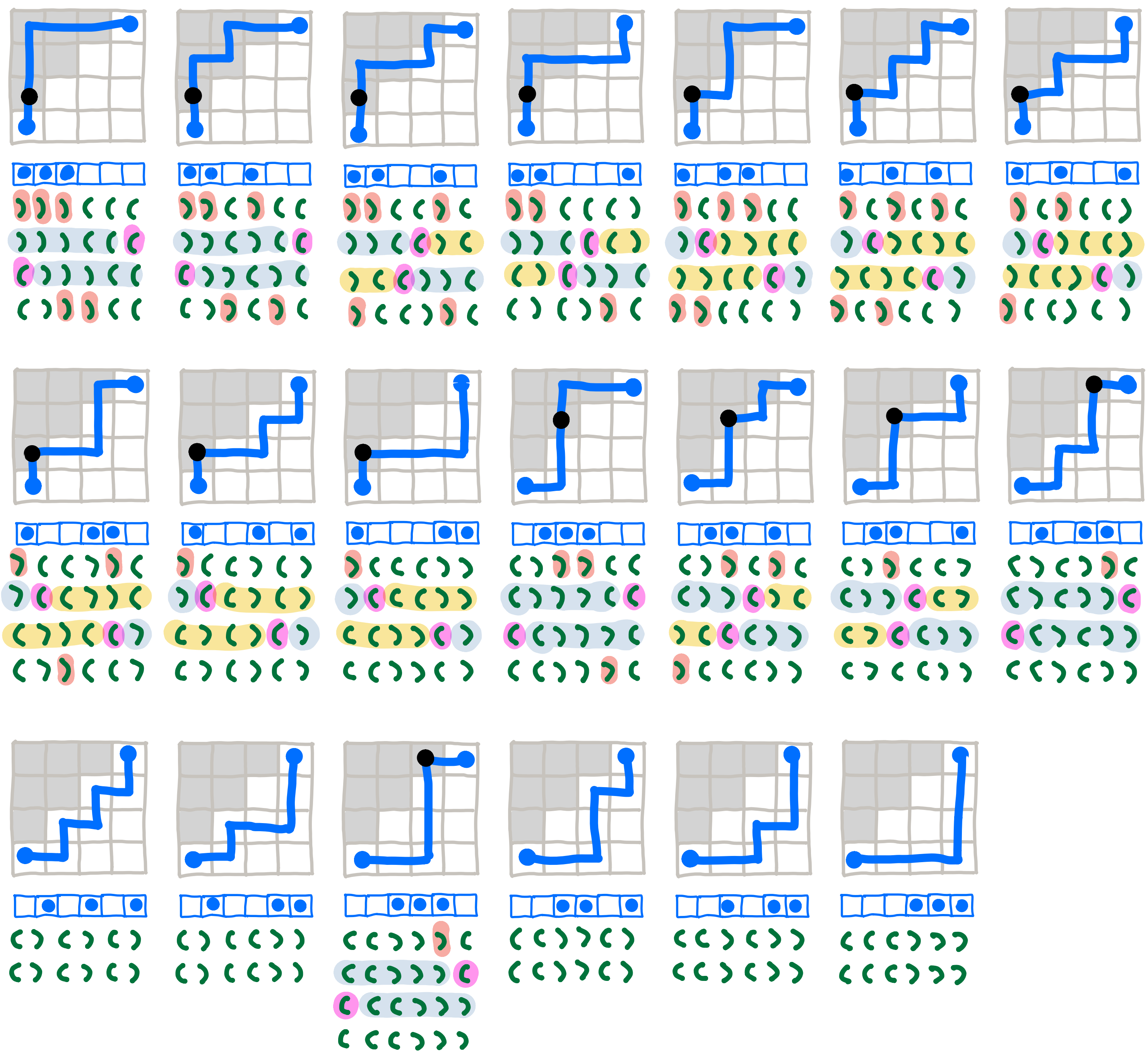
All paths $\binom{6}{3}$ ups

Bad paths flip to $\binom{6}{4}$ ups

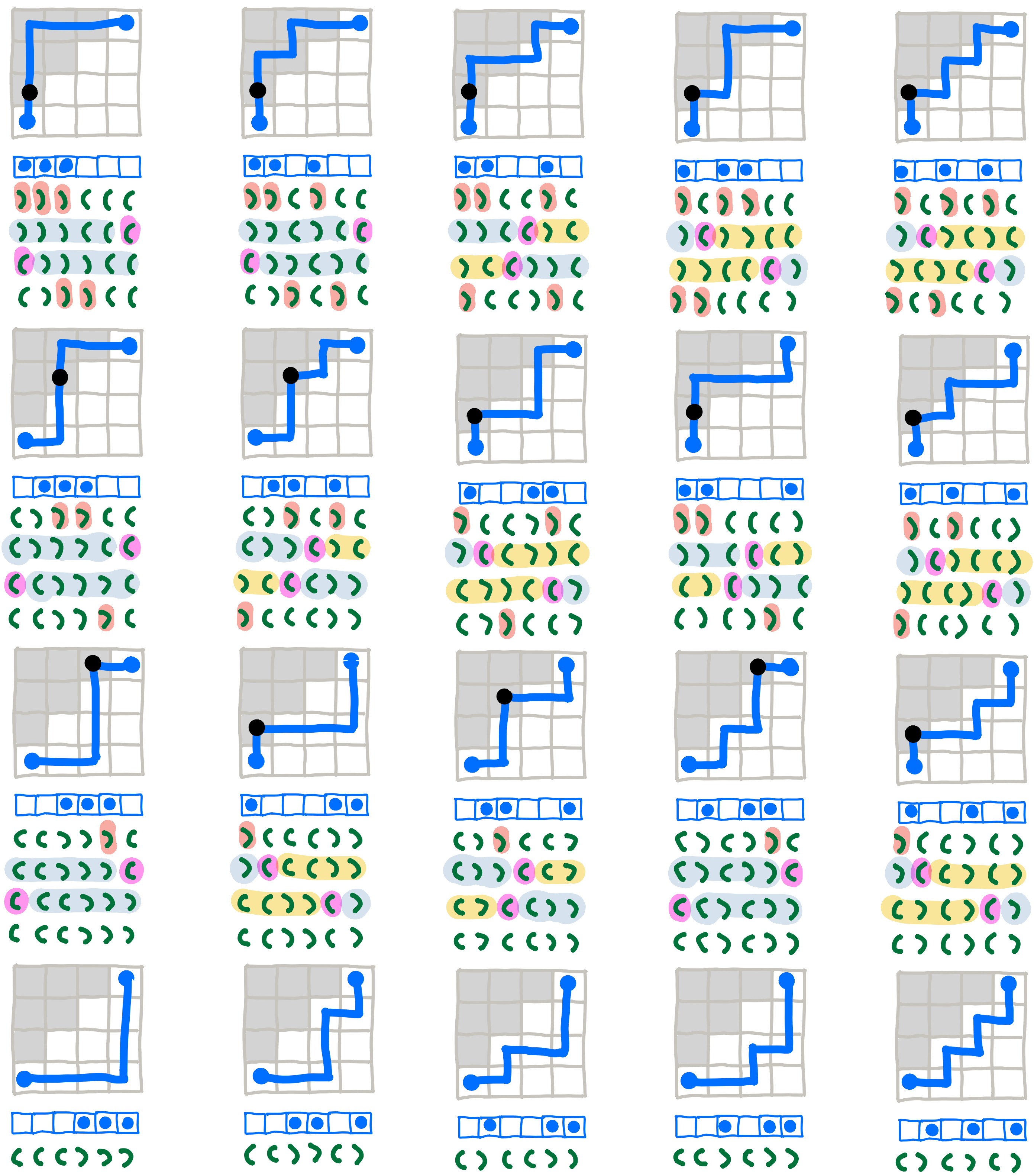
$$\binom{6}{3} - \binom{6}{4} = 20 - 15 = \boxed{5}$$

Second proof, explain the denominator $\frac{1}{n+1}\binom{2n}{n}$

One of many equal sized groups. Find the others...



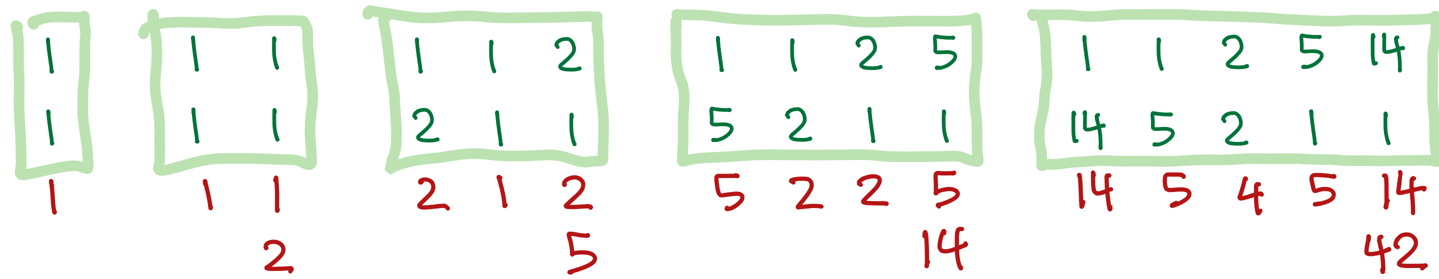
Rearrang in strands: $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{4}\binom{6}{3} = \frac{20}{4} = 5$



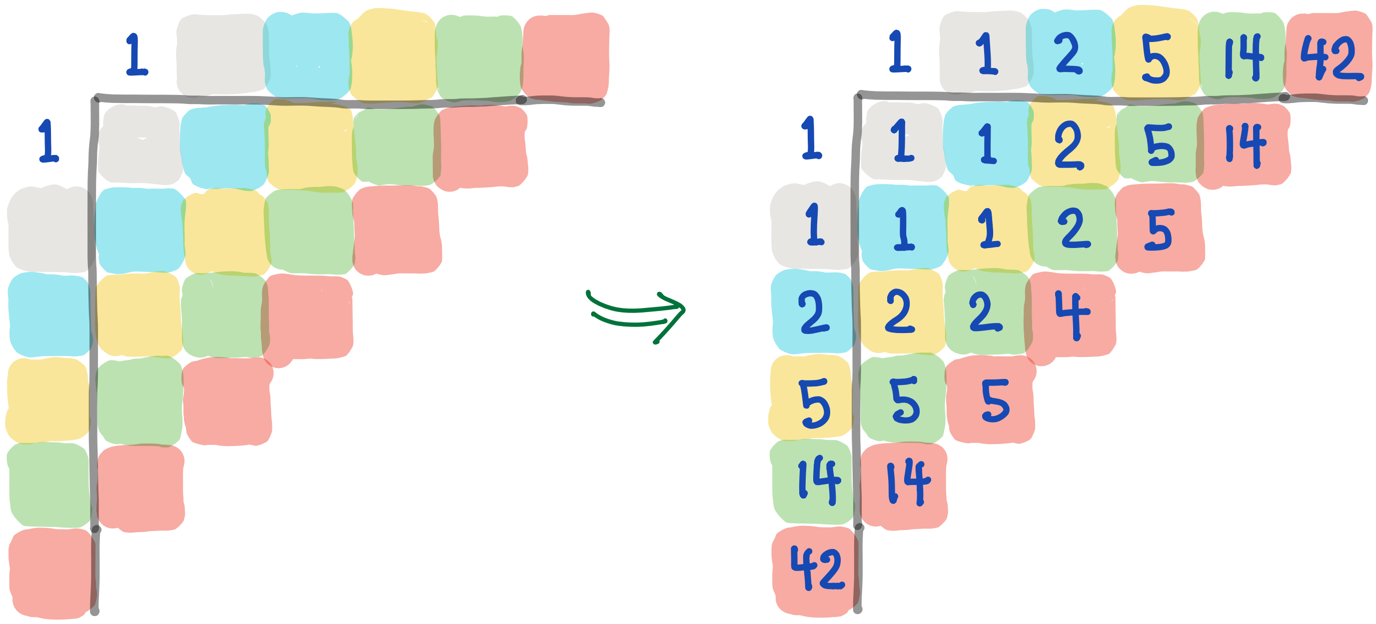
What about generating function?

Play with different ways to present recursion.

From last class: 1 | 1 | 2 | 5 | 14 | 42 | ...



Flip numbers so far, take dot product.



Let $g(t) = \sum_{n=0}^{\infty} C_n t^n$

Then $g(t) = 1 + t\, g(t)^2$

$t\, g(t)^2 - g(t) + 1 = 0$

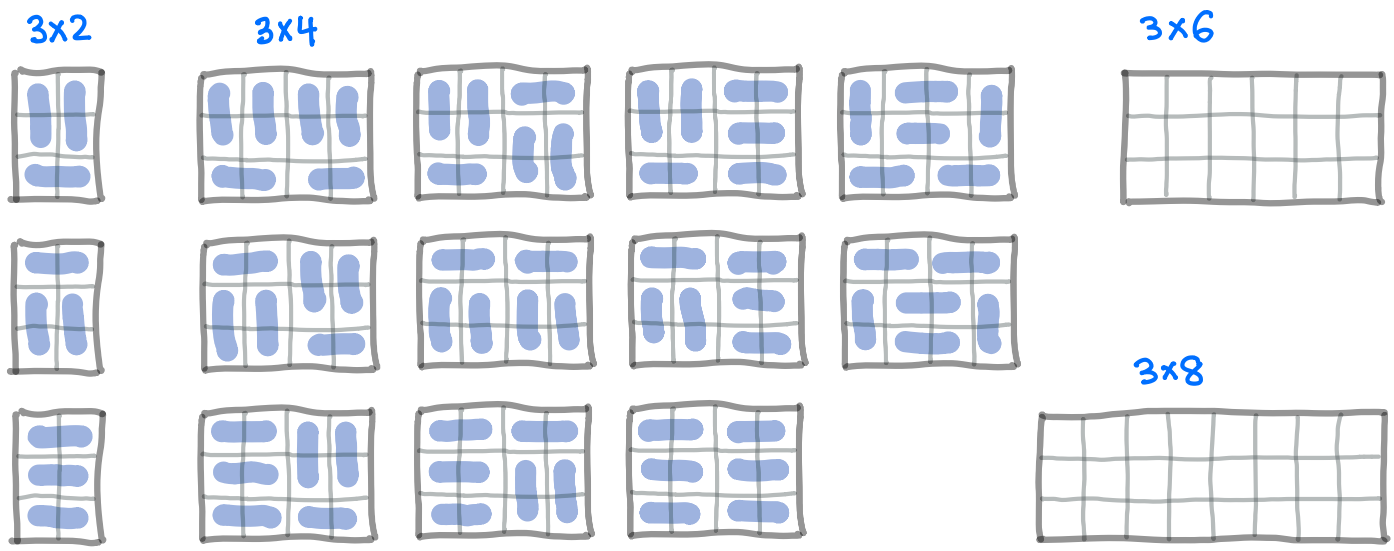
$\boxed{ax^2 + bx + c \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \Rightarrow g(t) = \frac{1 \pm \sqrt{1-4t}}{2t}$

Domino tilings

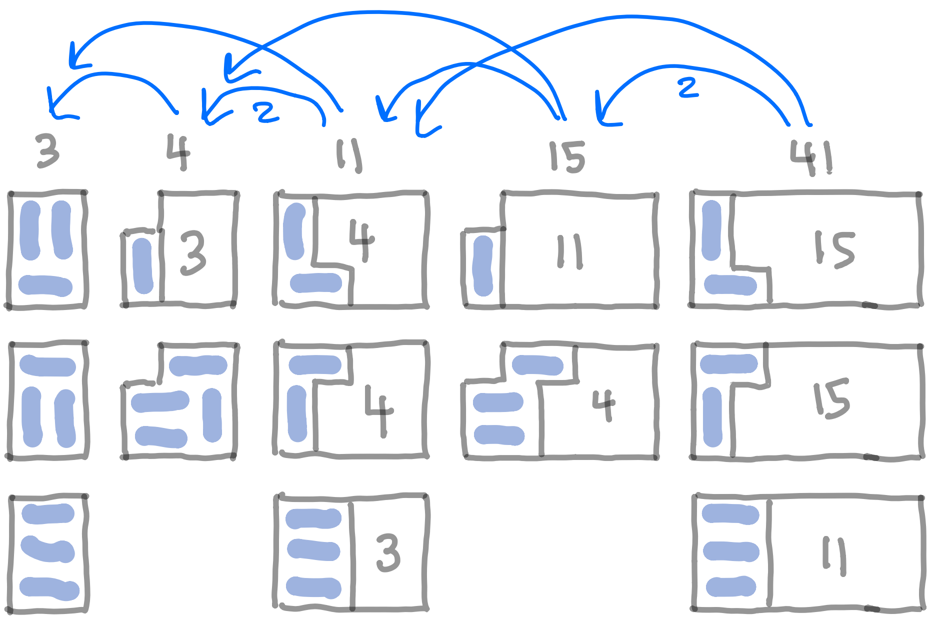
Let $f(n) = \#$ of domino tilings of a $3 \times 2n$ grid.

$f(1) = 3, \quad f(2) = 11$

Find $f(3)$, $f(4)$, generating function.

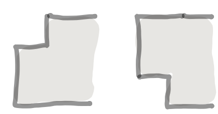


First try (scrap work) break into cases from the left



Use symmetry

Treat

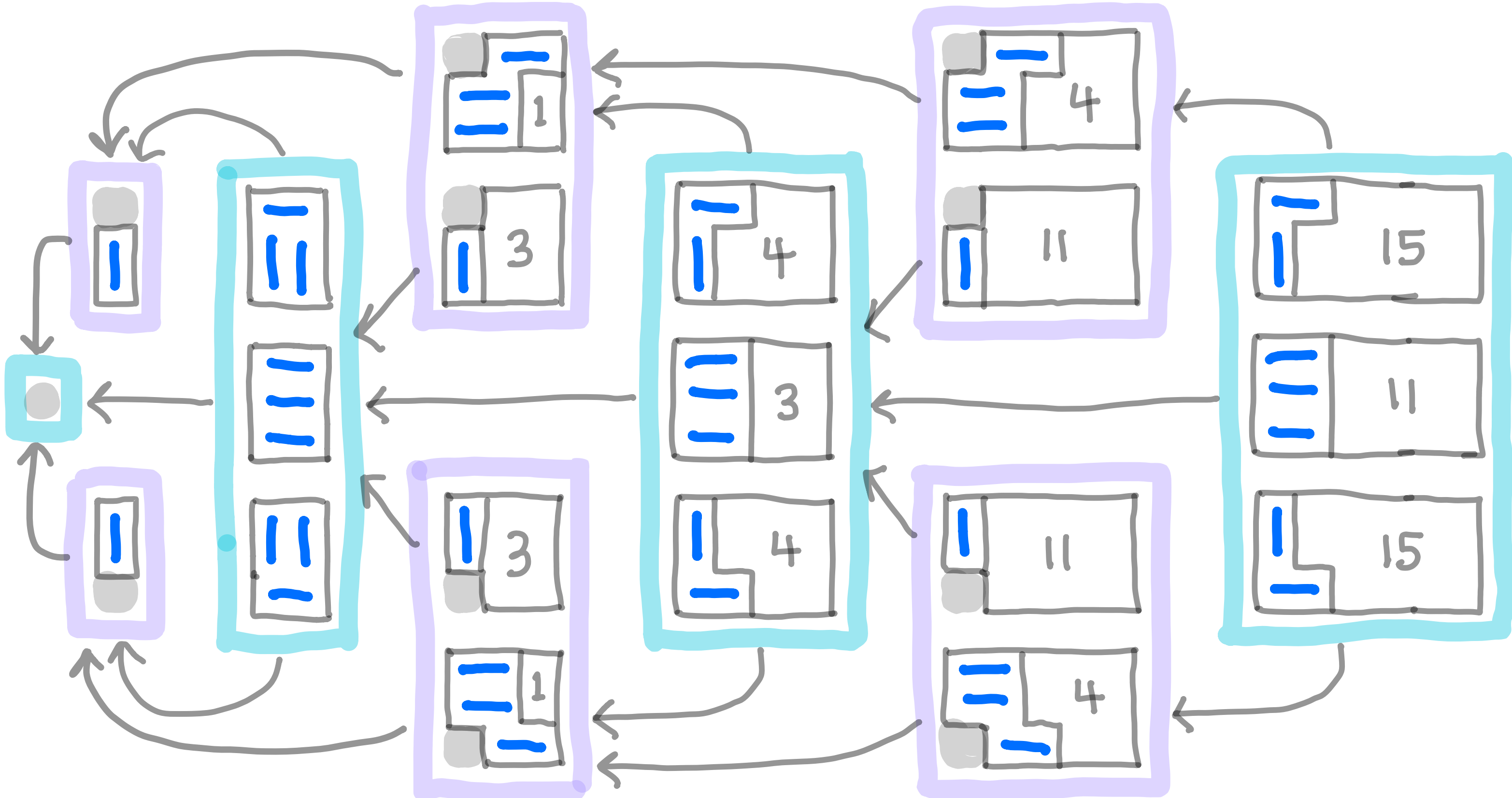


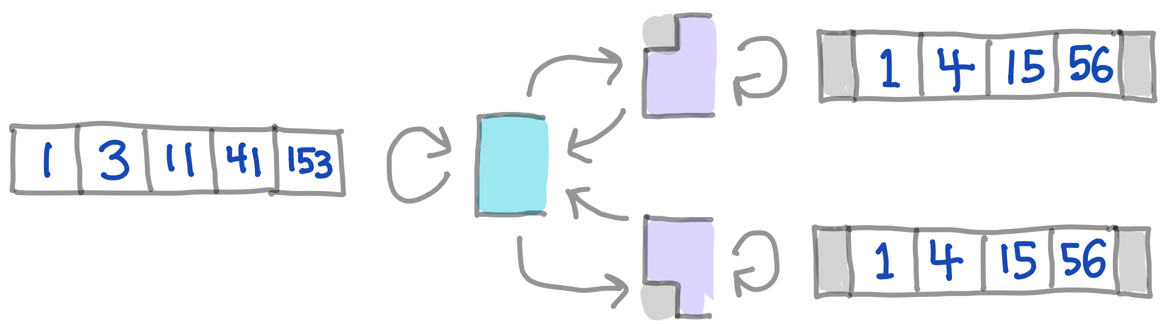
as same case

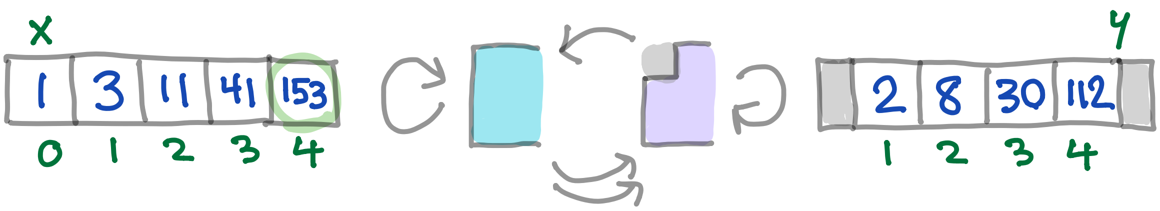
3 4 11 15 41 56 153

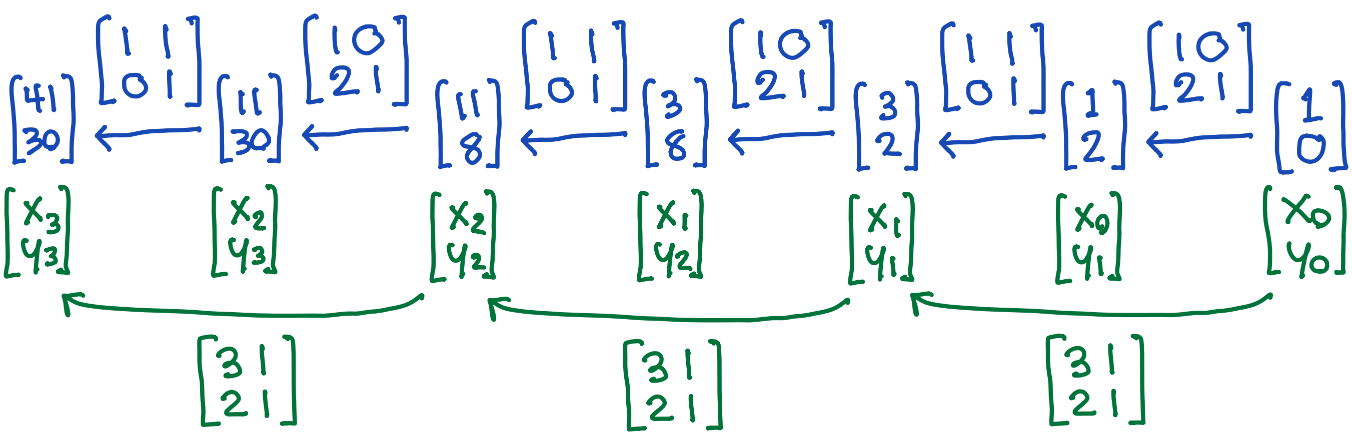
$\boxed{f(3) = 41}$ $\boxed{f(4) = 153}$

Redraw more carefully:









$$\begin{bmatrix}3&1\\2&1\end{bmatrix}^2=\begin{bmatrix}11&4\\8&3\end{bmatrix},\quad \begin{bmatrix}11&4\\8&3\end{bmatrix}^2=\begin{bmatrix}153&56\\112&41\end{bmatrix},\quad \begin{bmatrix}153&56\\112&41\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}153\\112\end{bmatrix}$$

$$\boxed{g(t)=\sum_{n=0}^{\infty} x_n}$$

$$h(t)=\sum_{n=0}^{\infty} y_n$$

$$\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+t\begin{bmatrix}3&1\\2&1\end{bmatrix}\begin{bmatrix}g(t)\\h(t)\end{bmatrix}$$

$$\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}-t\begin{bmatrix}3&1\\2&1\end{bmatrix}\right)\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$(1-3t)(1-t)-2t\cdot t$$
$$=1-4t+t^2$$

$$\begin{bmatrix}1-3t&-t\\-2t&1-t\end{bmatrix}\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}g(t)\\h(t)\end{bmatrix}=\begin{bmatrix}1-3t&-t\\-2t&1-t\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1-t&t\\2t&1-3t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\Big/(1-4t+t^2)=\begin{bmatrix}1-t\\2t\end{bmatrix}\Big/(1-4t+t^2)$$

So $\boxed{g(t)=\dfrac{1-t}{1-4t+t^2}}$

$$1-4t+t^2=0$$
$$1=4t-t^2$$

n	0	1	2	3	4	...
$1/(1-4t+t^2)$	1	4	15	56	209	...
$-t/(1-4t+t^2)$	0	1	4	15	56	...
x_n	1	3	11	41	153	...

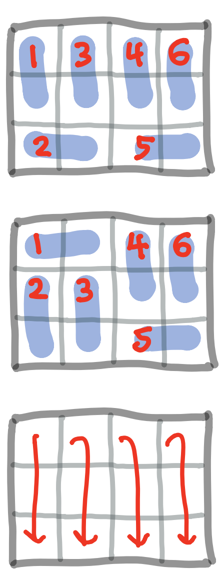
use recurrence $1=4t-t^2$

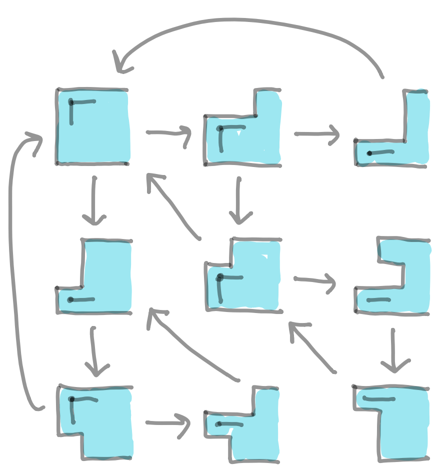
shift by t

OEIS A001835

Second approach: Remove dominos in canonical order.

What does the frontier look like?

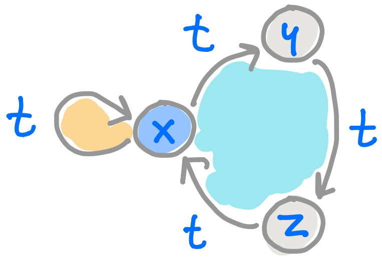




First attempt at graph

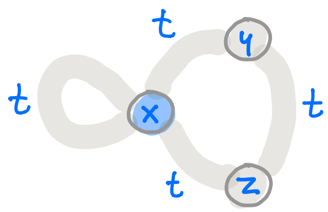
We need a "calculus" for walks on graphs, to get generating function.

Take simpler example: Represent each path by product of edge labels.

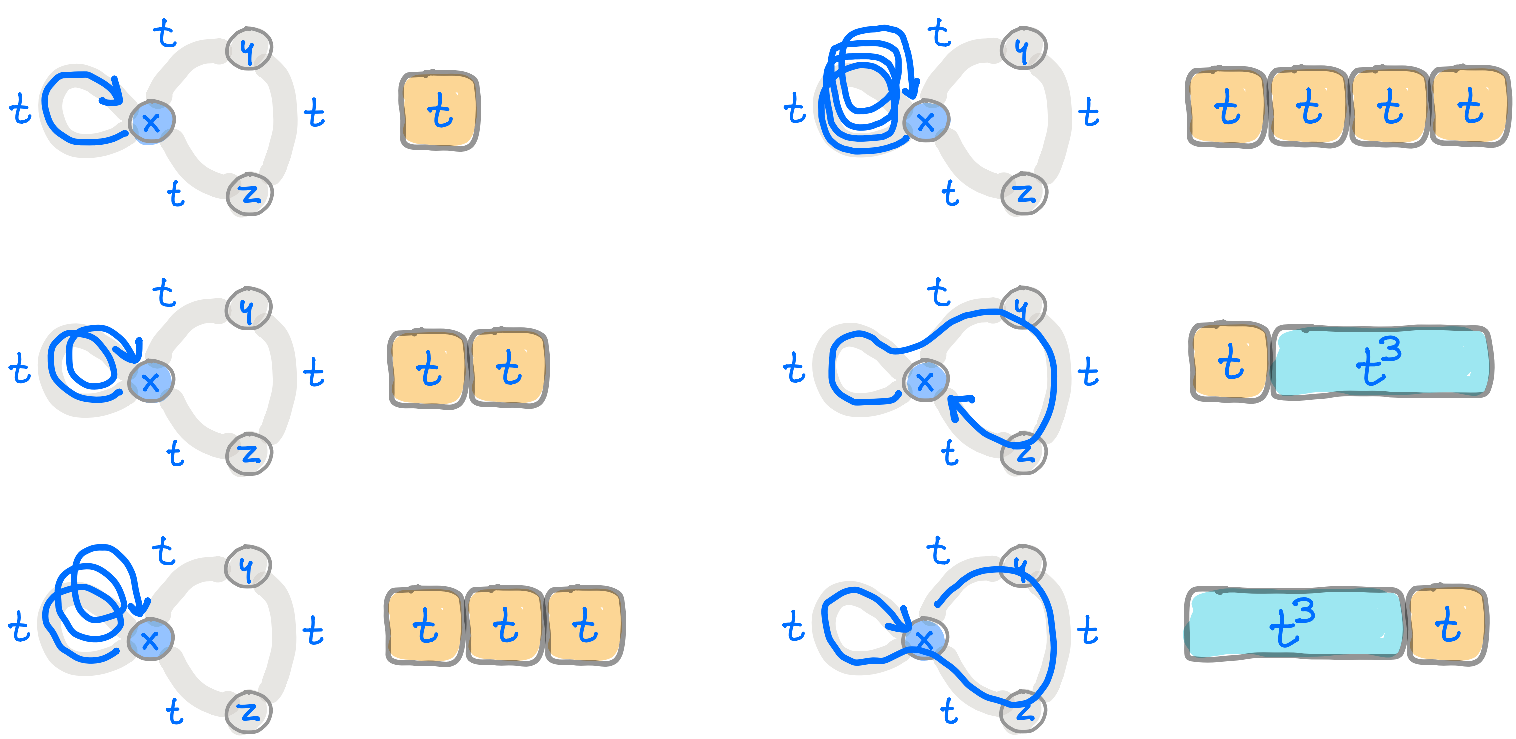


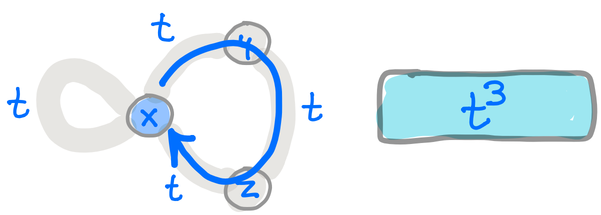
Generating functions are sums of all paths with given start, end vertices.

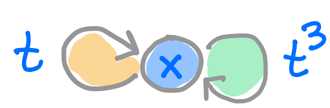
We can simplify graph if it gives same generating function. What are rules?



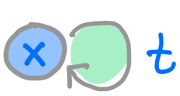
paths x to itself: $\boxed{1 + t + t^2 + 2t^3 + 3t^4 + \ldots = \frac{1}{1-(t+t^3)}}$





same: 

same: 

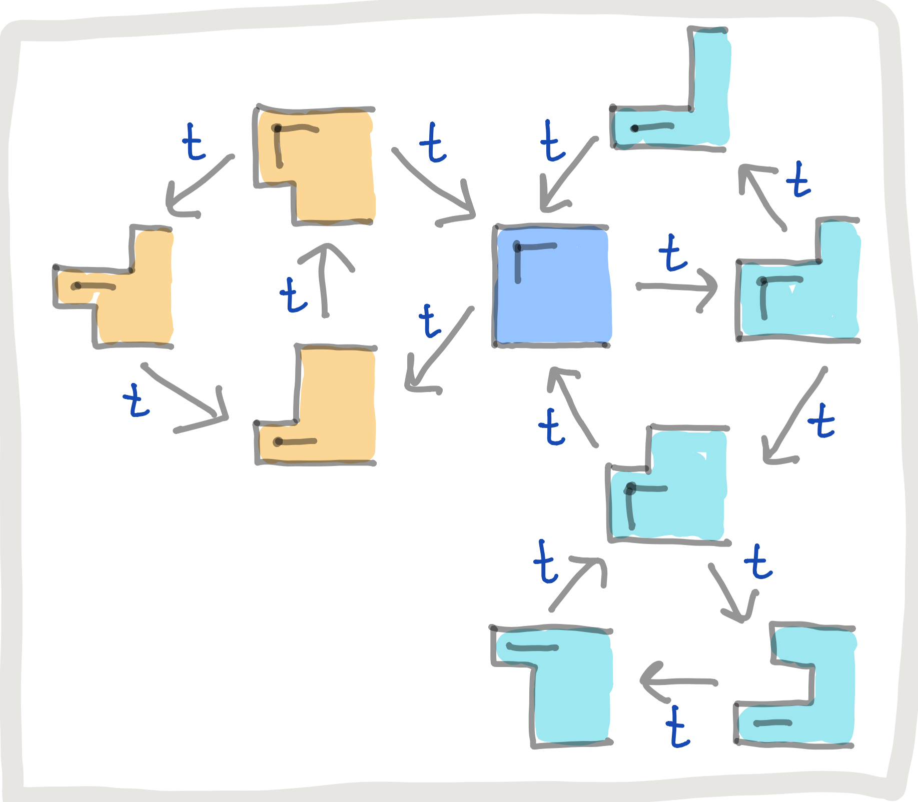


$1 + t + t^2 + t^3 + \ldots = \frac{1}{1-t}$

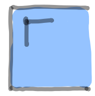
$1 + (t+t^3) + (t+t^3)^2 + (t+t^3)^3 + (t+t^3)^4 + \ldots = \frac{1}{1-(t+t^3)}$

$1 \quad t \quad t^3 \quad t^2 \quad 2t^4 \quad t^6 \quad t^3 \quad 3t^5 \quad 3t^7 \quad t^9 \quad t^4 \quad 4t^6 \ldots$

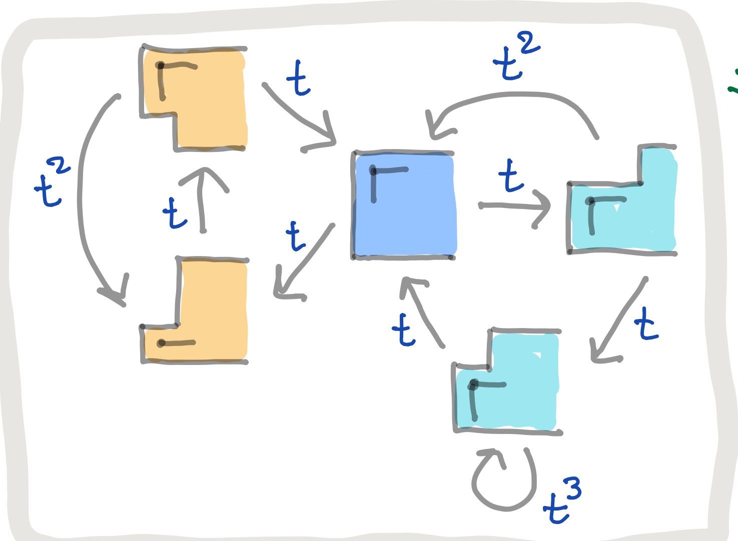
$1 + t + t^2 + 2t^3 + 3t^4 + \ldots$

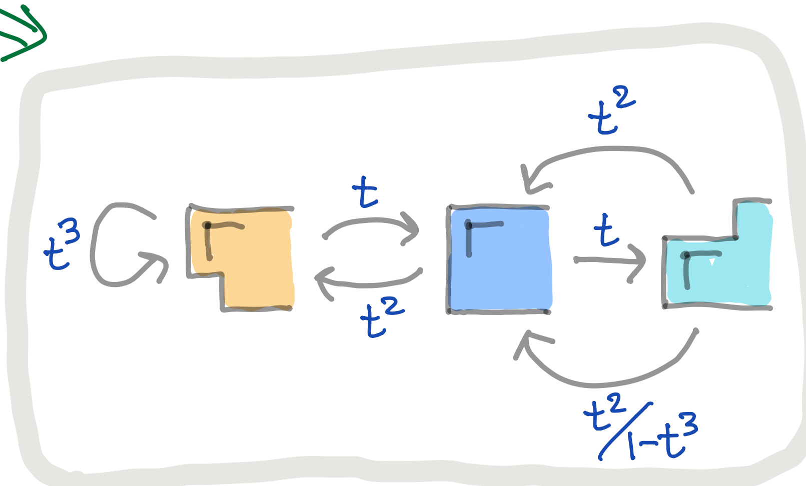


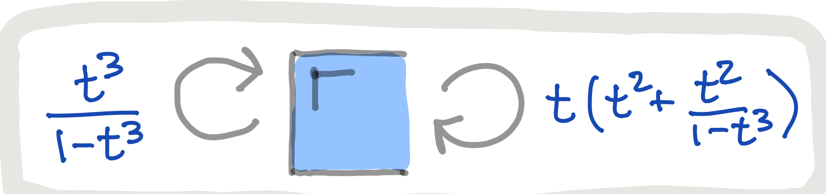
Redrawn.

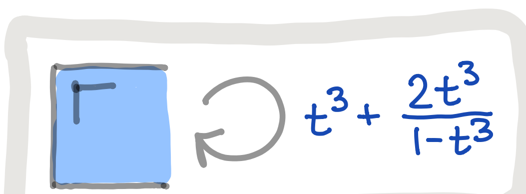
We want to sum all paths from  to itself, taking product of labels for each path.

Now simplify.







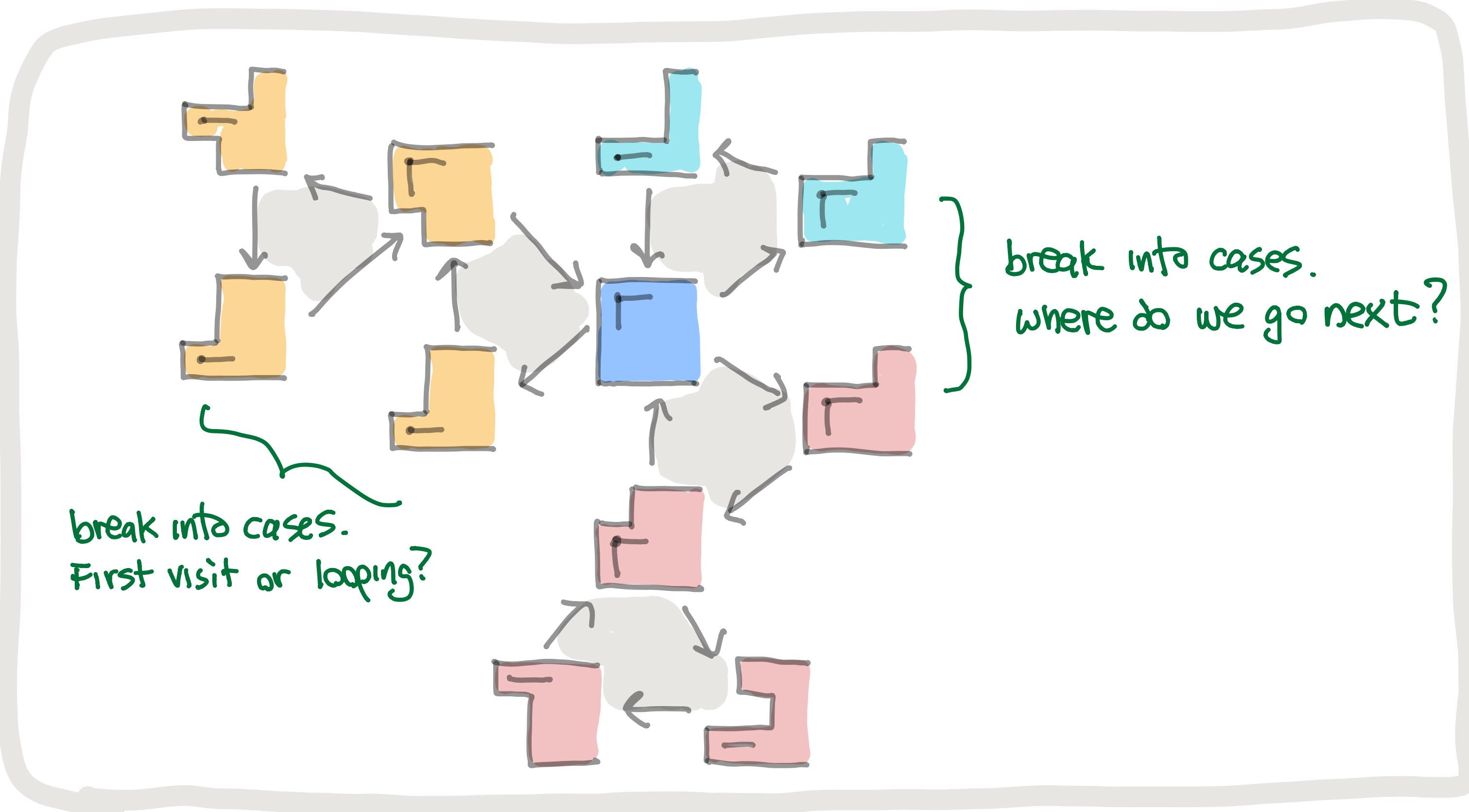


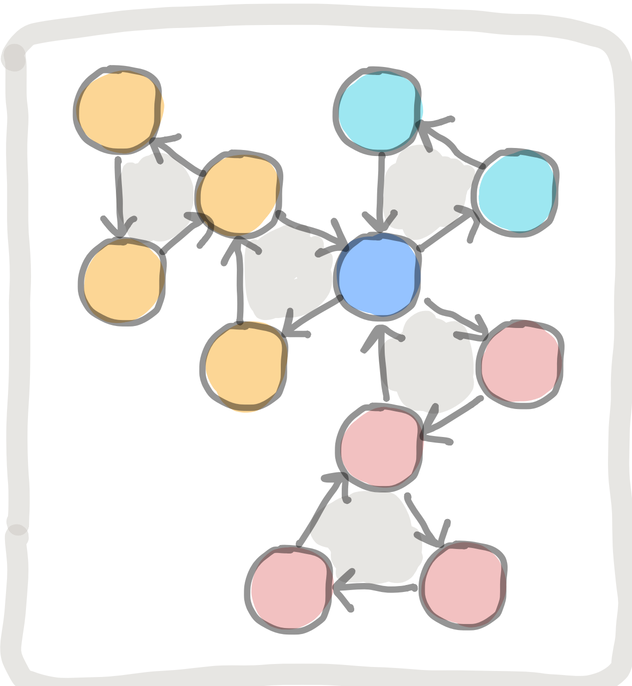
$$\boxed{1\Big/\left(1-t^3-\frac{2t^3}{1-t^3}\right)} = \frac{1-t^3}{(1-t^3)-t^3(1-t^3)-2t^3} = \frac{1-t^3}{1-4t^3+t^6}$$

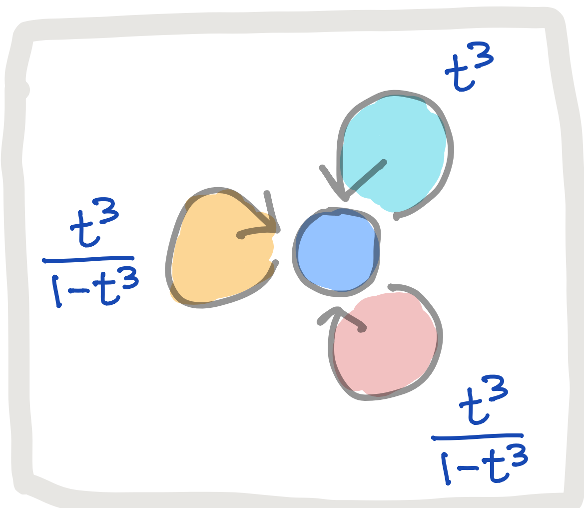
$$\boxed{\frac{1-t}{1-4t+t^2}}$$

3 dominos per n

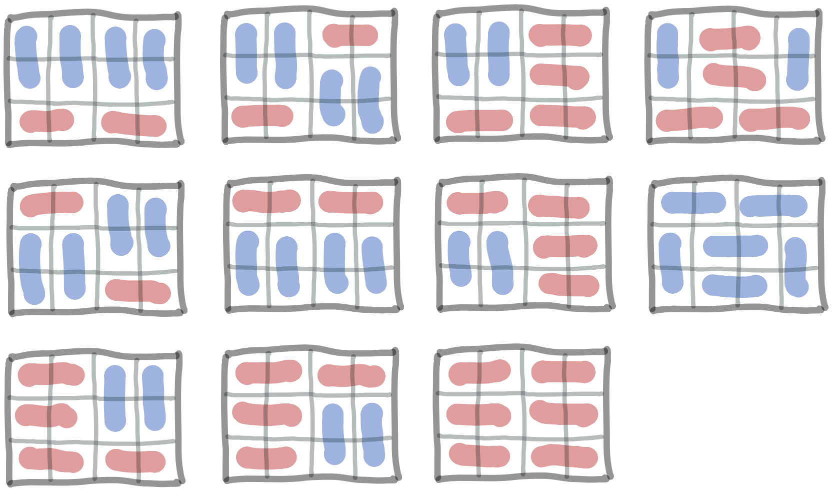
Revised second approach: Can we make this easier to see?





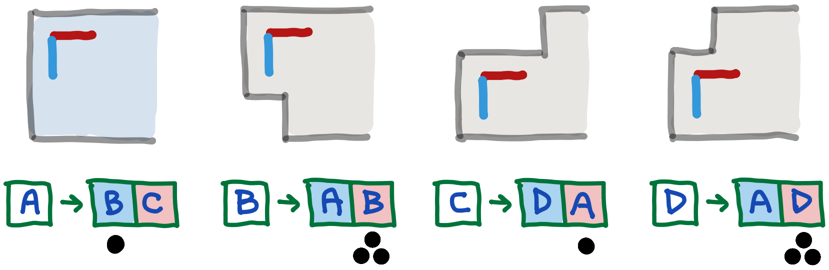


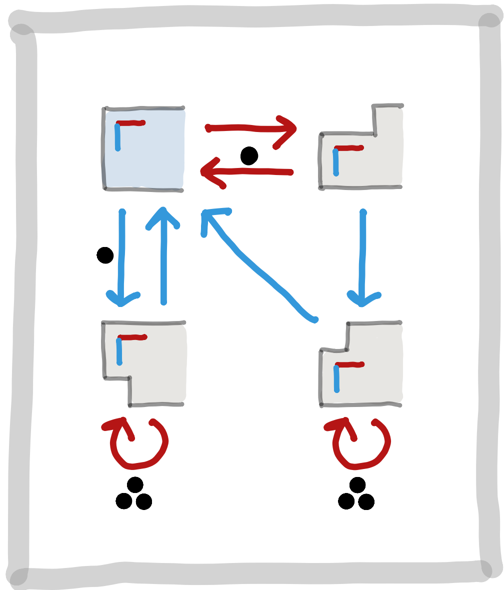
$$\boxed{1/\left(1-t^3-\frac{2t^3}{1-t^3}\right)} = \boxed{\frac{1-t}{1-4t+t^2}}$$



Better to stop only when there is a choice.

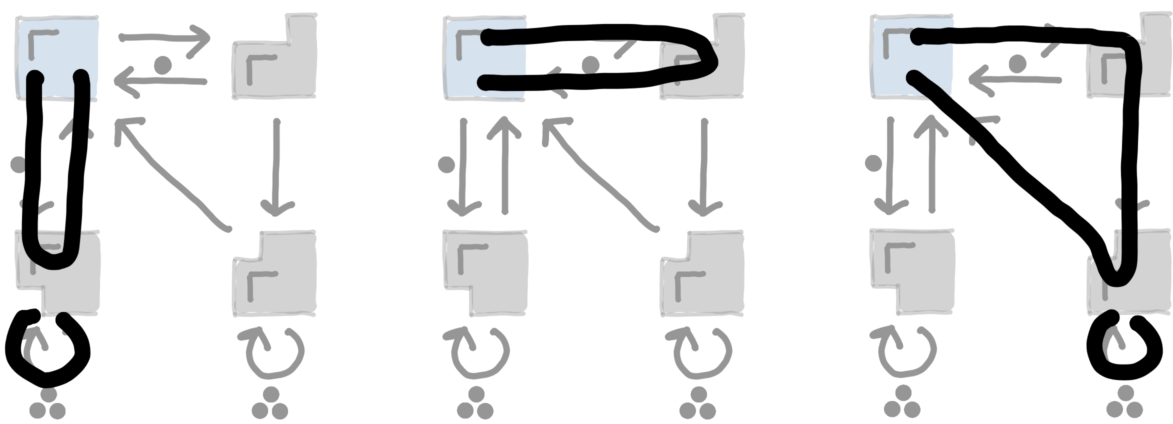
Color-code edges.





(• marks extra steps to get to a choice.)

Now classify "irreducible" walks A to A:



$t^3/1-t^3$ $\qquad$ t^3 $\qquad$ $t^3/1-t^3$

$$\boxed{1\Big/\left(1-t^3-\frac{2t^3}{1-t^3}\right)} = \frac{1-t^3}{(1-t^3)-t^3(1-t^3)-2t^3} = \frac{1-t^3}{1-4t^3+t^6}$$

$$\Downarrow$$

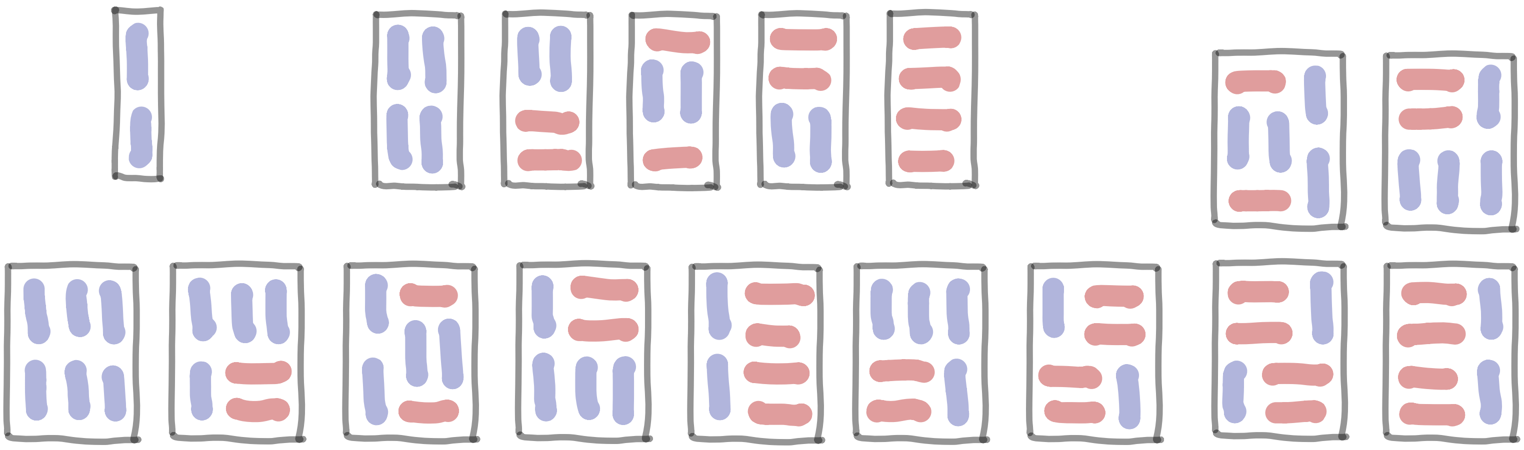
$$\boxed{\frac{1-t}{1-4t+t^2}}$$

3 dominos per n

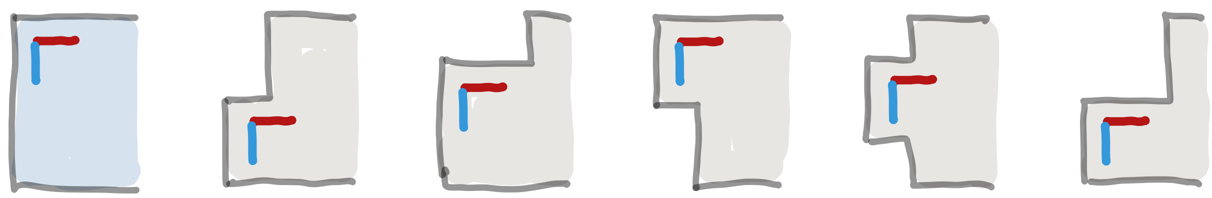
Payoff: Can we do a harder case?

Let $F(n)$ = # of domino tilings of a $4 \times n$ grid.

$F(1)=1$, $F(2)=5$, $F(3)=11$. Find $F(4)$, $F(5)$, generating function.

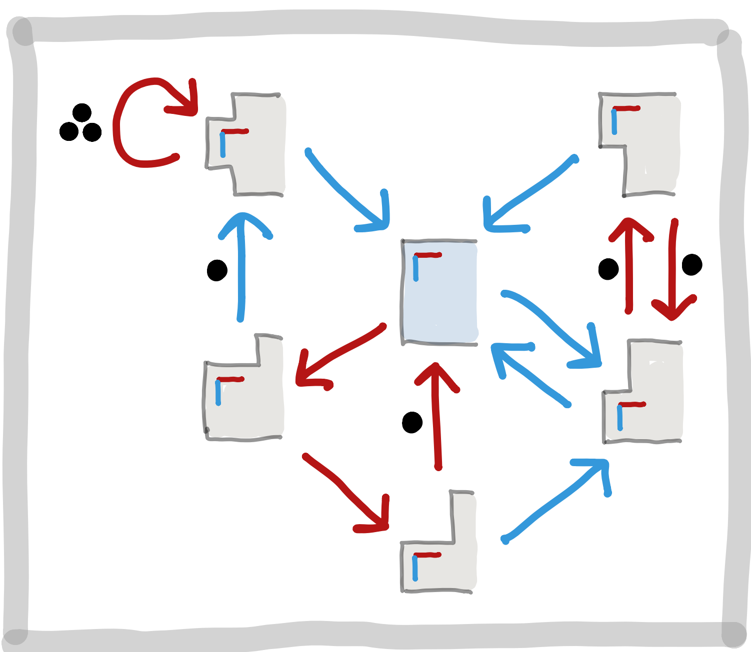


what are frontier shapes? Stop only where there's a choice.

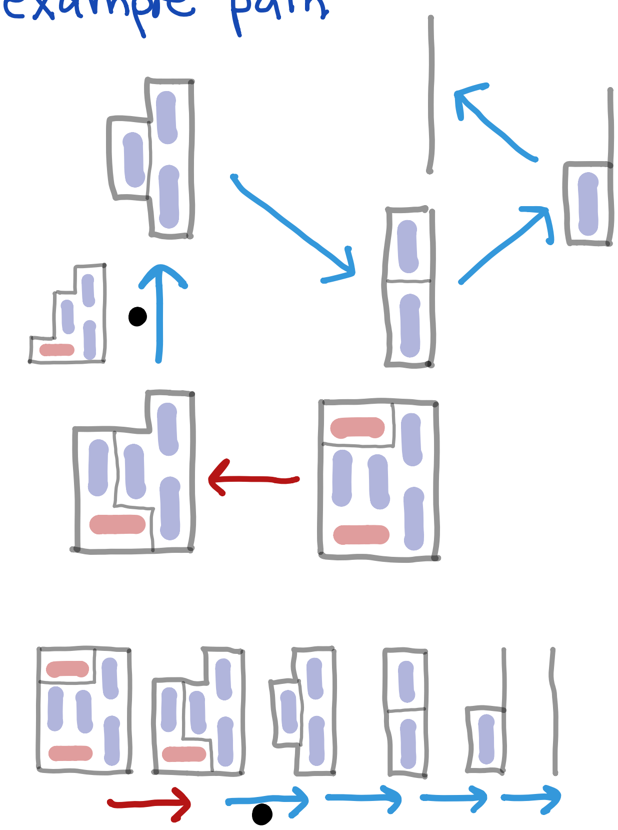


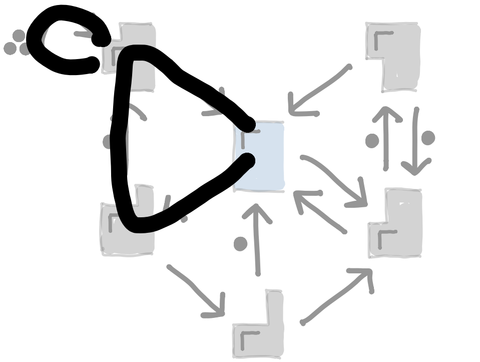
A → B C   B → A D   C → E F   D → A B   E → A E   F → B A

graph

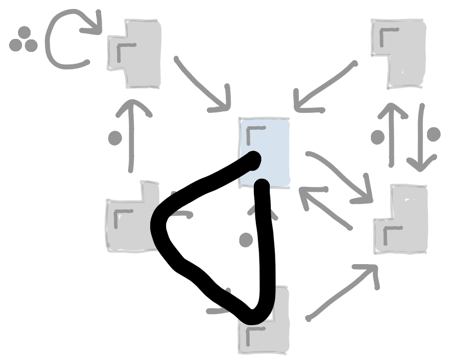


example path

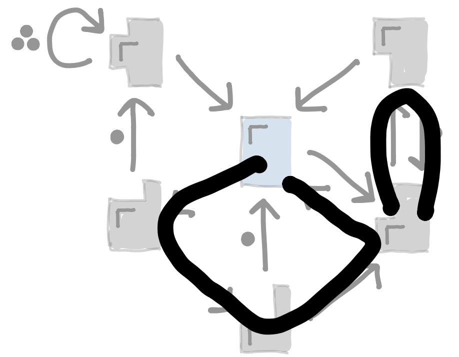




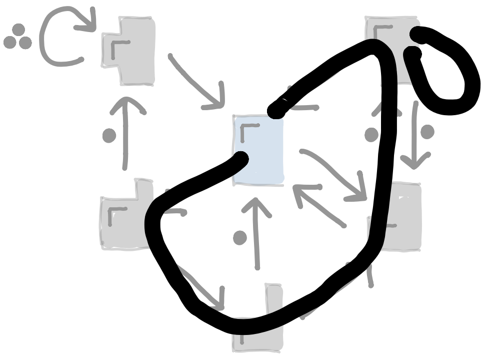
$t^4/1-t^4$



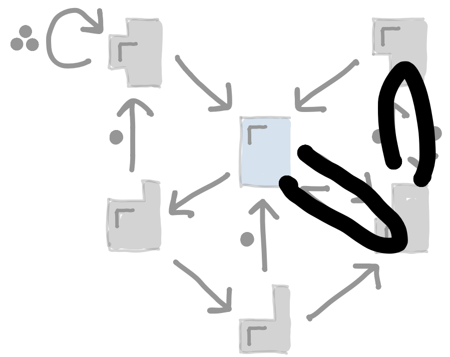
t^4



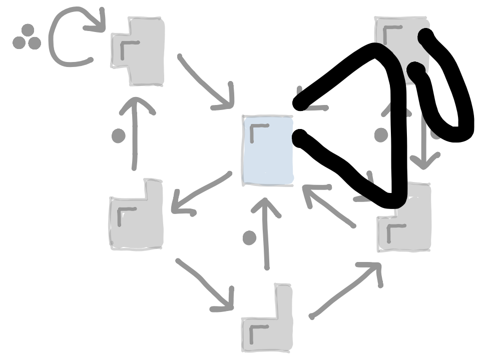
$t^4/1-t^4$



$t^6/1-t^4$



$t^2/1-t^4$



$t^4/1-t^4$

Two dominos per n, so substitute t for t^2 everywhere.

$$\boxed{\frac{1}{1-\left(t^2+\frac{t+3t^2+t^3}{1-t^2}\right)}}$$

$$= \boxed{\frac{1-t^2}{1-t-5t^2-t^3+t^4}}$$

dominos.nb

```
In[1]:= g = 1 / (1 - (t^2 + (t + 3 t^2 + t^3) / (1 - t^2)))
```

$$\text{Out[1]=} \quad \frac{1}{1-t^2-\frac{t+3\,t^2+t^3}{1-t^2}}$$

```
In[2]:= g // Simplify
```

$$\text{Out[2]=} \quad \frac{1-t^2}{1-t-5\,t^2-t^3+t^4}$$

```
In[3]:= Series[g, {t, 0, 5}]
```

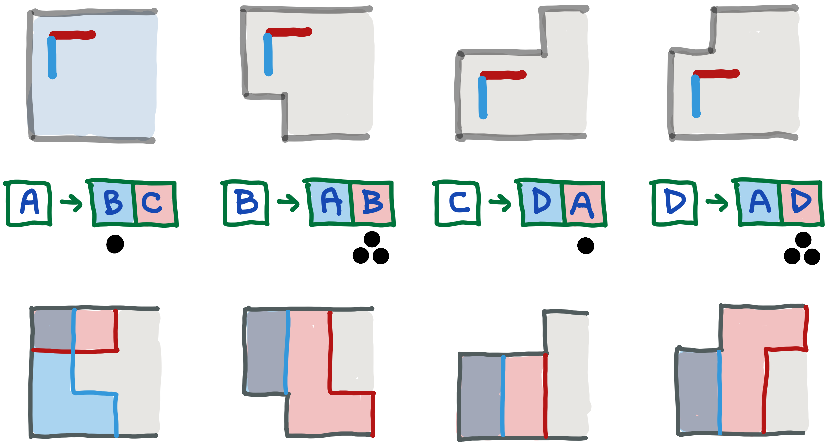
$$\text{Out[3]=} \quad 1+t+5\,t^2+11\,t^3+36\,t^4+95\,t^5+O[t]^6$$

or we multiply by $\frac{1-t^2}{1-t^2}$ $\Longleftrightarrow$ Mathematica code

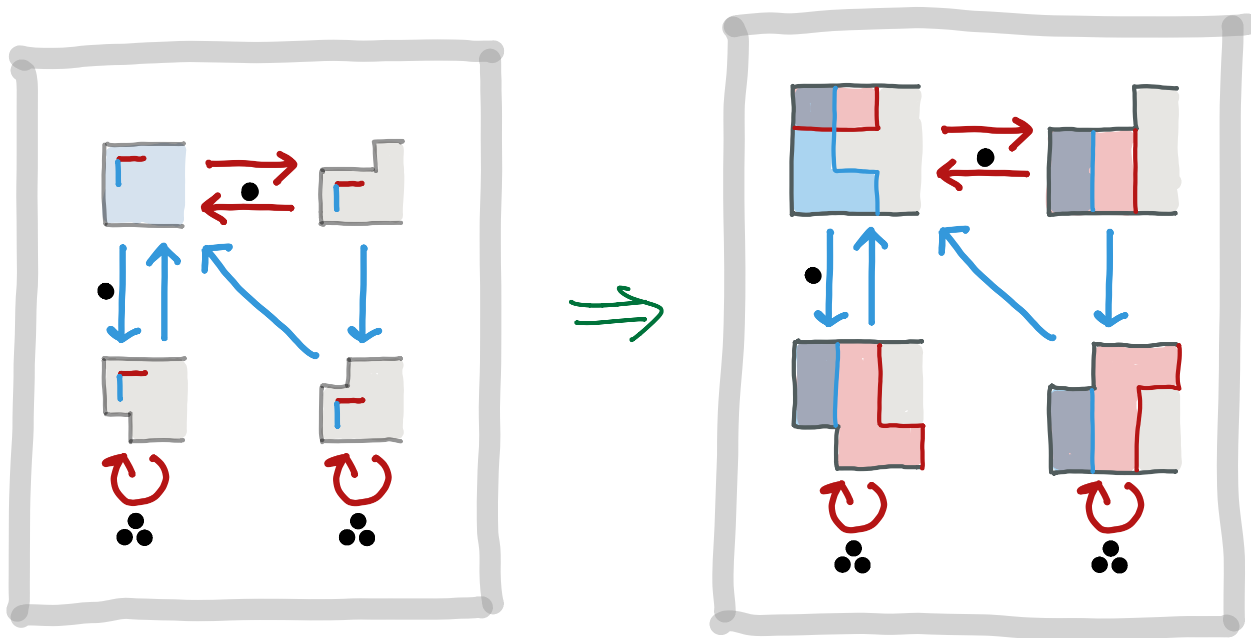
n	0	1	2	3	4	5	...
$1/(1-t-5t^2-t^3+t^4)$	1	1	6	12	42	107	...
$-t^2/(1-t-5t^2-t^3+t^4)$	0	0	1	1	6	12	...
$f(n)$	1	1	5	11	36	95	...

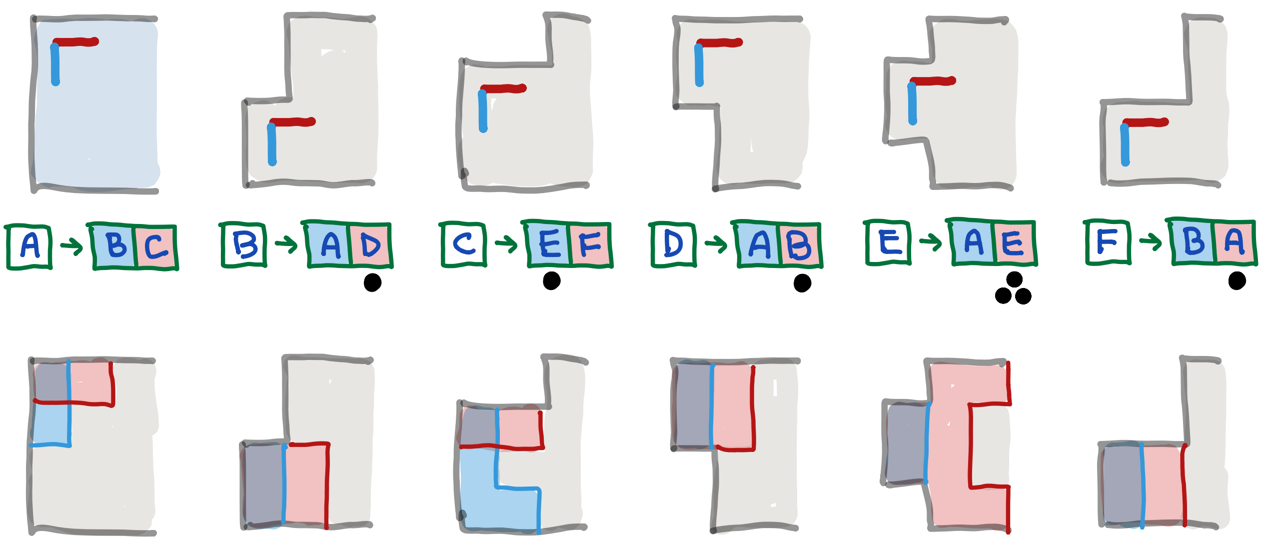
use recurrence
$1 = t+5t^2+t^3-t^4$
shift by t^2

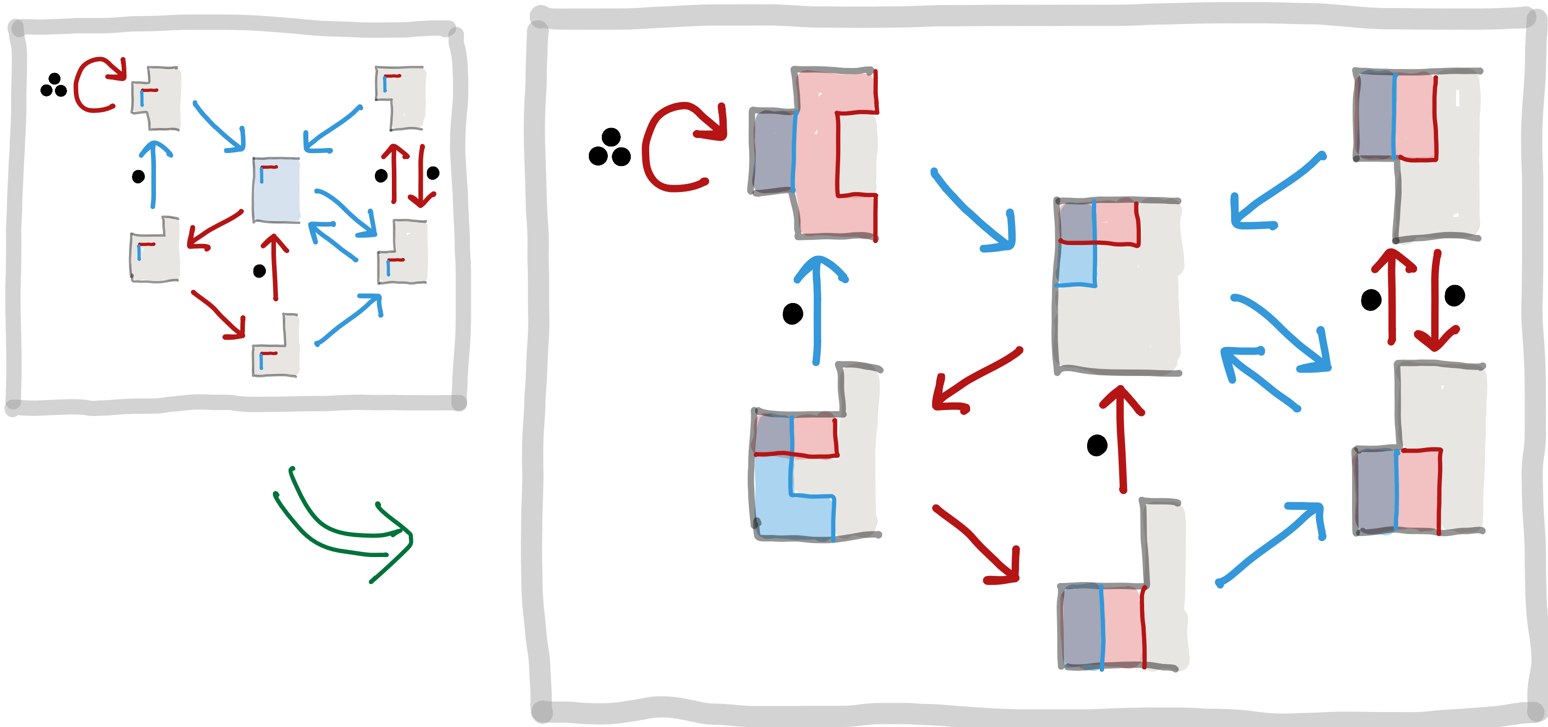
OEIS A005178

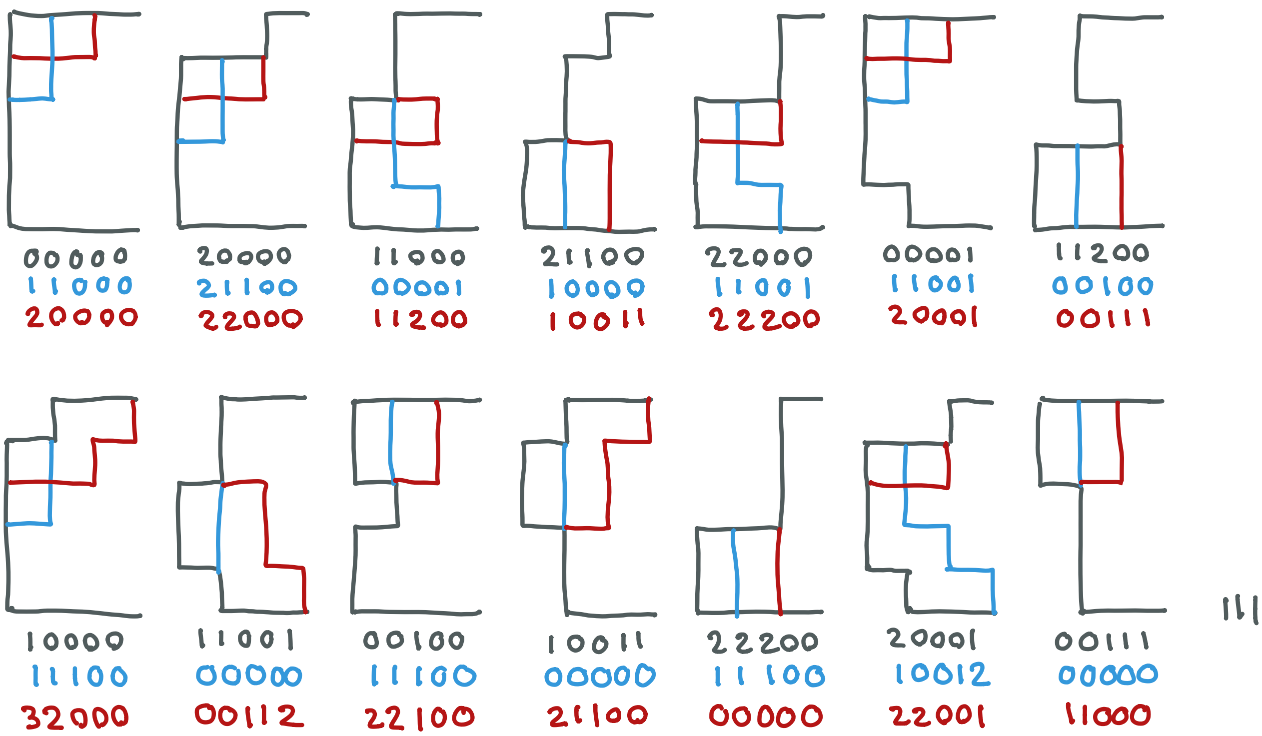


Processing these tables is a strain. Can we make this easier to see?









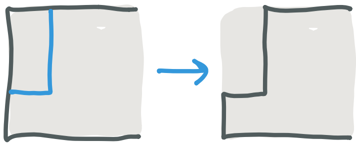
Easier to learn rules for 00000 representation without diagrams

Need to track # of moves for ∴ labels

(Of course we could also switch to a computer...)

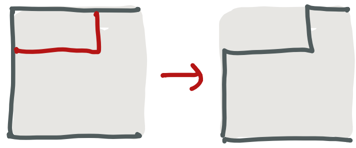
Relearn 4xn case in preparation for 5xn case.

What are rules?



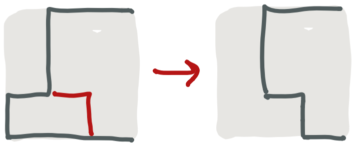
0 0 0
1 1 0

Increment first pair of zeros



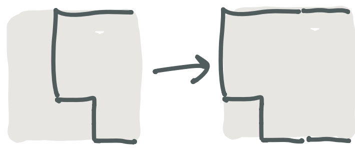
0 0 0
2 0 0

Add 2 to first zero



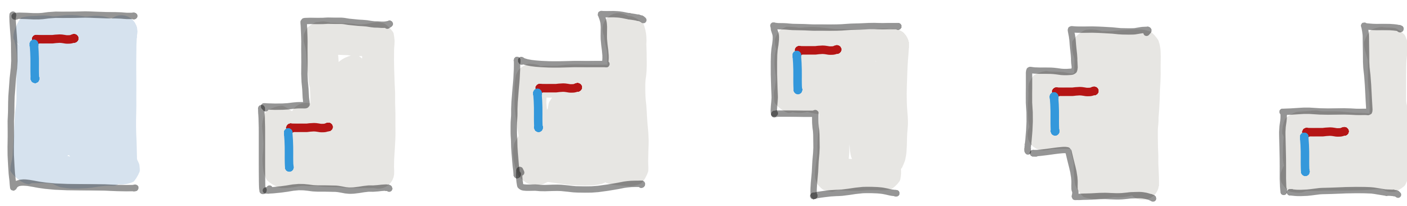
1 1 0
1 1 2 •

Isolated zero is forced move



1 1 2
0 0 1

Decrement if no zeros



A → B C

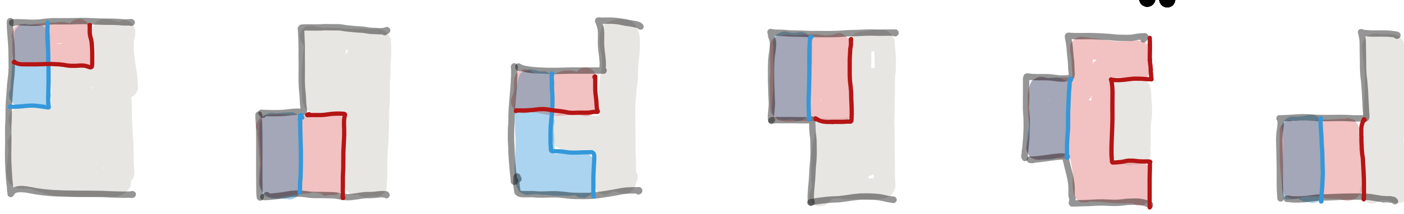
B → A D •

C → E F •

D → A B •

E → A E •••

F → B A •



A	B	C	D	E	F
0000	1100	2000	0011	1001	2200
1100	0000	1001 2	0000	0000	1100
2000	0011 2	2200	1100 2	1001 4	0000 2

B	C	D	E	F
1100	2000	0011	1001	2200
1120	2110	2011	1201	2220
1122 •	2112 •	2211 •	1221 •	2222 •
0011	1001	1100	0110	1111
			2110 •	0000
			2112 •	
			1001	

Now, 5xn case (confirms we've reached that table.)

0	0	0	0	0
1	1	0	0	0
2	0	0	0	0

0	0	0	0	1
1	1	0	0	1
2	0	0	0	1

0	0	0	1	1
0	0	1	0	0
2	0	0	1	1

2

0	0	1	0	0
1	1	1	0	0
2	2	1	0	0

2

0	0	1	1	1
0	0	0	0	0
1	1	0	0	0

2

0	0	1	1	2
0	0	0	0	1
1	1	0	0	1

2

1	0	0	0	0
1	1	1	0	0
1	2	0	0	0

1	0	0	1	1
0	0	0	0	0
2	1	1	0	0

3

1	0	0	1	2
0	0	0	0	1
1	0	0	1	2

5

1	1	0	0	0
0	0	0	0	1
1	1	2	0	0

2

1	1	0	0	1
0	0	0	0	0
0	0	1	1	2

3

1	1	1	0	0
0	0	0	0	0
0	0	0	1	1

2

1	1	2	0	0
0	0	1	0	0
0	0	1	1	1

2

1	2	0	0	0
2	1	0	0	1
1	2	2	0	0

3

1	2	2	0	0	
2	1	1	0	0	2
1	0	0	0	0	3

2	0	0	0	0
2	1	1	0	0
2	2	0	0	0

2	0	0	0	1
1	0	0	1	2
2	2	0	0	1

3

2	0	0	1	1
1	0	0	0	0
1	1	1	0	0

2

2	1	0	0	1
1	0	0	0	0
2	1	0	0	1

5

2	1	1	0	0
1	0	0	0	0
1	0	0	1	1

2

2	2	0	0	0
1	1	0	0	1
2	2	2	0	0

2

2	2	0	0	1
1	1	0	0	0
0	0	0	0	1

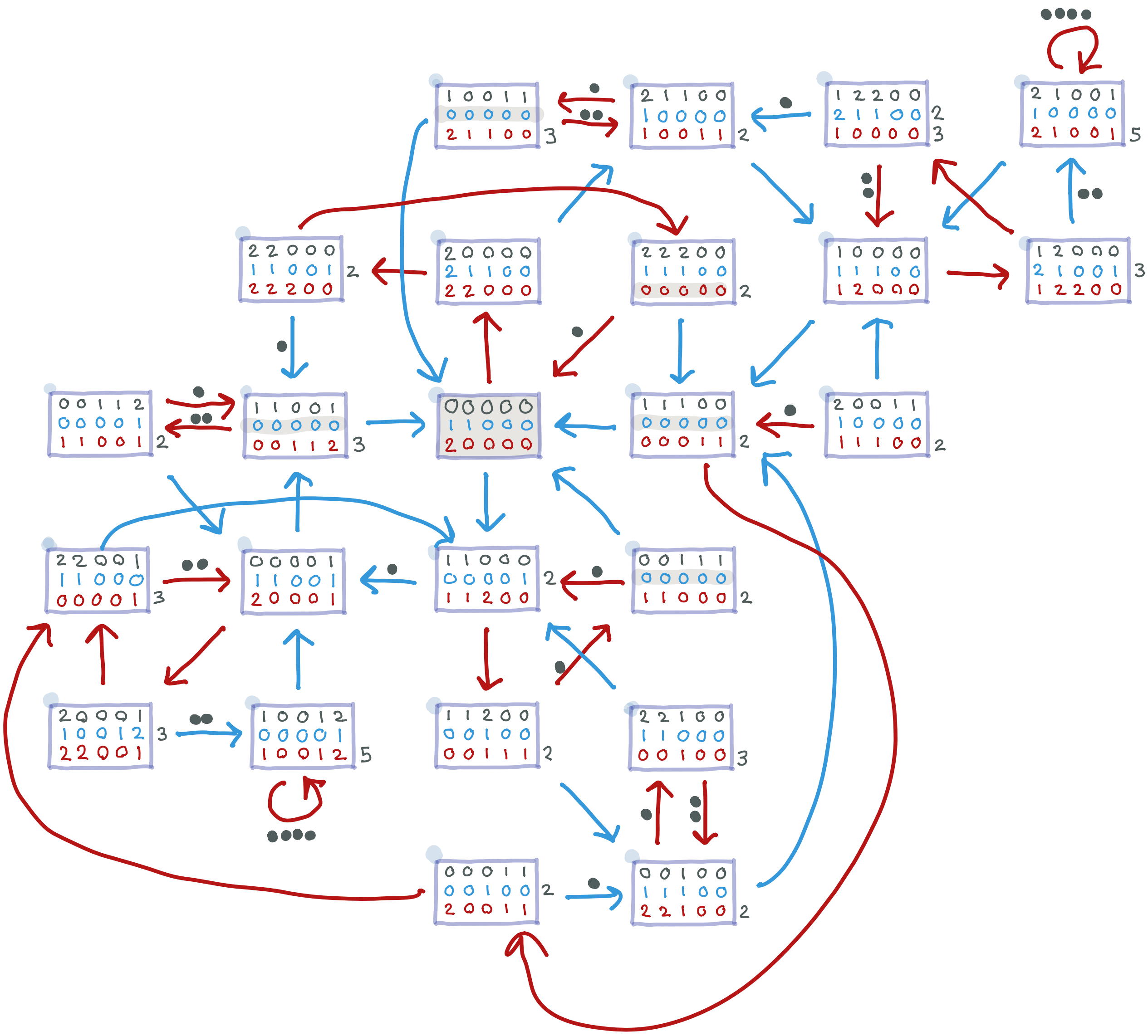
3

2	2	1	0	0
1	1	0	0	0
0	0	1	0	0

3

2	2	2	0	0
1	1	1	0	0
0	0	0	0	0

2



We're well past the point where we should have switched to a computer.

At least the process is in a form that's easy to program.

This wasn't easy, but the fact it is possible shows us this kind of counting problem can be mechanized.

Generalization: **Finite State Automata**

Aigner, p242 Burnside's lemma

$\boxed{\text{Counting with symmetry}}$

$$\sum_{x \in X} |G_x| = \sum_{g \in G} |X_g| . \tag{1}$$

Lemma 6.1. *Let G act on X. Then for any $x \in X$,*

$$|M(x)| = \frac{|G|}{|G_x|} . \tag{2}$$

Lemma 6.2 (Burnside–Frobenius). *Let the group G act on X, and let $\mathcal{M}$ be the set of patterns. Then*

$$|\mathcal{M}| = \frac{1}{|G|} \sum_{g \in G} |X_g| . \tag{3}$$

We need to understand how to read this.

X = raw set of objects

G = symmetries acting on X

M = patterns, equivalence classes of objects up to symmetry

X_g = elements of X fixed by $g \in G$

Example: X = length 2 lists from $\{a,b\}$ $G = \{ \boxed{1}, \boxed{\leftrightarrow} \}$

do nothing flip

$X = \{ \boxed{a,a}\ \boxed{a,b}\ \boxed{b,a}\ \boxed{b,b} \}$

$M = \{ \boxed{\boxed{a,a}}\ \boxed{\boxed{a,b}\ \boxed{b,a}}\ \boxed{\boxed{b,b}} \}$

$|X| = 4$

$|G| = 2$

$|M| = 3$

$\boxed{\boxed{a,a}}$ $\boxed{\boxed{a,b}\ \boxed{b,a}}$ $\boxed{\boxed{b,b}}$

M = "orbits" of action of G on X

$X_1 = \{ \boxed{a,a}\ \boxed{a,b}\ \boxed{b,a}\ \boxed{b,b} \}$

$X_{\leftrightarrow} = \{ \boxed{a,a}\ \boxed{b,b} \}$

$|X_1| = 4$

$|X_{\leftrightarrow}| = 2$

$$\boxed{\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{2}\left(|X_1| + |X_{\leftrightarrow}|\right) = \frac{1}{2}(4+2) = 3 = |M|}$$

Example: $X =$ length 3 lists from $\{a, b, c\}$

$G = \{ \boxed{1}, \boxed{\leftrightarrow} \}$

do nothing   flip

$|X| = 27$ $\qquad |X_1| = 27$

$|G| = 2$ $\qquad |X_{\leftrightarrow}| = 9$

$|M| = 18$

$$\boxed{\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{2}\left(|X_1| + |X_{\leftrightarrow}|\right) = \frac{1}{2}(27+9) = 18 = |M|}$$

Example: X = length k lists from $\{a_1, \dots, a_n\}$

$G = \{ \boxed{1}, \boxed{\leftrightarrow} \}$

do nothing   flip

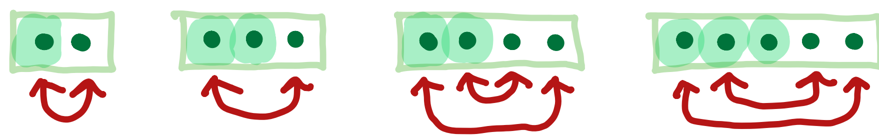
$|X| = n^k = |X_1|$

$|G| = 2$

$|X_{\leftrightarrow}| = n^{\lceil \frac{k}{2} \rceil}$

substep: do a counting problem

$|X_{\leftrightarrow}| = n^{\lceil \frac{k}{2} \rceil}$ ($\lceil \frac{k}{2} \rceil$ = round up $k/2$)



1 choice	2 choices	2 choices	3 choices
n	n^2	n^2	n^3

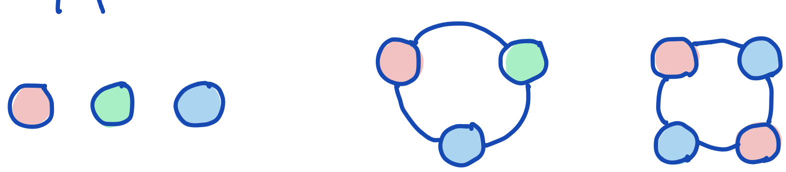
$$\boxed{\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{2}(|X_1| + |X_{\leftrightarrow}|) = \frac{1}{2}(n^k + n^{\lceil \frac{k}{2} \rceil}) = |M|}$$

$n = k = 2$ $\frac{1}{2}(2^2 + 2) = 3$ ☑

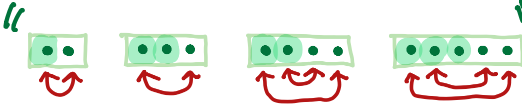
$n = k = 3$ $\frac{1}{2}(3^3 + 3^2) = 18$ ☑

Example: "Necklace" problems

Make an n-bead necklace using k possible colors of beads
Two patterns are the same if they agree after rotation.
How many patterns?



For each n, there will be a version of the "



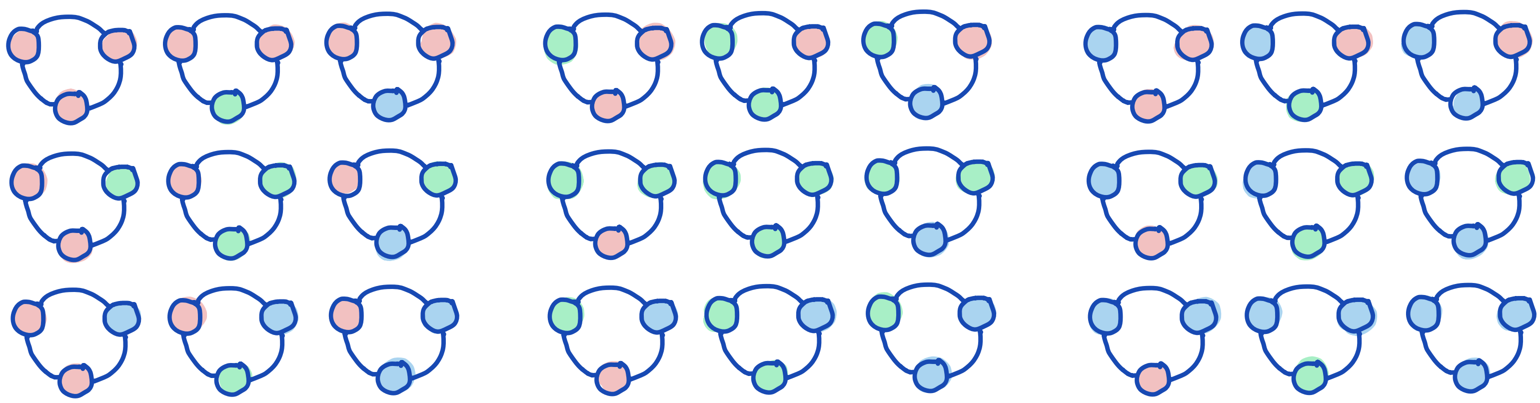
"

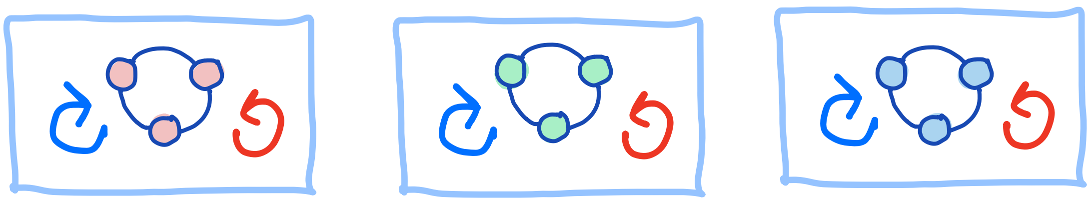
$n^{\lceil \frac{k}{2} \rceil}$ counting problem

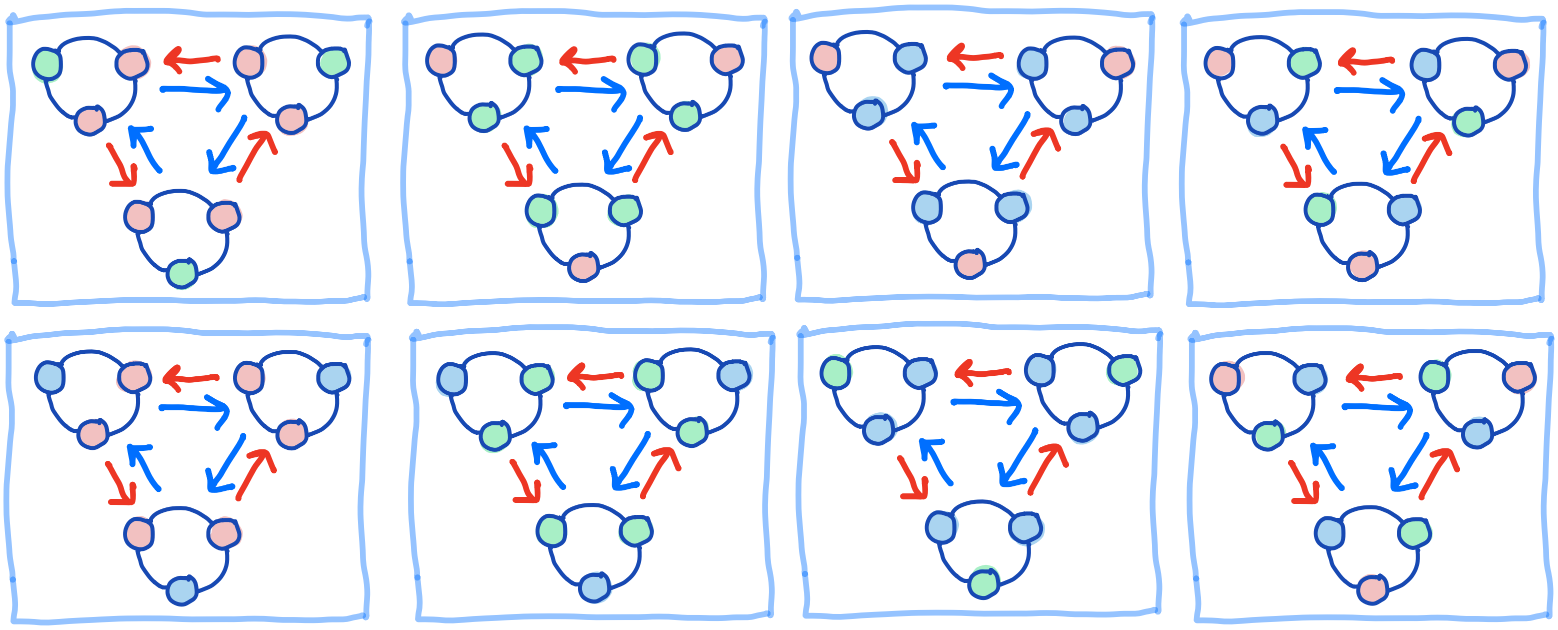
Divisibility = more symmetry

$n = K = 3$ $\qquad G = \left\{ \boxed{1}, \boxed{2}, \boxed{\circlearrowleft} \right\}$

$\boxed{1}$: do nothing $\qquad \boxed{2}$: ⅓ turn → $\qquad \boxed{\circlearrowleft}$: ⅓ turn →



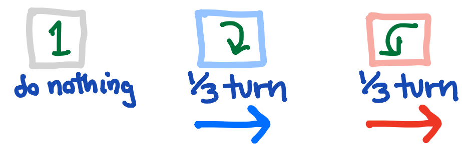




$|G| = 3 \qquad |X| = 27 = |X_1| \qquad |X_2| = |X_{\circlearrowleft}| = 3$

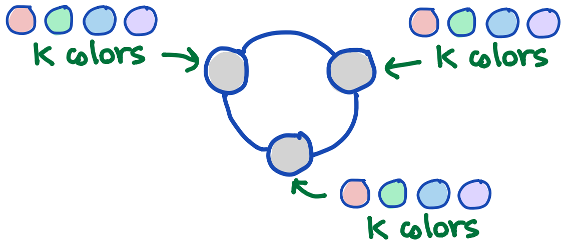
$$\boxed{\frac{1}{|G|}\sum_{g \in G} |X_g| = \frac{1}{3}\left(|X_1| + |X_2| + |X_{\circlearrowleft}|\right) = \frac{1}{3}(27+3+3) = 11}$$ ☑

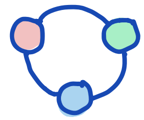
$n=3$ any k

$G = \{$ 

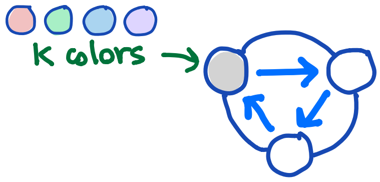
$\}$

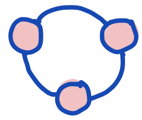
$|X| = |X_1| = k^3$



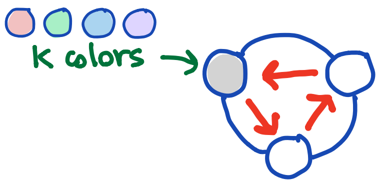


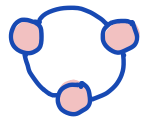
$|X_{\circlearrowright}| = k$





$|X_{\circlearrowleft}| = k$



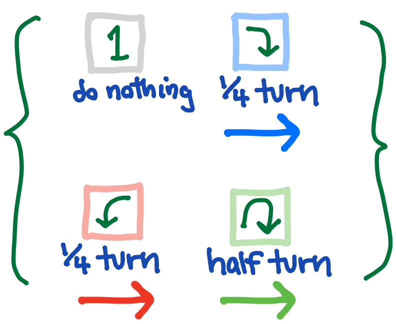


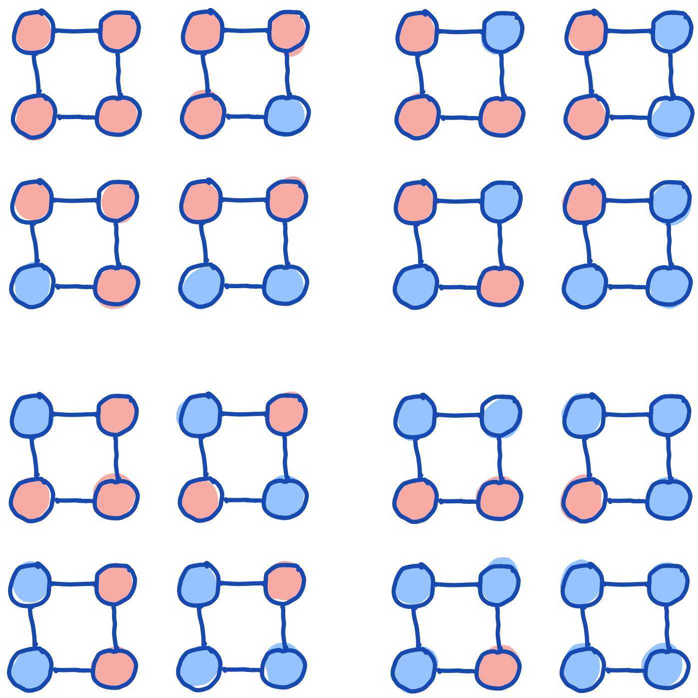
$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{3}\left(|X_1| + |X_{\circlearrowright}| + |X_{\circlearrowleft}|\right) = \frac{1}{3}\left(k^3 + k + k\right)}$$

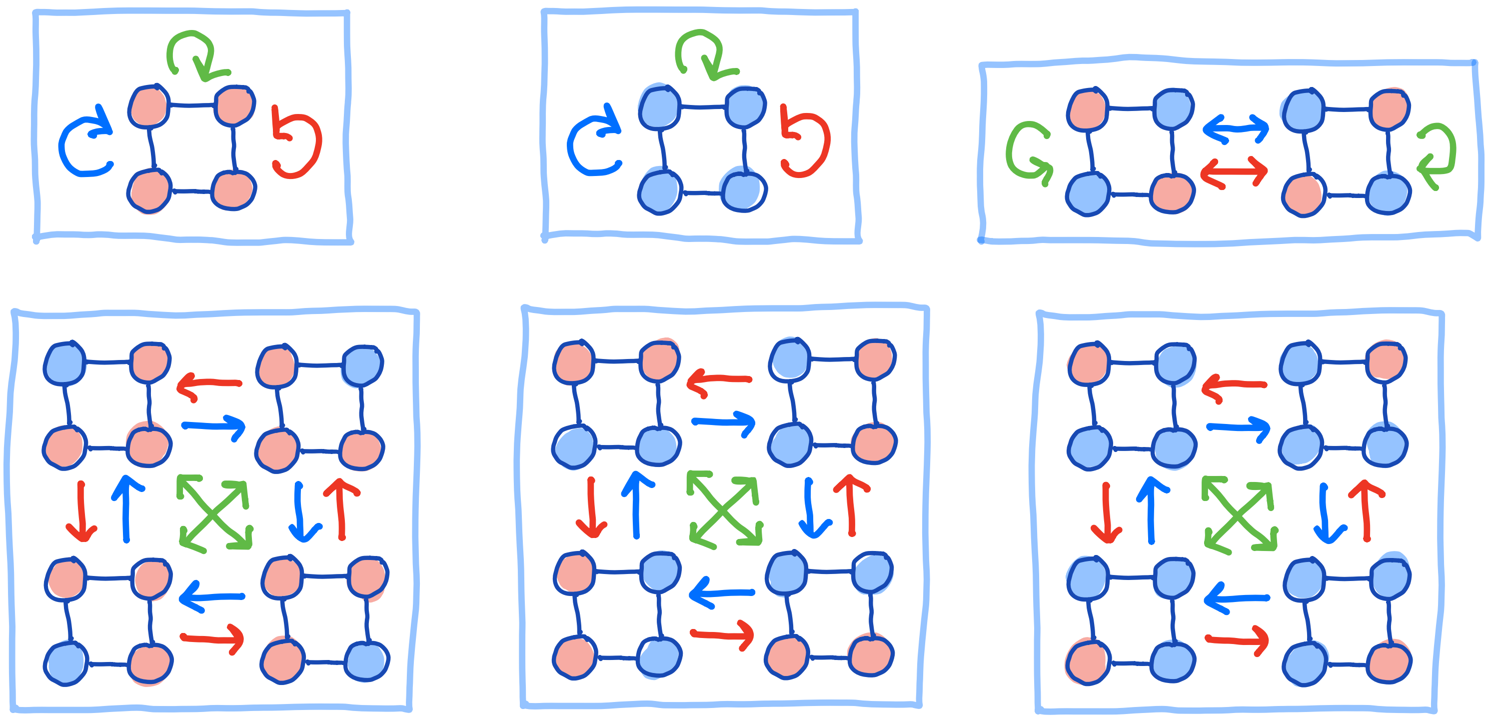
Check: $k=3$ $\quad \frac{1}{3}\left(k^3+k+k\right) = \frac{1}{3}(27+3+3) = 11$

$n=4 \quad k=2$

$G =$







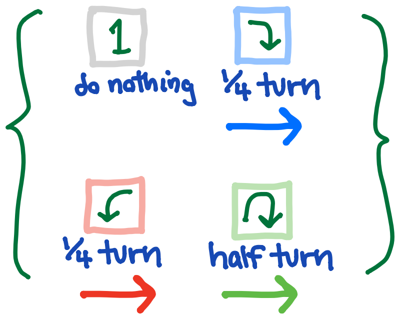
$|G| = 4 \qquad |X| = 16 = |X_1| \qquad |X_{\curvearrowright}| = |X_{\curvearrowleft}| = 2 \qquad |X_{\circlearrowright}| = 4$

$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{4}\left(|X_1|+|X_{\curvearrowright}|+|X_{\curvearrowleft}|+|X_{\circlearrowright}|\right) = \frac{1}{4}(16+2+2+4) = 6}$$

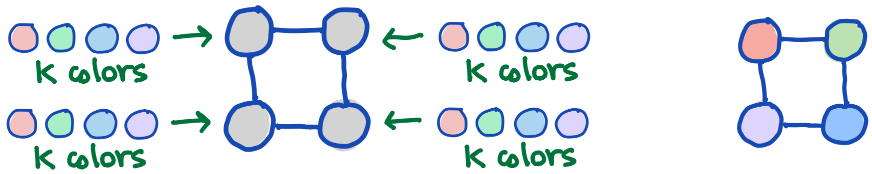
☑

$n = 4 \qquad \text{any } k$

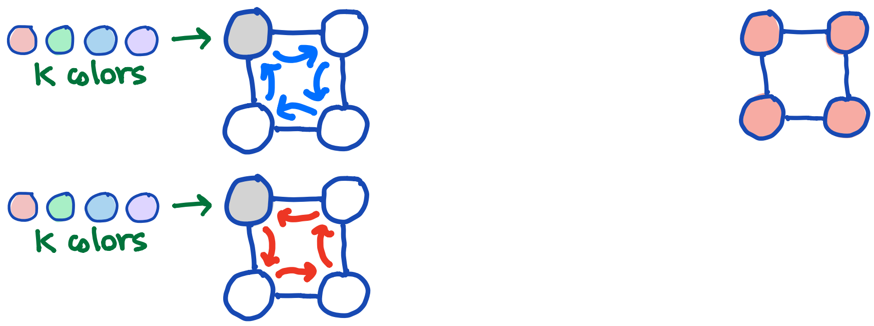
$G =$



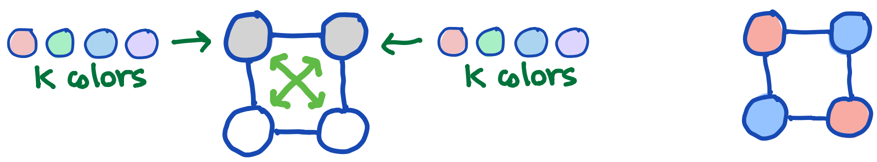
$|X| = |X_1| = k^4$



$|X_{\curvearrowright}| = |X_{\curvearrowleft}| = k$



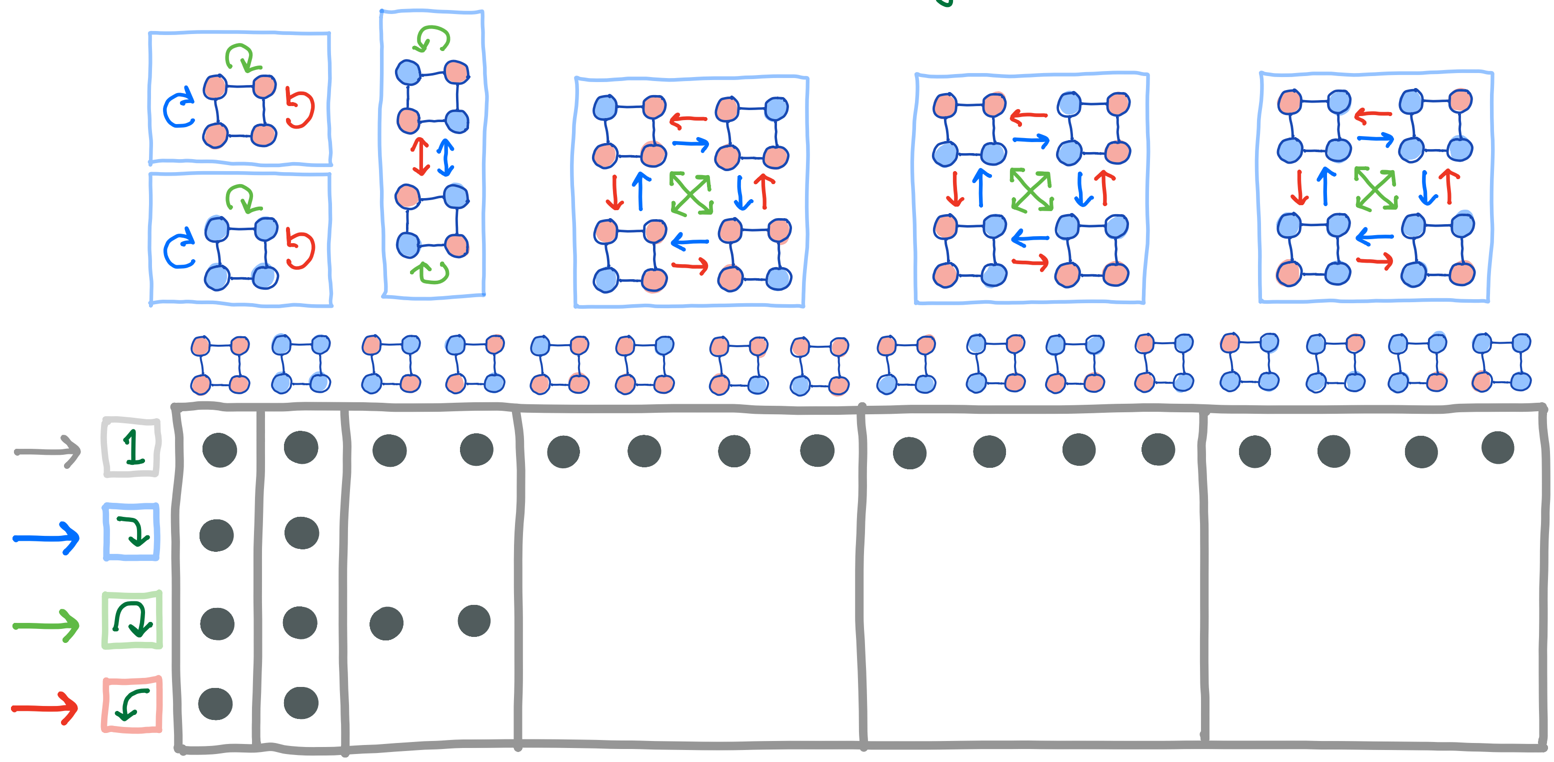
$|X_{\circlearrowright}| = k^2$



$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{4}\left(|X_1|+|X_{\curvearrowright}|+|X_{\curvearrowleft}|+|X_{\circlearrowright}|\right) = \frac{1}{4}\left(k^4+k+k+k^2\right)}$$

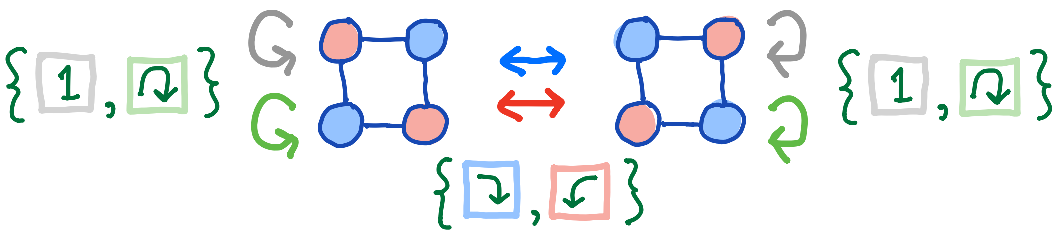
Check: $k = 2 \qquad \frac{1}{4}\left(k^4+k+k+k^2\right) = \frac{1}{4}(16+2+2+4) = 6$ ☑

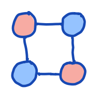
Why does this work? $\frac{1}{|G|}\sum_{g\in G}|X_g| = |M|$



Each dot ● marks an object fixed b a group element.
Each box is a pattern up to symmetry.
The row sums are $|X_1|, |X_{↷}|, |X_{↻}|, |X_{↶}|$.
If we can figure out why each box gets $|G|$ dots, we're done.

Group Theory in a nutshell: things divide up evenly.
Look more closely at each orbit. This one is interesting:



$G_{\square} = \{\boxed{1}, \boxed{↻}\}$ = elements of G that fix 

$\boxed{↷}\, G_{\square} = \boxed{↷}\{\boxed{1}, \boxed{↻}\} = \{\underbrace{\boxed{↷}\boxed{1}}_{\boxed{↷}}, \underbrace{\boxed{↷}\boxed{↻}}_{\boxed{↶}}\} = \{\boxed{↷}, \boxed{↶}\}$

$$\boxed{\left|\{\boxed{1}, \boxed{↻}\}\right| \left|\{\square, \square\}\right| = \left|\{\boxed{1}, \boxed{↷}, \boxed{↻}, \boxed{↶}\}\right| = |G|}$$

Combinatorics Feb23

What is a group? One operation $*$ or $+$
Identity and inverses
Associative: $(ab)c = a(bc)$

$\mathbb{Z}_2$:

+	0	1
0	0	1
1	1	0

mod 2

$\approx$

+	even	odd
even	even	odd
odd	odd	even

$\approx$

*	1	-1
1	1	-1
-1	-1	1

$\approx$

*	1	2
1	1	2
2	2	1

mod 3

$\mathbb{Z}_3$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

mod 3

$\mathbb{Z}_4$:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

mod 4

$\approx$

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

mod 5

+		*
0	$\leftrightarrow$	1
1	$\leftrightarrow$	2
2	$\leftrightarrow$	4
3	$\leftrightarrow$	3

$\mathbb{Z}_2 \times \mathbb{Z}_2$:

+	0,0	0,1	1,0	1,1
0,0	0,0	0,1	1,0	1,1
0,1	0,1	0,0	1,1	1,0
1,0	1,0	1,1	0,0	0,1
1,1	1,1	1,0	0,1	0,0

mod 2,2

$\mathbb{Z}_5$:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\mathbb{Z}_6$:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

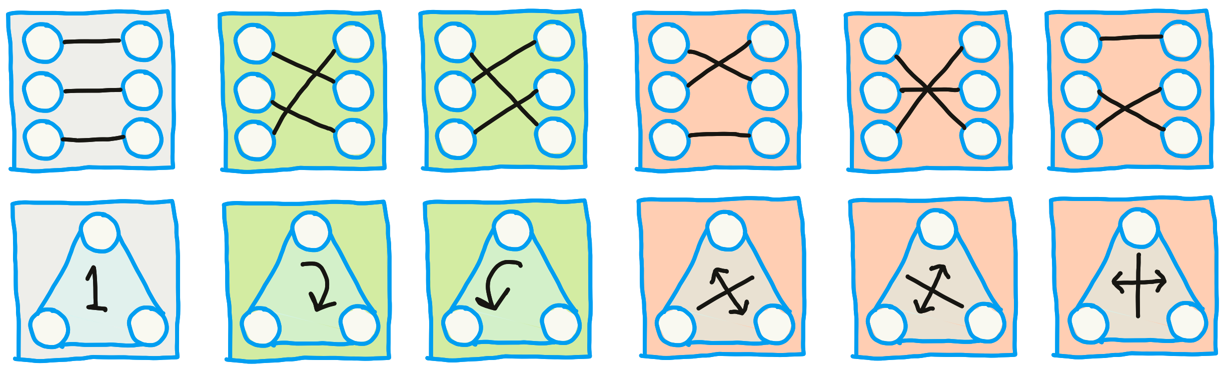
$\mathbb{Z}_2 \times \mathbb{Z}_3$:

+	0,0	0,1	0,2	1,0	1,1	1,2
0,0	0,0	0,1	0,2	1,0	1,1	1,2
0,1	0,1	0,2	0,0	1,1	1,2	1,0
0,2	0,2	0,0	0,1	1,2	1,0	1,1
1,0	1,0	1,1	1,2	0,0	0,1	0,2
1,1	1,1	1,2	1,0	0,1	0,2	0,0
1,2	1,2	1,0	1,1	0,2	0,0	0,1

mod 2,3

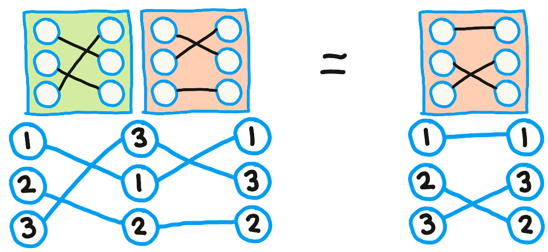
Inverses $\Leftrightarrow$ Each row is a permutation of the first row
Each col is a permutation of the first col

The symmetric group S_3: Permutations of $\{1,2,3\}$
Symmetries of a triangle

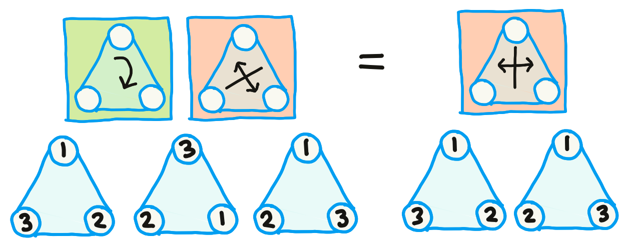


How to multiply?

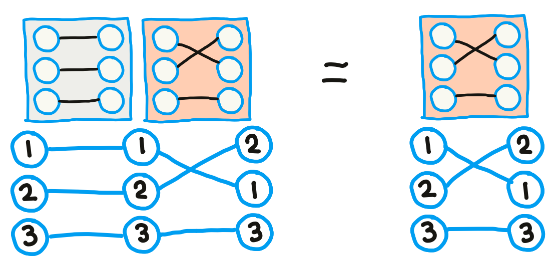
Pull tight



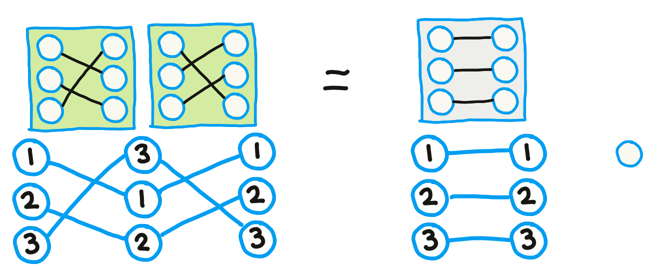
Watch test triangle



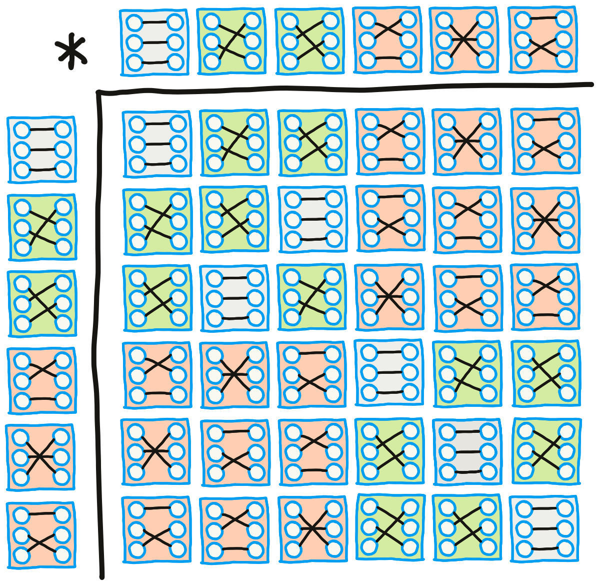
Identity

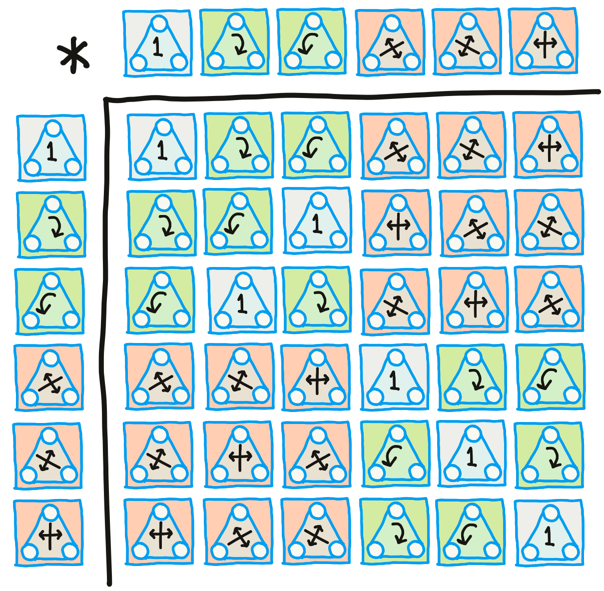


Inverses

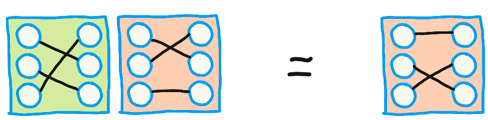


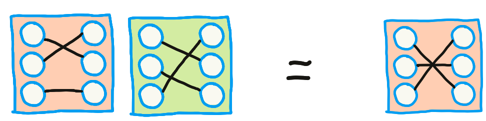
S_3 multiplication tables





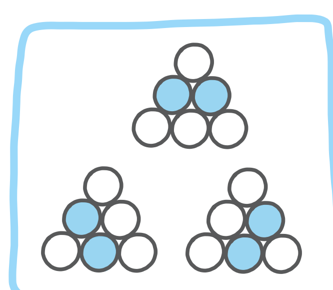
Not commutative

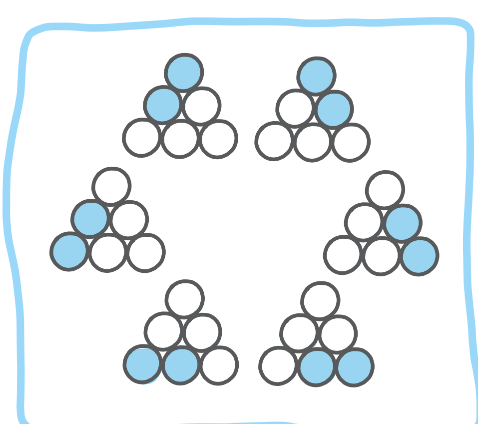


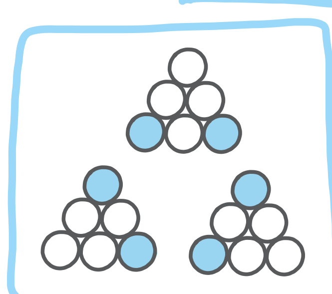


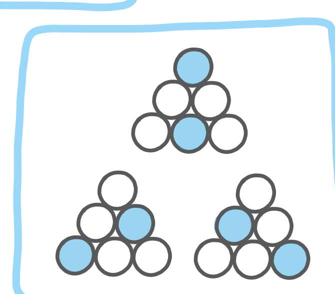
Counting problem: Mark k cells in a triangular grid
How many patterns, up to S_3 symmetry?

$k=2$

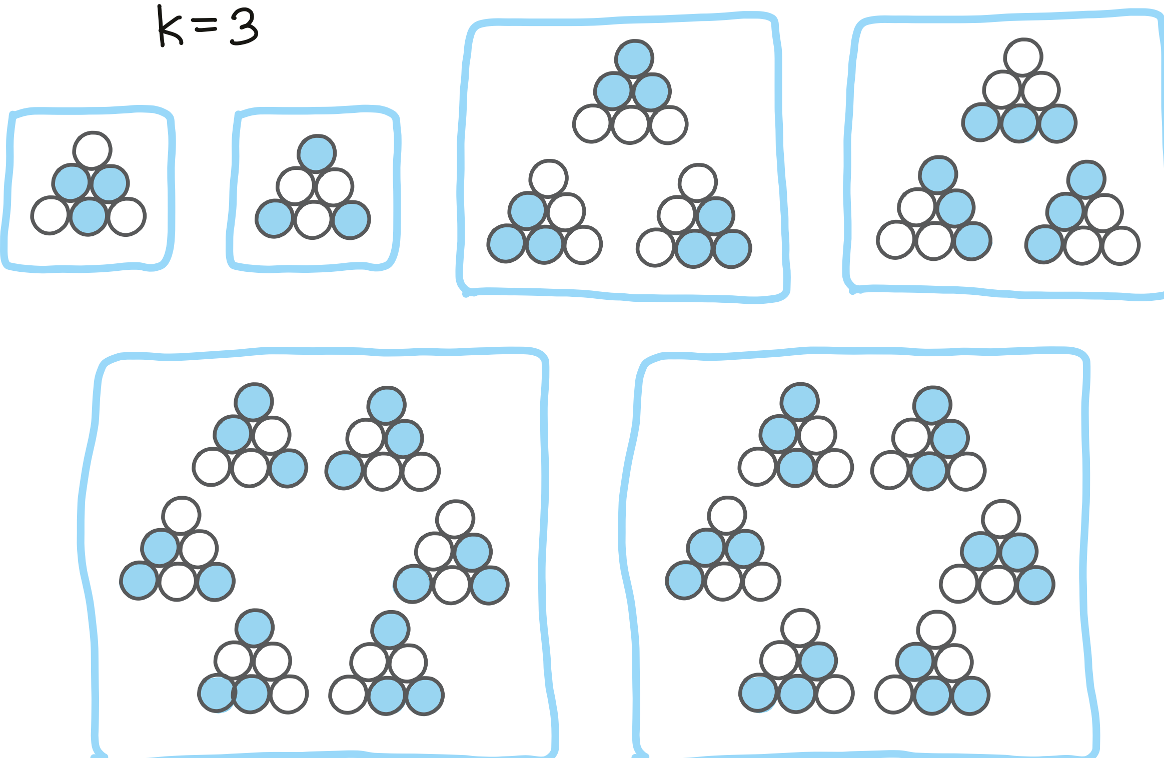




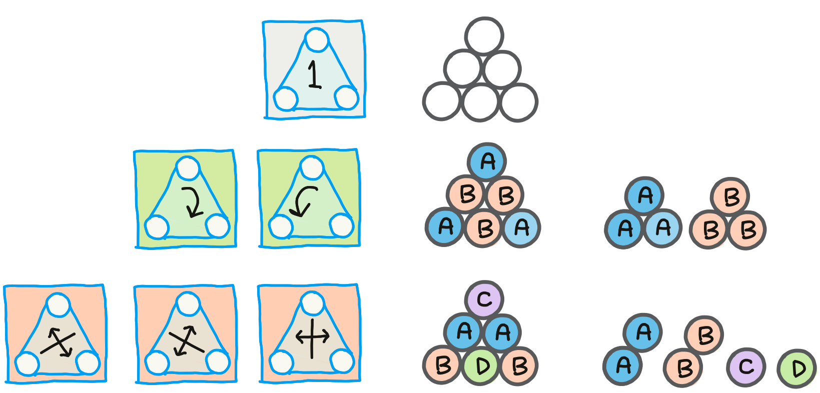




$k=3$



$$|P| = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

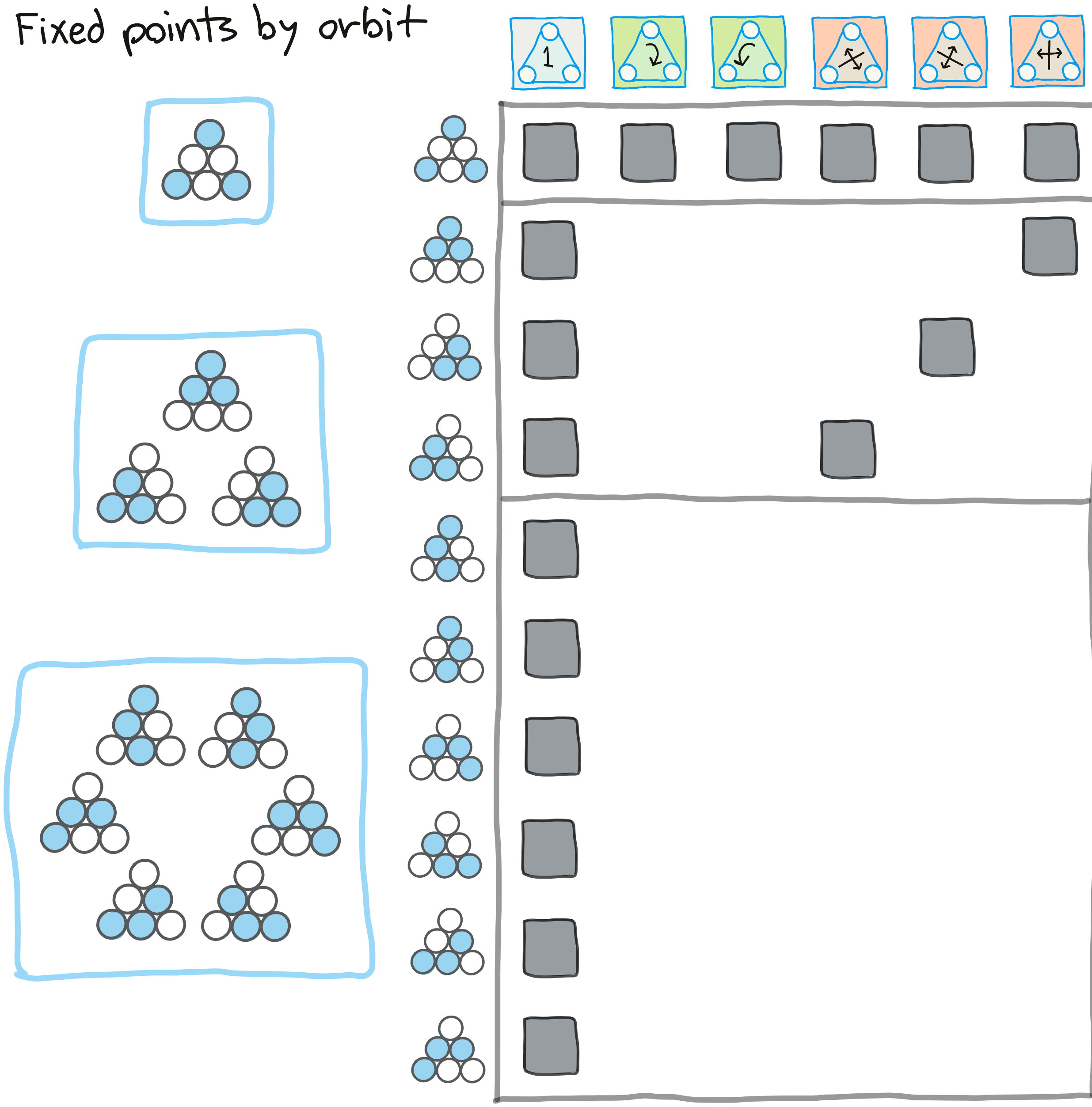


$k=2$	$k=3$
$\binom{6}{2}$	$\binom{6}{3}$
0	$\binom{2}{1}$
$\binom{2}{1}+\binom{2}{2}$	$\binom{2}{1}\binom{2}{1}$

$k=2$: $\frac{1}{6}\left[\binom{6}{2}+3\left(\binom{2}{1}+\binom{2}{2}\right)\right] = \frac{1}{6}(15+3\cdot 3) = 4$ ☑

$k=3$: $\frac{1}{6}\left[\binom{6}{3}+2\binom{2}{1}+3\binom{2}{1}\binom{2}{1}\right] = \frac{1}{6}(20+2\cdot 2+3\cdot 4) = 6$ ☑

Fixed points by orbit

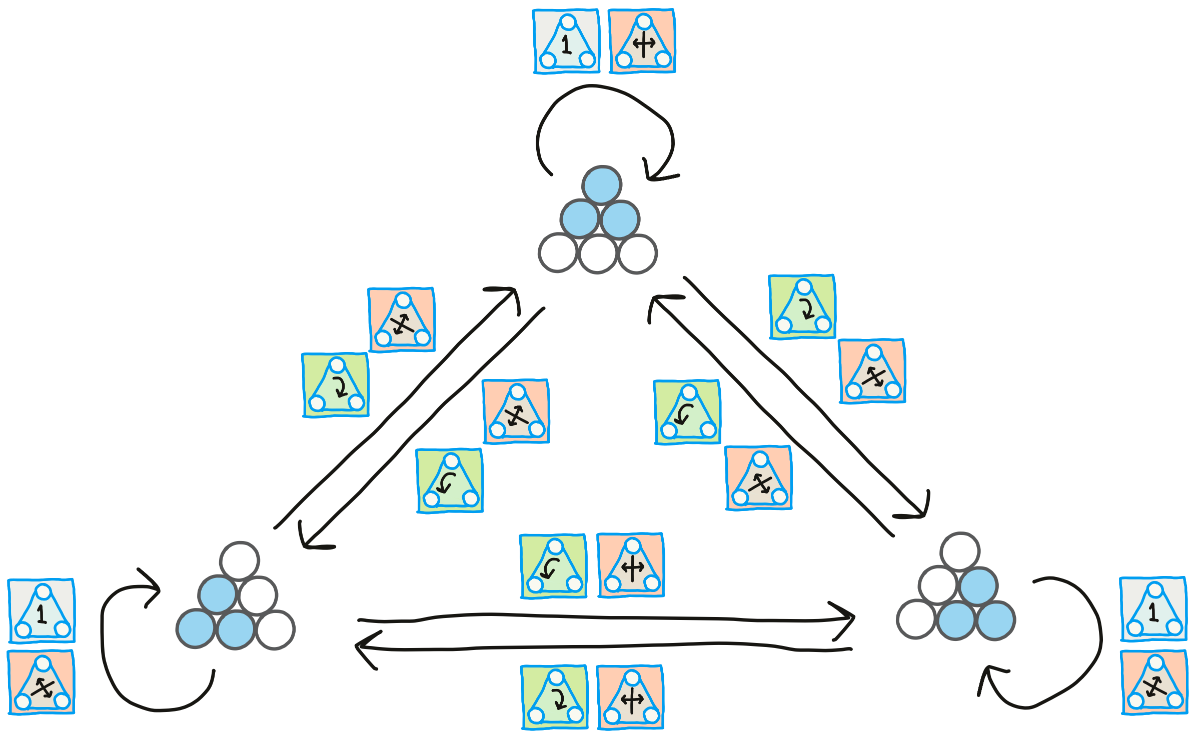


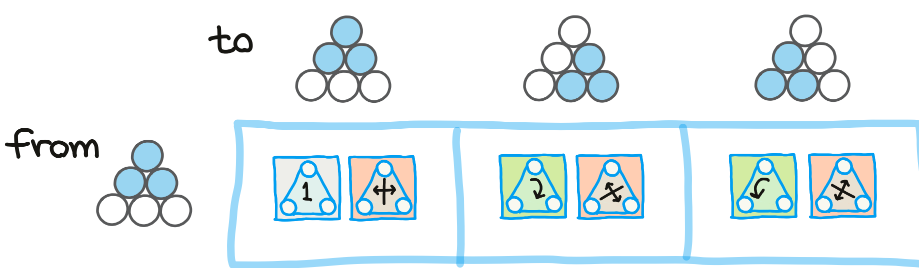
$\sum_{g \in G} |X_g|$ counts all fixed points (g,x) where $gx = x$

If we can understand why there are $|G|$ fixed points per orbit,

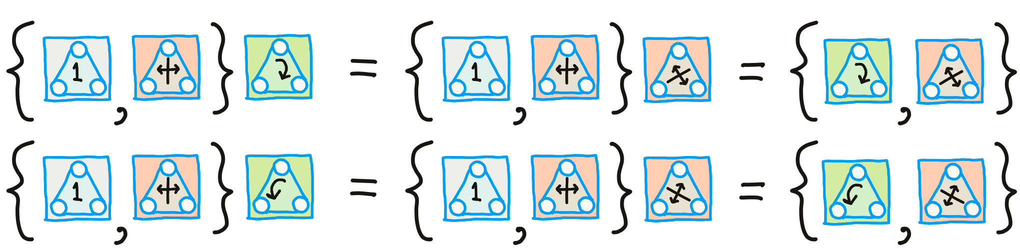
then we understand $|P| = \frac{1}{|G|} \sum_{g \in G} |X_g|$

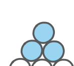
Look closely at how G acts on a particular orbit

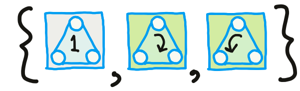


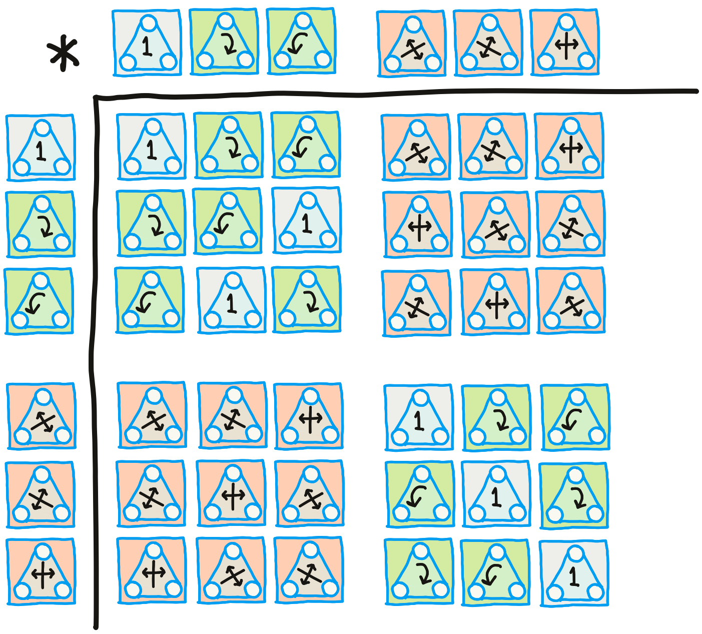


These subsets of G (cosets) are always in 1:1 correspondence with each other, so they divide G into equal sized subsets.



(# Fixed points of )(size of orbit) $= |G|$

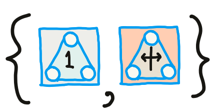
Quotients: mod out by "normal subgroup" 



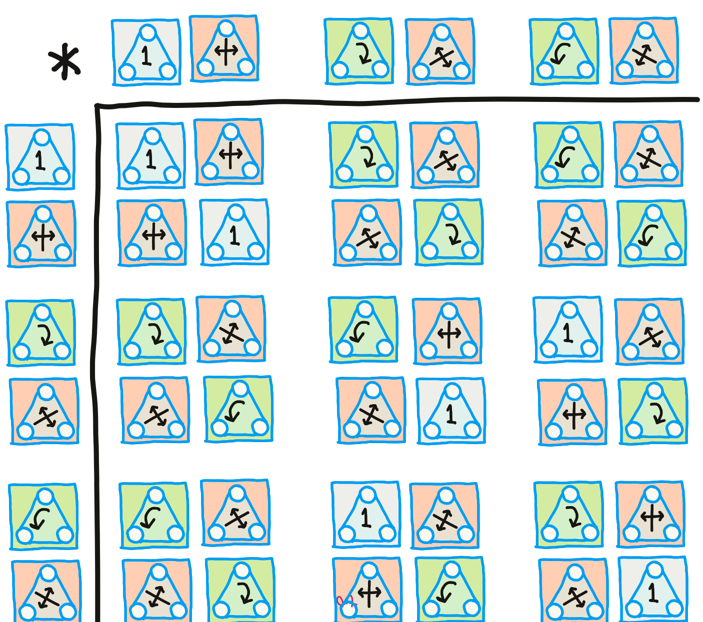
$\Longrightarrow$

+	0	1
0	0	1
1	1	0

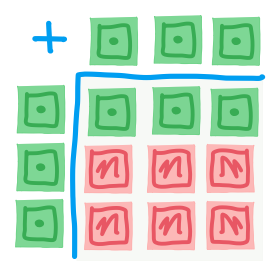
mod 2

 is not normal

and we don't get a coherent table when we try to mod out.



$\Longrightarrow$

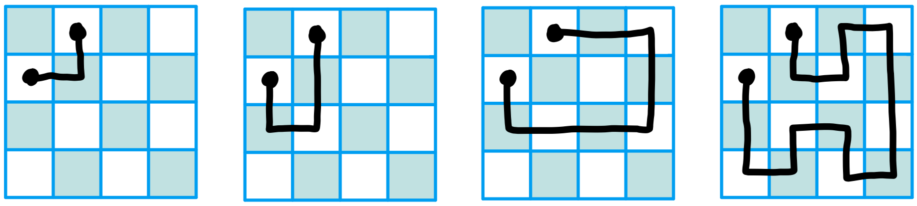


Various entries are inconsistent

Expand on class questions:

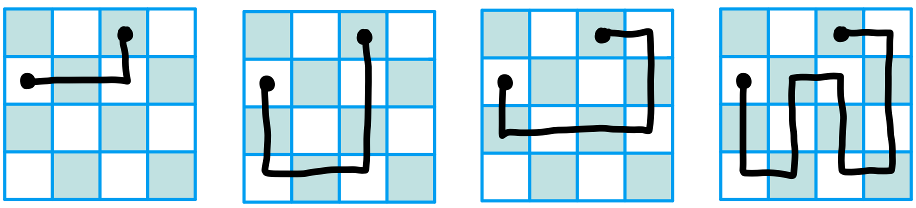
Even-odd parity.

Walks alternate square colors



2 4 8 14

Walks between squares of the same color: even # steps

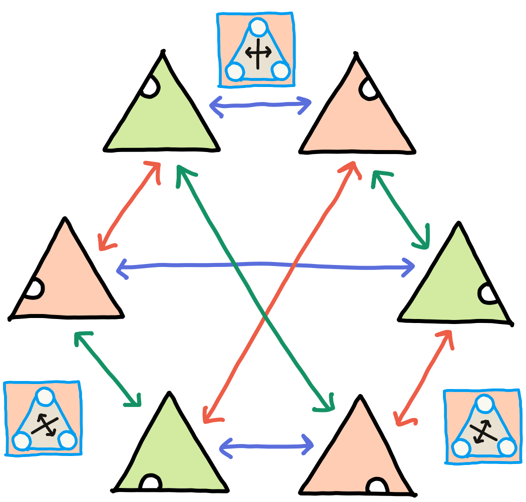


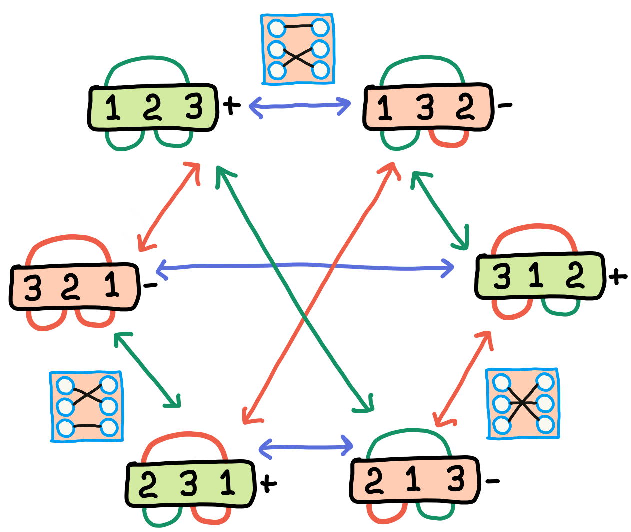
3 7 7 13

Walks between squares of the opposite color: odd # steps

We can checkerboard the graph of all triangle positions.

Flips all change checkerboard color

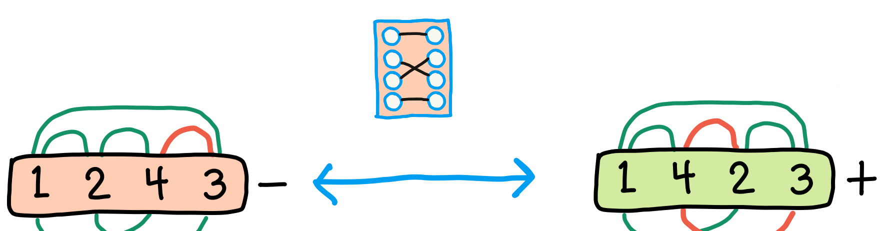




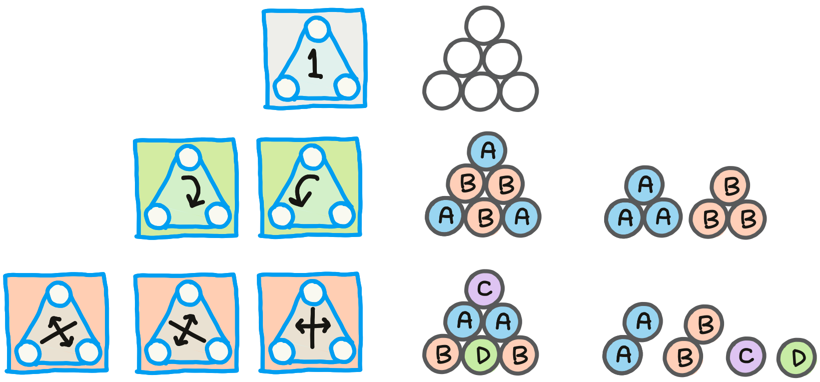
We can checkerboard the graph of all permutations of $\{1,\dots,n\}$

Even-odd: How many pairs are out of order?

Adjacent pair swaps change this count by 1



k colors $\quad |P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$

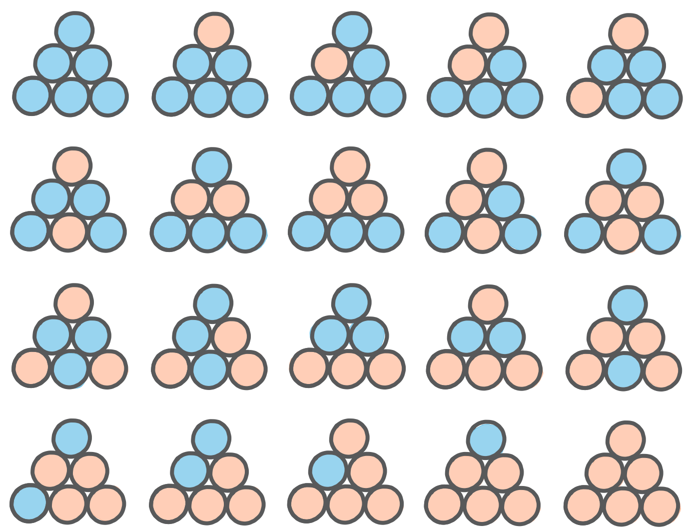


k^6

k^2

k^4

$k=2$



$$|P| = \frac{1}{6}(k^6+2k^2+3k^4)$$

$$= \frac{1}{6}(64\underbrace{+2\cdot4+}_{12}\underbrace{3\cdot16}_{8})$$

$$= 20$$

$k=3 \quad |P| = \frac{1}{6}(k^6+2k^2+3k^4) = \frac{1}{6}(729+2\cdot9+3\cdot81) = 165$

use 1 color: $\quad 3$

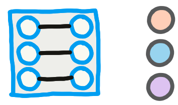
use 2 colors: $\quad \binom{3}{2}18 \quad$ (From above)

$\Rightarrow$ use 3 colors: $\quad 165-3-\binom{3}{2}18 = 108$

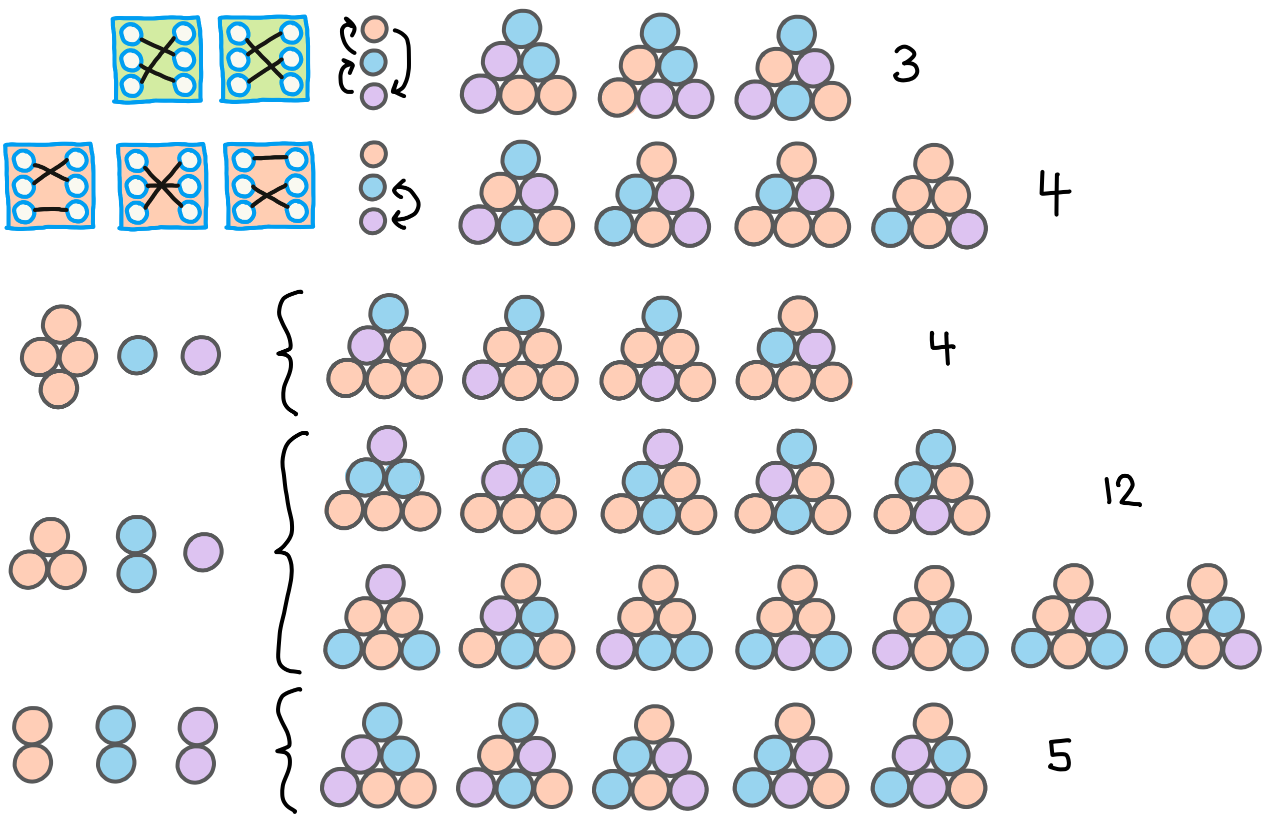
Not easily checked

(This way lies madness)

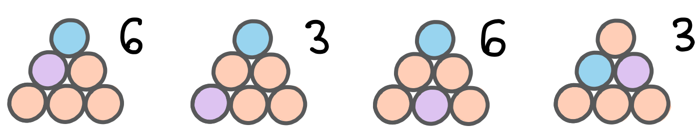
Let S_3 act on the colors, for this $|X| = 108$



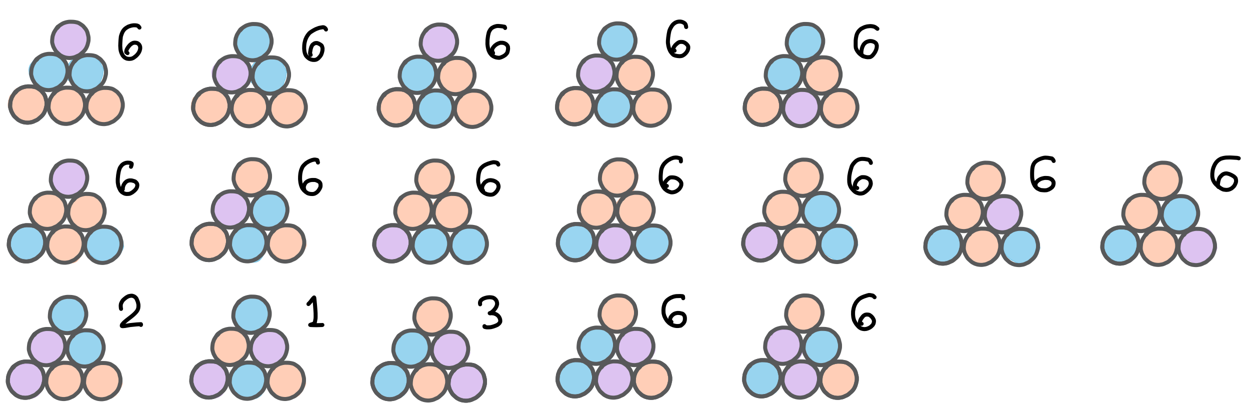
108 $\qquad \frac{1}{6}\left(\underset{18}{108}+\underset{1}{2\cdot 3}+\underset{2}{3\cdot 4}\right) = 21$



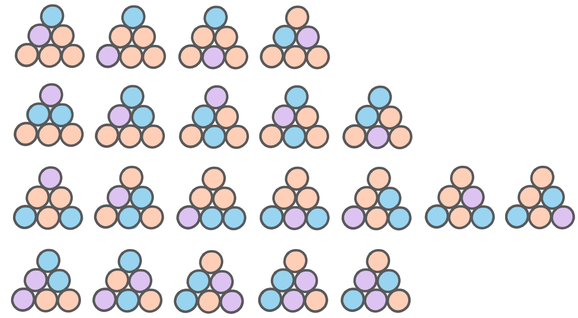
Now count orbit sizes by S_3 acting on colors

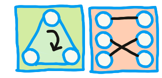


$$\boxed{\underset{96}{16\cdot 6}+\underset{9}{3\cdot 3}+\underset{2}{2\cdot 1}+\underset{1}{1\cdot 1} = 108}$$

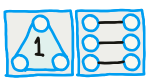
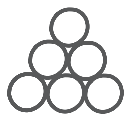


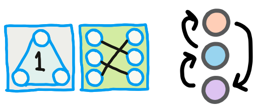
More systematic way to get 21 ways to color  using 3 interchangeable colors up to triangle symmetries:



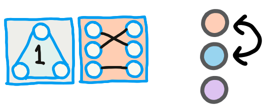
Let $G = S_3 \times S_3$, group of pairs of actions of form  acting on triangle and then color choices

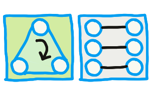
$|G| = |S_3||S_3| = 6 \cdot 6 = 36$

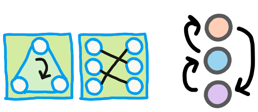
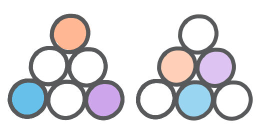
15 1   $3^6 - 3\cdot 2^6 + 3 = 729 - 192 + 3 = 540$

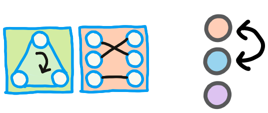
2  none

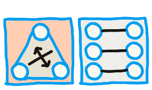
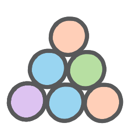
$$\frac{1}{36}(\underset{15}{540} + \underset{1}{4\cdot 9} + \underset{3}{3\cdot 36} + \underset{2}{9\cdot 8}) = 21$$

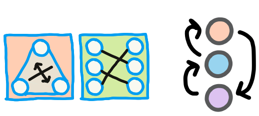
3  none

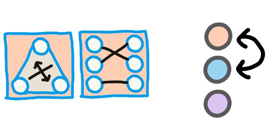
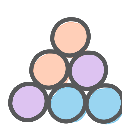
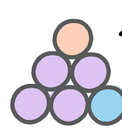
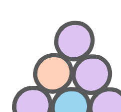
2  none

1 4   $3\cdot 3 = 9$

6  none

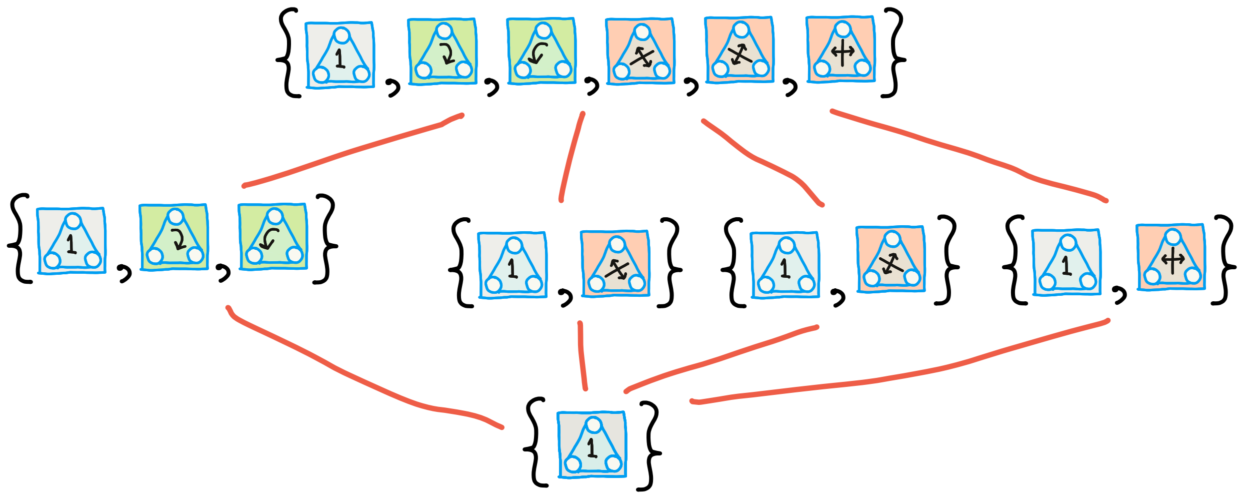
3 3   4 zones
color using all 3 colors
$3^4 - 3\cdot 2^4 + 3 = 81 - 48 + 3 = 36$

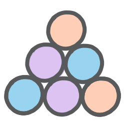
6  none

2 9   4  2  2 8

21 36

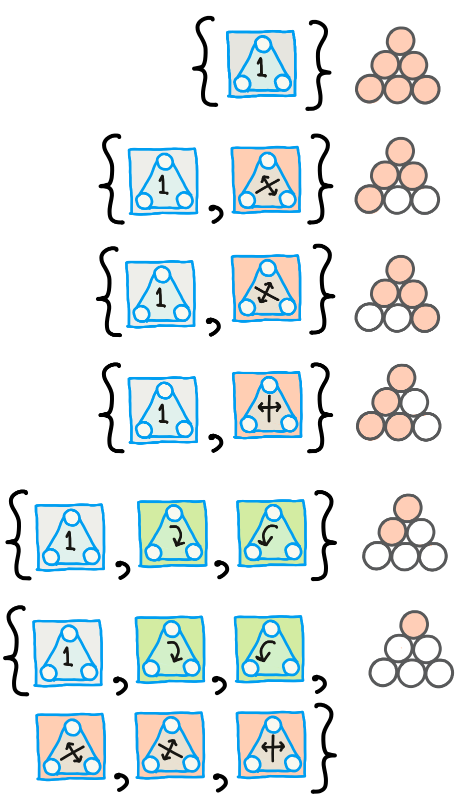
Can we use inclusion-exclusion instead of Burnside's lemma? Need to consider poset of subgroups of S_3. Möbius inversion.





k colors

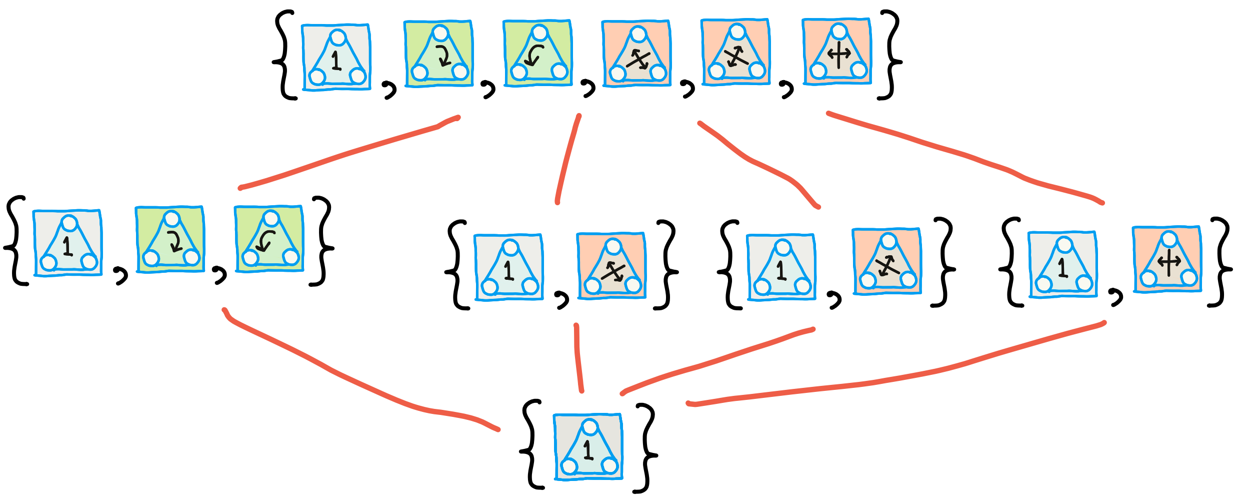
$$|P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$$

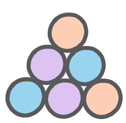


type of symmetry	at least	exactly, divided by symmetries	$(k^6$	k^4	k^2	$k)/6$
	k^6	$\frac{1}{6}(k^6-3k^4-k^2+3k)$	1	-3	-1	3
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^4	$\frac{1}{3}(k^4-k)$		2		-2
	k^2	$\frac{1}{2}(k^2-k)$			3	-3
	k	k				6
			1	3	2	0

$\frac{1}{6}(k^6+2k^2+3k^4)$ ☑

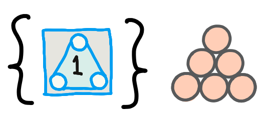
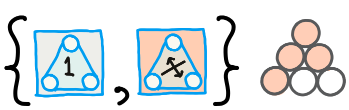
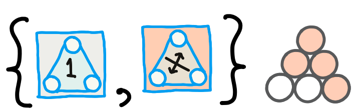
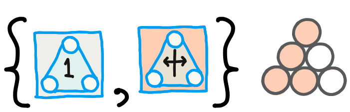
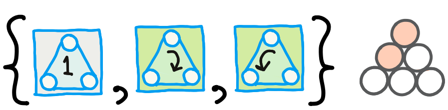
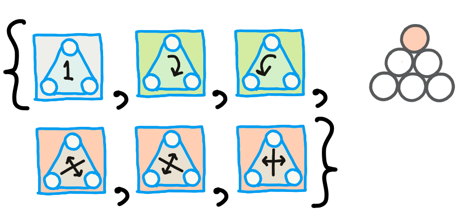
Better approach: Skip Möbius inversion to compute "exactly".
Rather, when a pattern has d versions, we want to count each one with weight $1/d$.
Work up the poset, adjusting weights based on count so far from below.





k colors

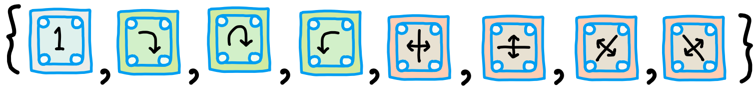
$$|P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$$

type of symmetry	at least	desired weight	subtract below	net contribution
	k^6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}k^6$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
	k^2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}k^2$
	k	1	0	

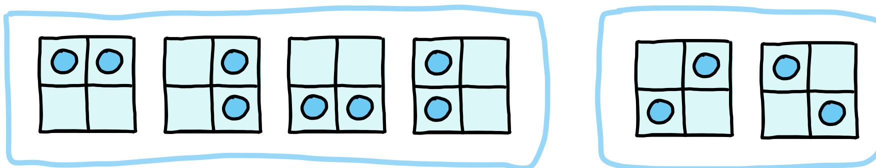
$\frac{1}{6}(k^6+2k^2+3k^4)$ ☑

This can be easier than Burnside's lemma.

Placing k markers on an n×n board, up to symmetry.

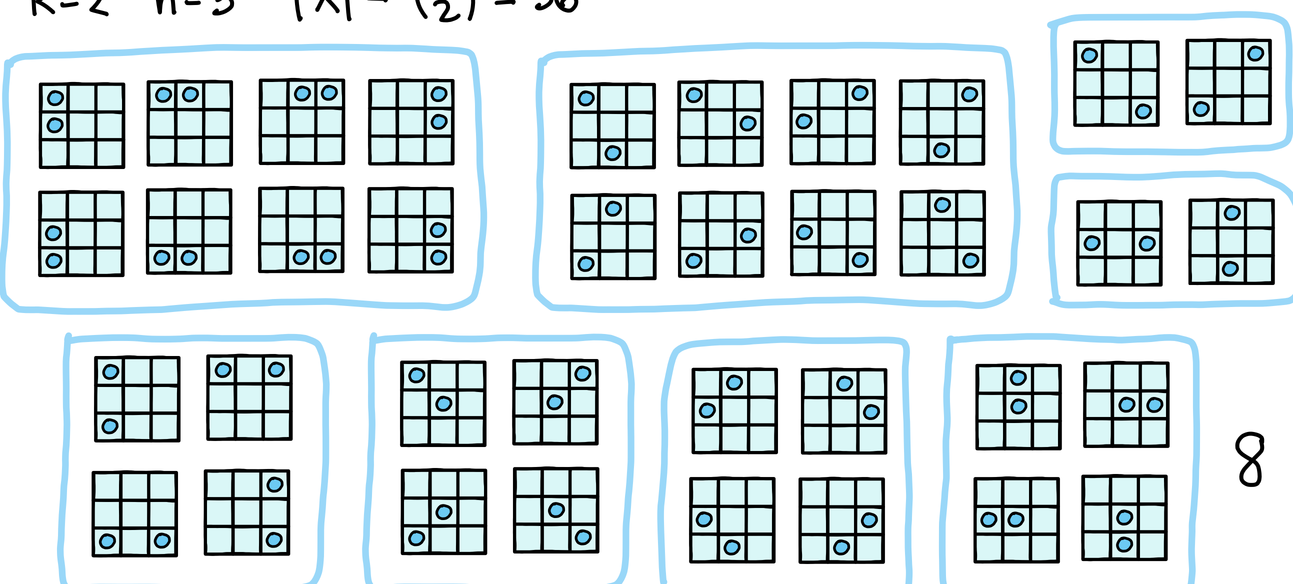
$G = \{$  $\}$ $|G| = 8$

$k=n=2$ $|X| = \binom{4}{2} = 6$

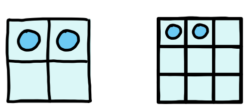


2

$K=2$ $n=3$ $|X| = \binom{9}{2} = 36$

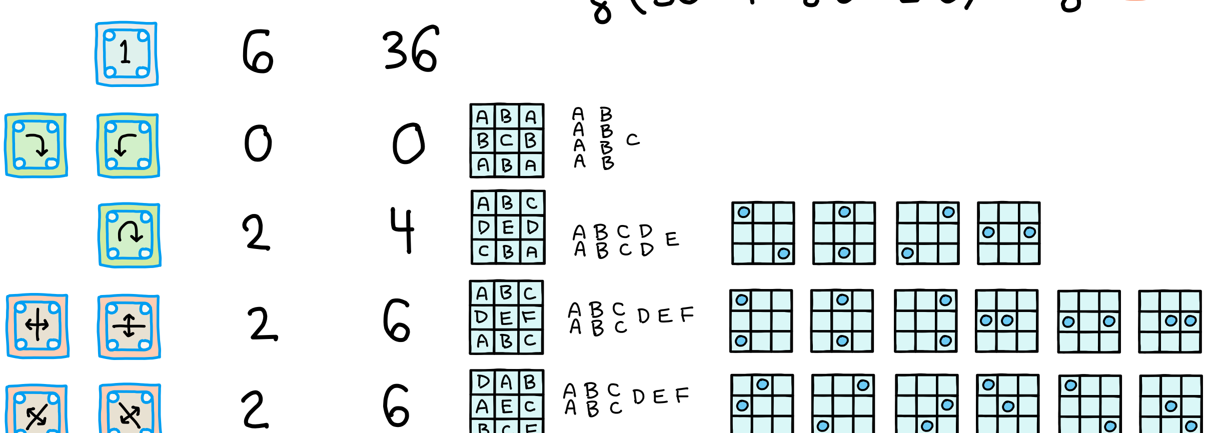


8



$\frac{1}{8}(6+2+2\cdot2+2\cdot2) = 2$ ☑

$\frac{1}{8}(36+4+2\cdot6+2\cdot6) = 8$ ☑



March 9, 2021 — Counting with symmetries on polytopes.

Symmetries of space

Linear Algebra — w/o angle, length; then add these <fig> f.g

xkcd.com

chirality

orthonormal basis $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

$v_1 \perp v_2$ $\quad |v_i| = 1$

$v_1 \perp v_3$ $\quad v_i \cdot v_i = 1$

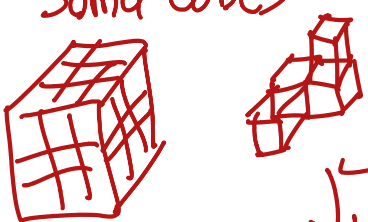
$v_2 \perp v_3$

$v_i \perp v_j = v_i \cdot v_j = 0$

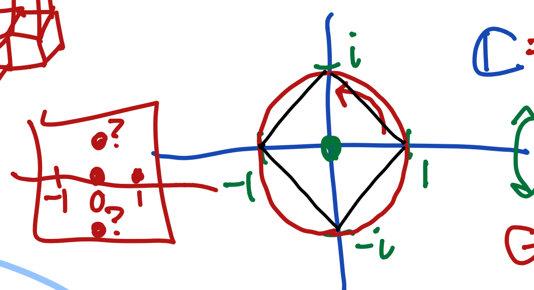
$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

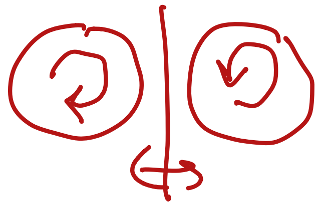
$A^{-1} = A^T$ $\quad \boxed{\det(A) = 1}$

Soma cubes



rotations in space





$\mathbb{C} = \mathbb{R}^2$

$\overline{a+bi} = a - bi$

$\overline{i} = -i$

$G = \{ c \in \mathbb{C} \mid |c| = 1 \} = SO(2)$

c·d rotations

$x^2 + 1 = 0$

$(x+i)(x-i) = 0$

$O(n) =$ orthonormal matrices $\mathbb{R}^n \to \mathbb{R}^n$

$SO(n) = \ldots \det = 1$

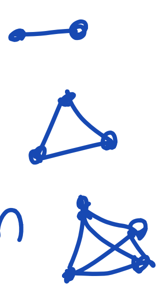
rotations

n-simplex

interval — 1-simplex

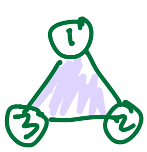
triangle — 2-simplex

tetrahedron — 3-simplex



symmetric group: permutations of $\{1, \ldots, n\}$ $\quad S_n$ all

A_n even



geometric view

permutation view

$|S_4| = 24 = 4!$

$|G| = 8$

$|S_4| = 24$

$|A_4| = 12$

S_5

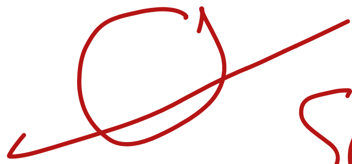
A_5 "can't be factored"

A_n $n \geq 5$

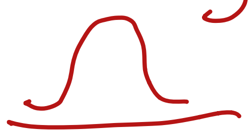
$$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$\sqrt{\ }$

$\overline{a+bi}$
$=a-bi$



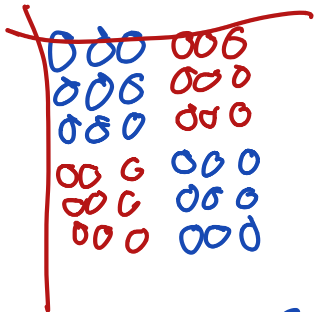
$\int e^{x^2}dx$



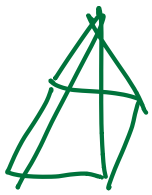
$\mathrm{erf}(x)$

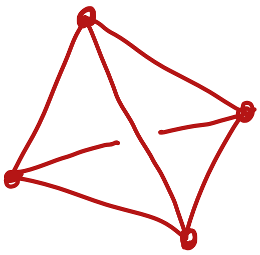
$\sqrt{\ }$ $\sqrt[3]{\ }$

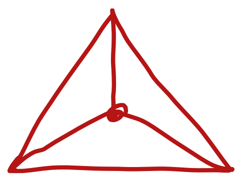
$\sqrt[4]{\ }$

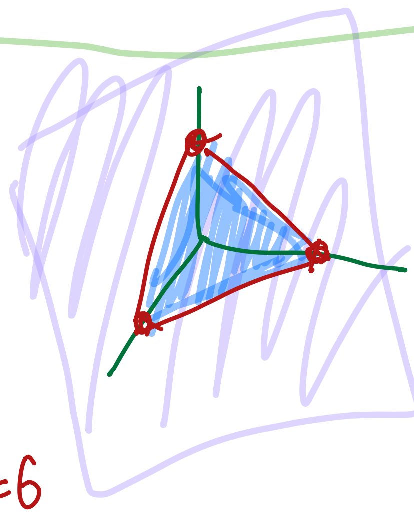


2 $\sqrt[2]{\ }$





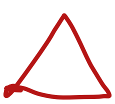


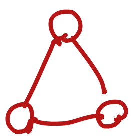


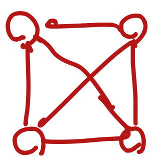
all $v=(x,y,z)$

$x,y,z\geq 0$

$x+y+z=1$

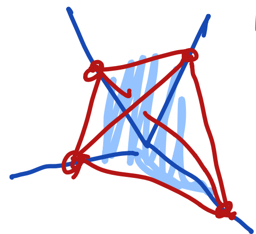


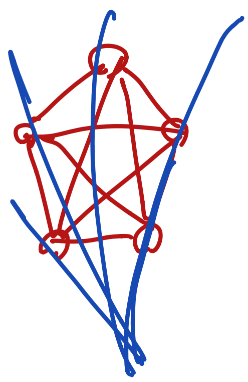




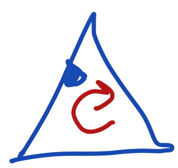
$\binom{4}{2}=6$

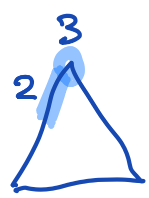
$\mathbb{R}^4$



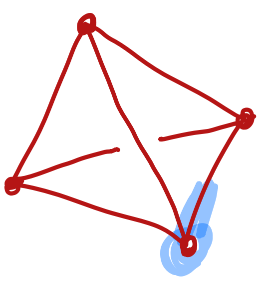


$\binom{5}{2}=10$





$3\cdot 2=6$



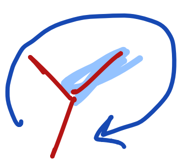
mark it to destroy symmetry

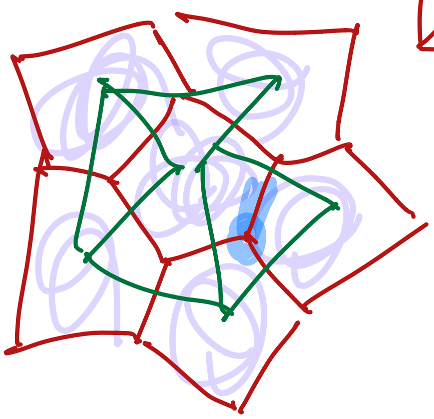
count choices

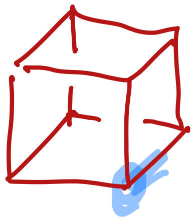
4 choices of corner

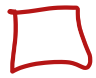
$\times$ 3 choices of edge meeting that corner

12









$|G| = 8$

$|S_4| = 24$

$|G| = 8 \cdot 3 = 24$

$|S_8| = 8!$

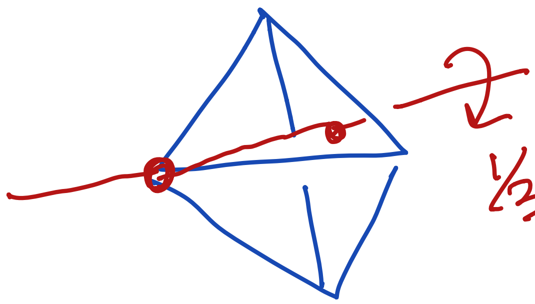
$12 \cdot 5/3 = 20$

$|G| = 20 \cdot 3 = 6$

$G = A_5$

\# ways k-color faces of a tetrahedron up to symmetry

$|G| = 12$



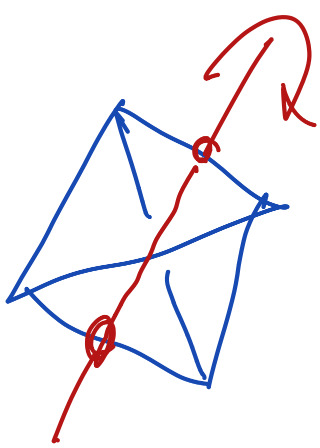
1/3 turn

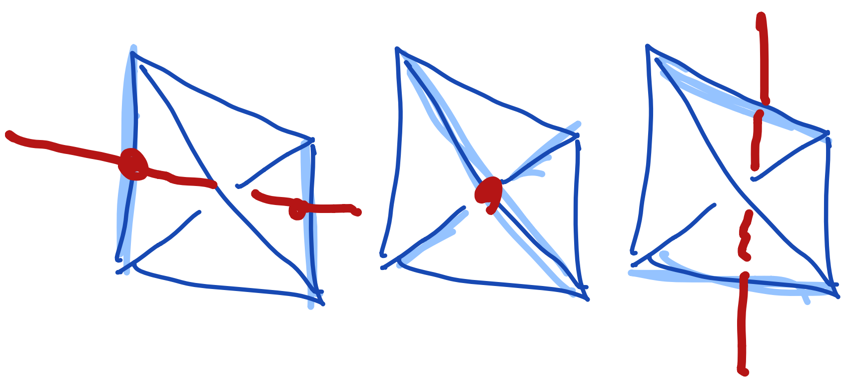
1 do nothing, identity

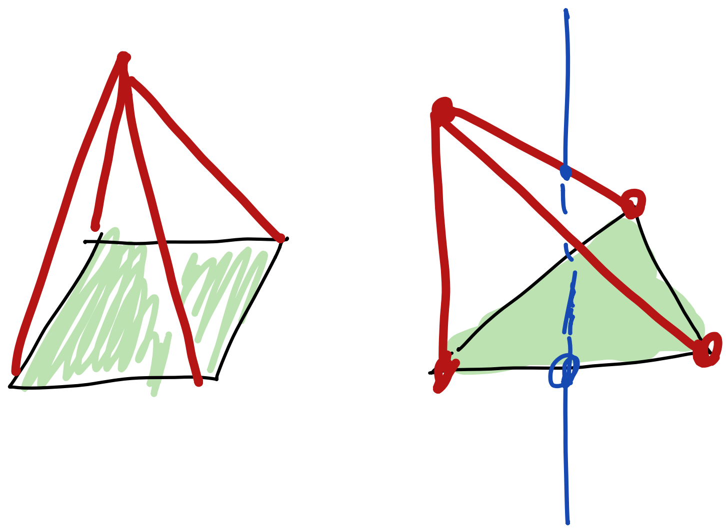
8 1/3 turns
4 vertices
x2 turns

3 1/2 turns

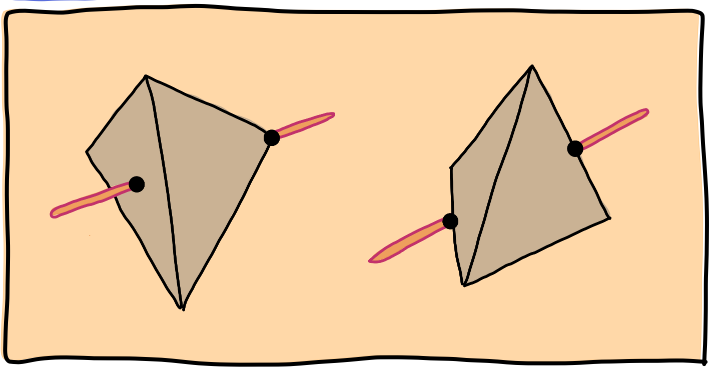
12

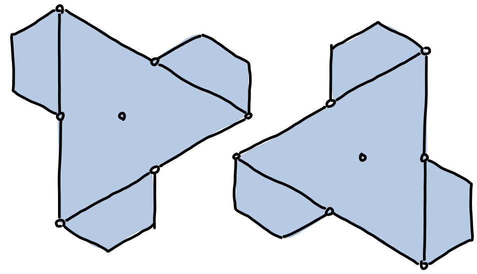






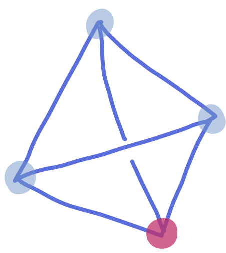
March 11



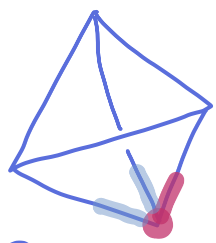


I have posted plans for the above model on our website.

The tetrahedron has 12 symmetries:



① Choose a corner
4 choices



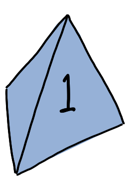
② choose an edge
meeting that corner
3 choices

G = group of symmetries of tetrahedron in $\mathbb{R}^3$
(we ignore flips through $\mathbb{R}^4$)

$|G| = 4 \cdot 3 \quad 12$

Can we find these 12 symmetries?

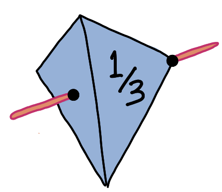
1



Identity

Do nothing

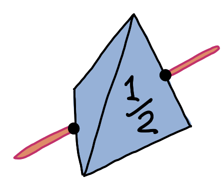
$2 \cdot 4$



1/3 turn either way

axis through
face and vertex

3



1/2 turn

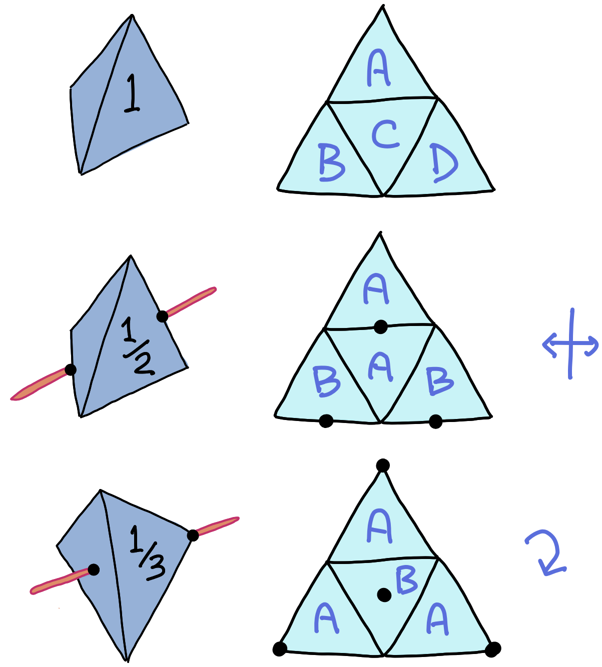
axis through
opposite edges

$1 + 2 \cdot 4 + 3 = 12$ ☑

Burnside's lemma:

$$\frac{1}{|G|} \sum_{g \in G} |X_g|$$

Example: How many ways can we color the sides of a tetrahedron, up to symmetry, using k colors?



1 $\quad k^4$

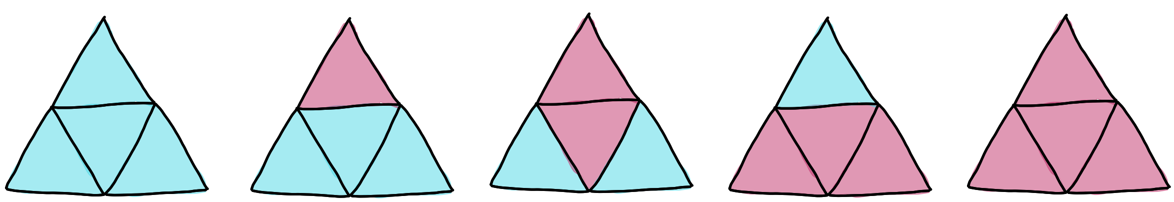
3 $\quad k^2$

8 $\quad k^2$

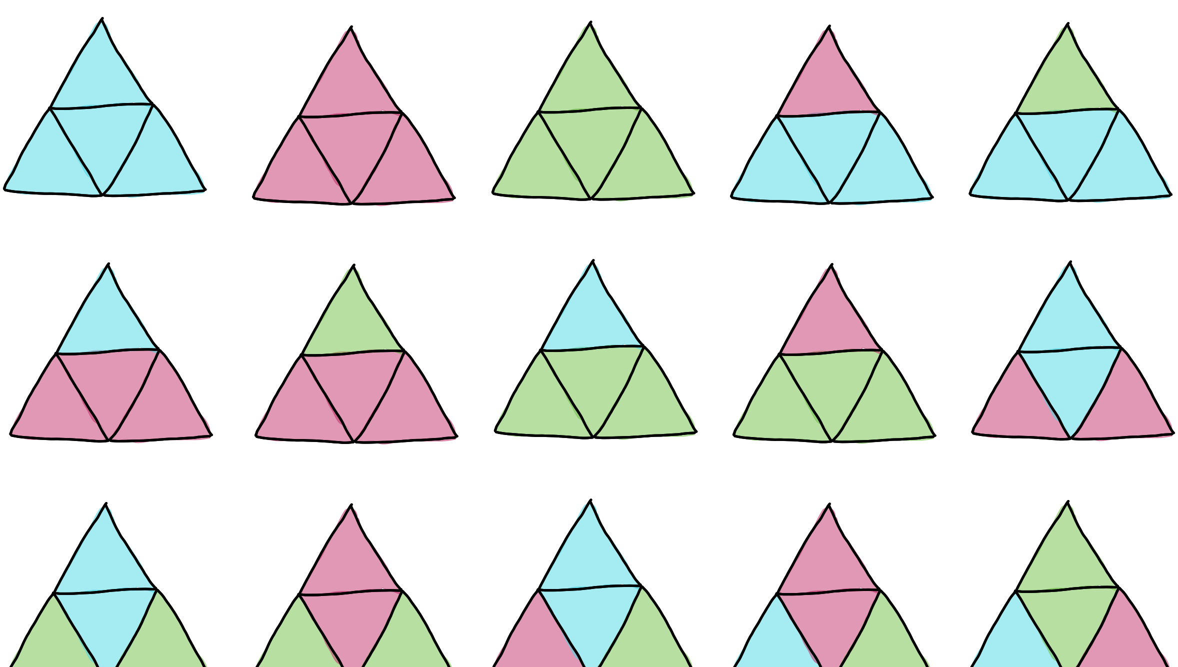
$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{12}\left(k^4+11k^2\right)}$$

k	#	
1	1	
2	5	16+44
3	15	81+99
4	36	256+176

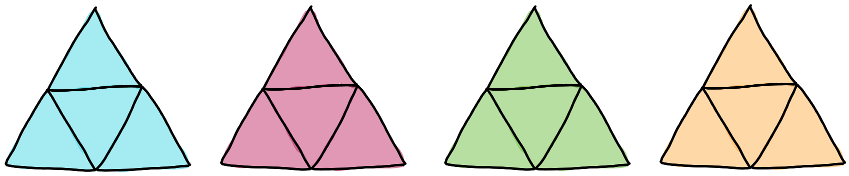
Check: k = 2  5 ☑



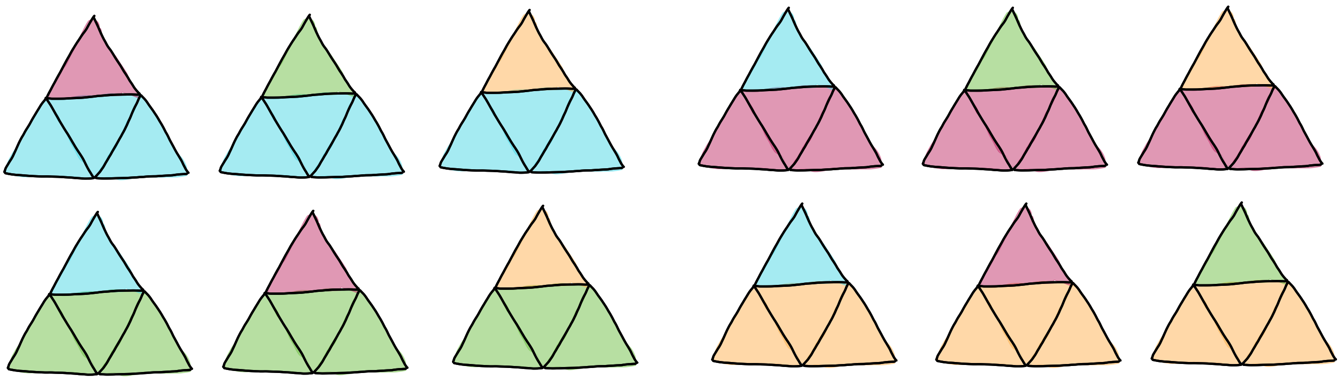
Check: k = 3  15 ☑



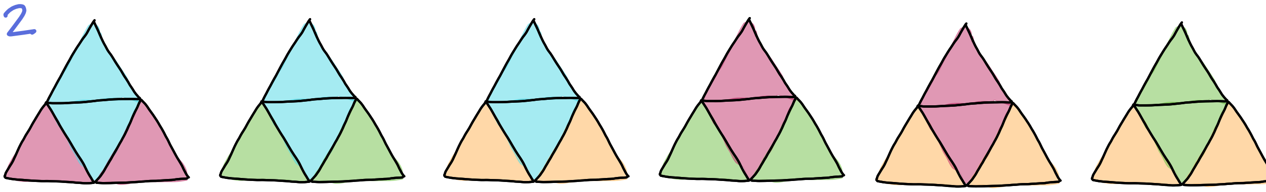
Check: k = 4  36 ☑

4 

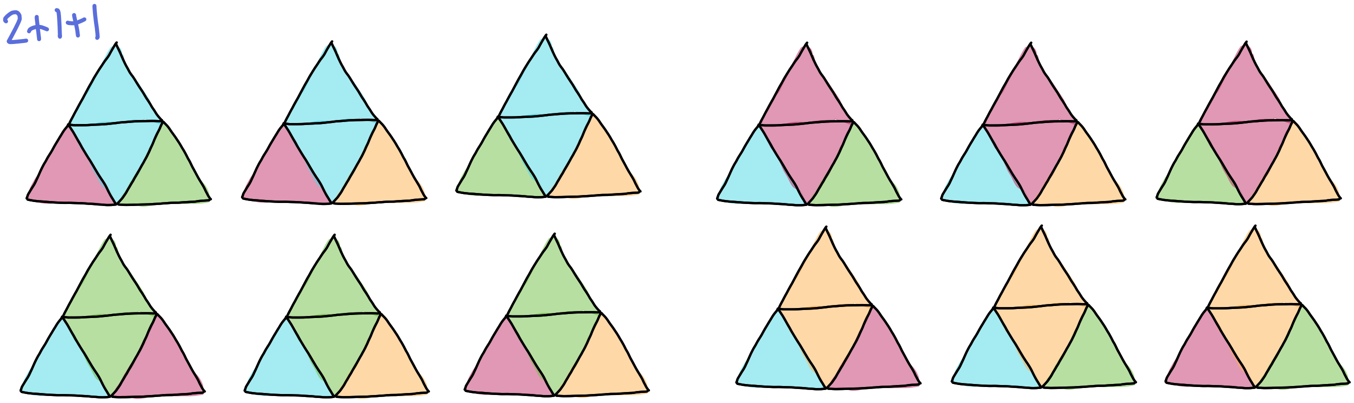
3+1

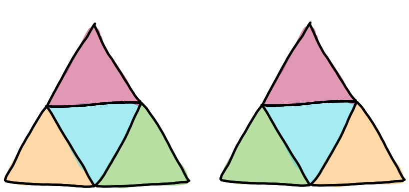


2+2



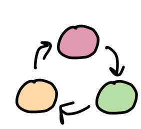
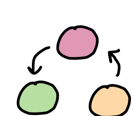
2+1+1





1+1+1+1

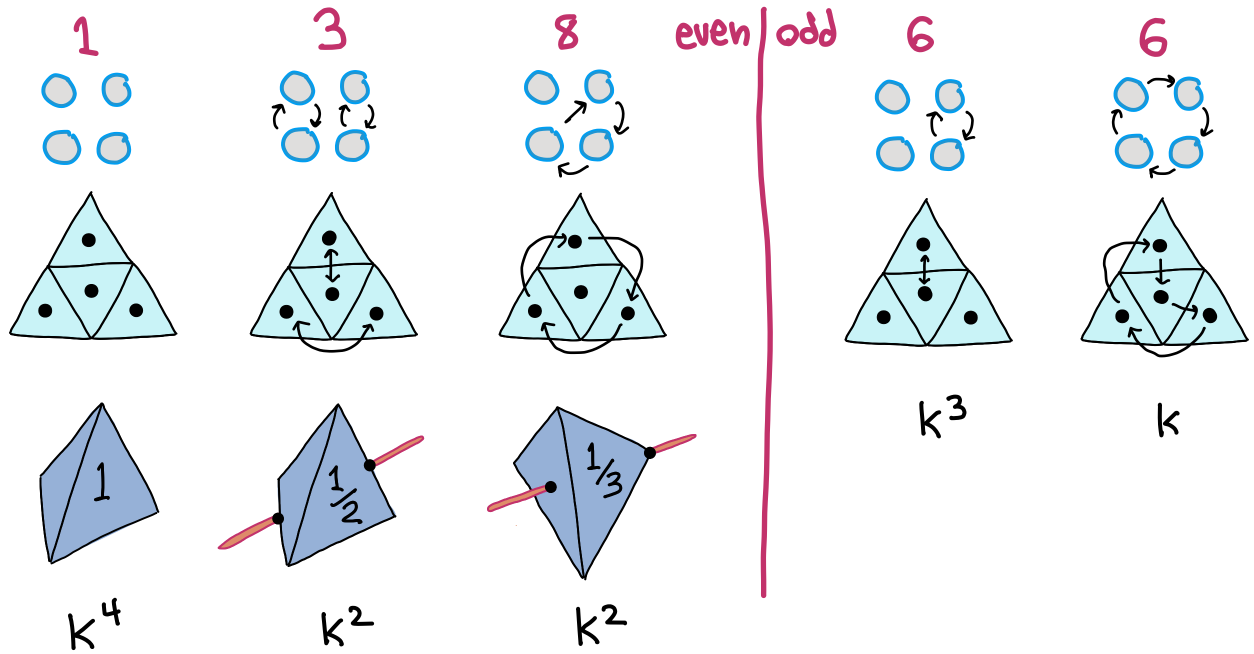
Finally a chiral pair

Look at , see  or 

This tells us that if we allow flips, we'll get

k	1	2	3	4	
G	1	5	15	36	(no flips)
S_4	1	5	15	35	(flips in $\mathbb{R}^4$)

$|S_4| = 4! = 24$ breaks up by cycle decomposition



$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}\left(k^4 + 6k^3 + 11k^2 + 6k\right)$$

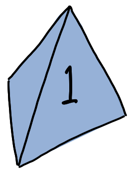
k	k^2	k^3	k^4	$6k$	$11k^2$	$6k^3$	k^4	Σ	#	
1	1	1	1	6	11	6	1	24	1	as before
2	4	8	16	12	44	48	16	120	5	
3	9	27	81	18	99	162	81	360	15	
4	16	64	256	24	176	384	256	840	35 ☑	not 36

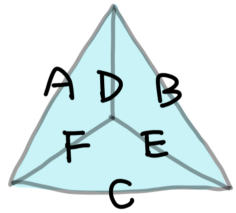
Choosing subsets of faces is restricted version of 2-coloring $\Rightarrow$ no chirality

Coloring vertices is dual to coloring faces, same problem

- Coloring edges?
- Coloring everything?

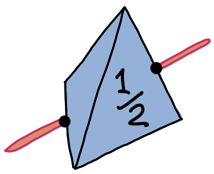
Coloring edges:

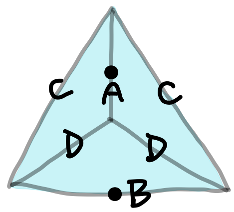
1 



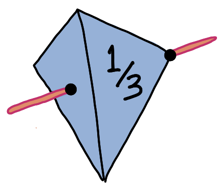
K^6

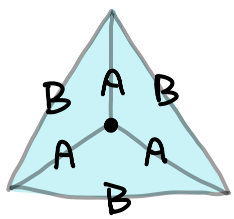
$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{12}\left(K^6+3K^4+8K^2\right)}$$

3 



$\leftrightarrow$ K^4

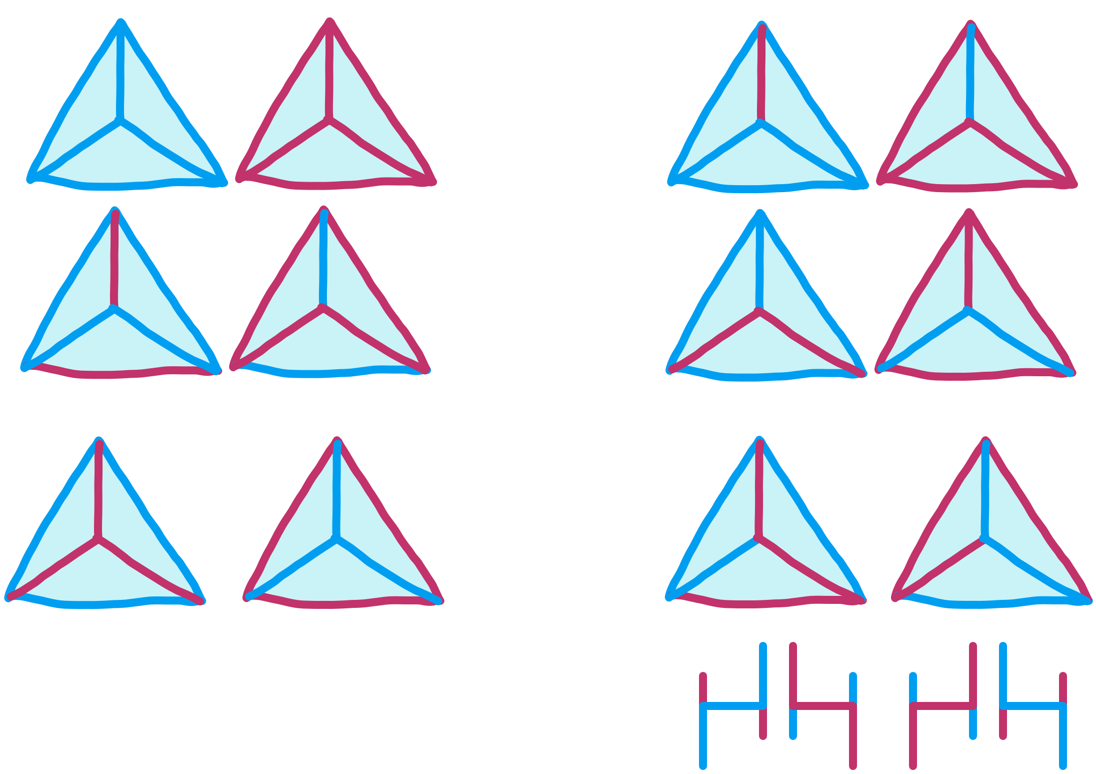
8 



2 K^2

K	1	2	3
K^2	1	4	9
K^4	1	16	81
K^6	1	64	729
$8K^2$	8	32	72
$3K^4$	3	48	243
K^6	1	64	729
Σ	12	144	1044
#	1	12	84

Check: $K=2$  12

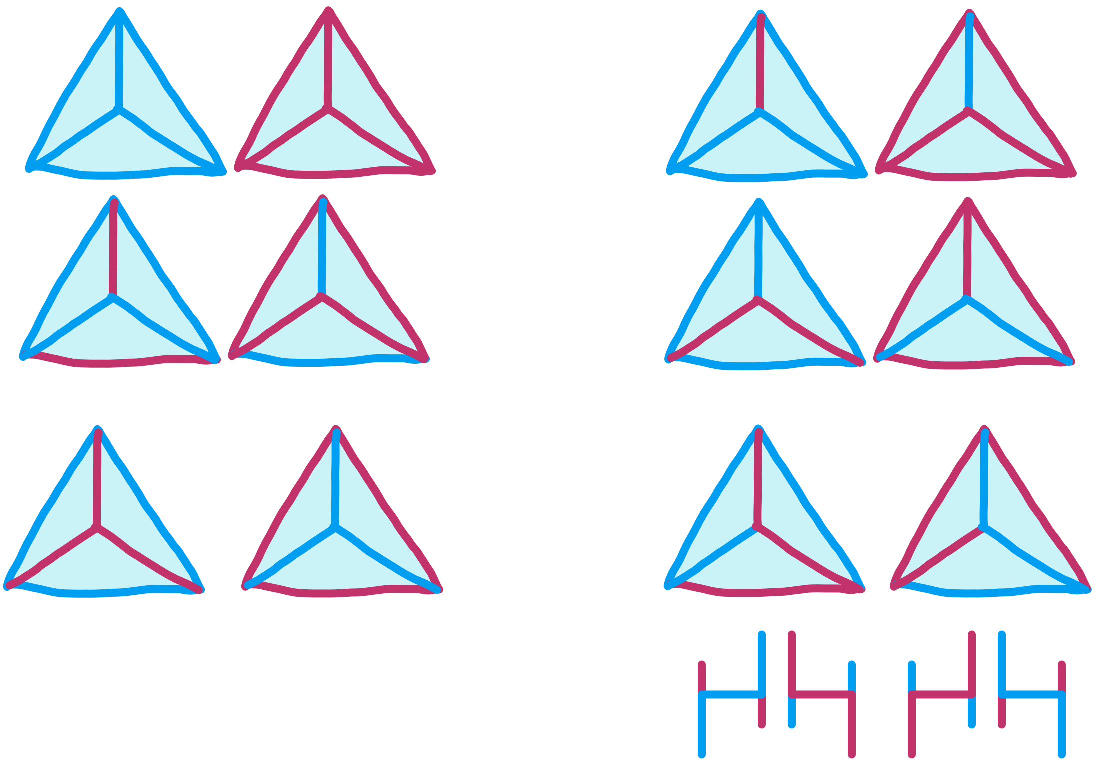


(Corrected from class)

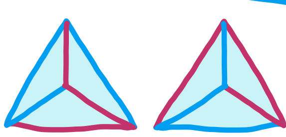
Tuesday, March 16

From last class: 12 ways to 2-color edges of a tetrahedron, up to symmetry

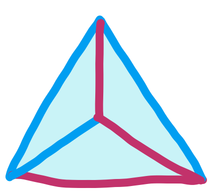
Check: k=2 12 ☑



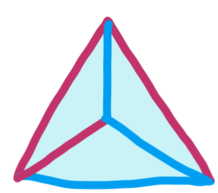
(Corrected from class)

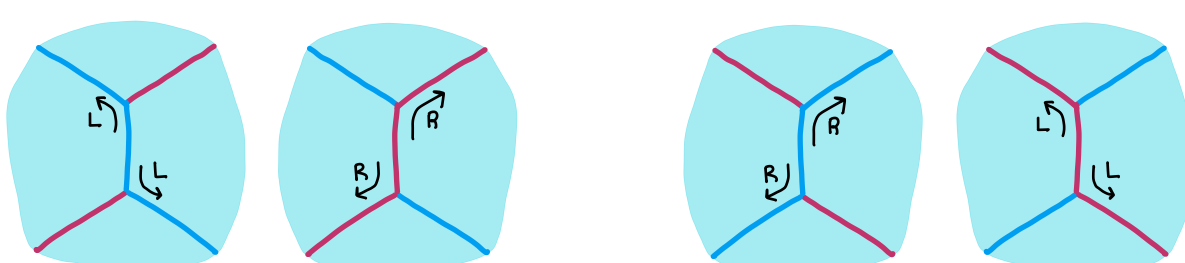
In class I had: 

These were actually the same.



The last two cases are chiral.



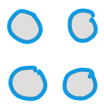


This tells us that including flips through $\mathbb{R}^4$, we should get 11 not 12

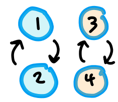
$|S_4| = 4! = 24$ breaks up by cycle decomposition

even

1

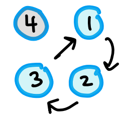


3

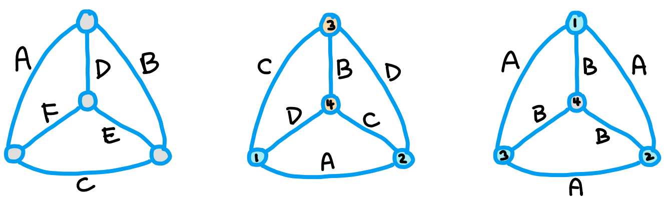


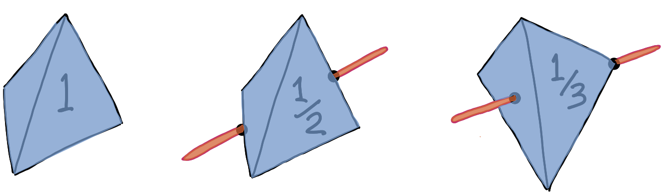
A (1,2)
B (3,4)
C (1,3) (2,4)
D (1,4) (2,3)

8



A (1,2) (2,3) (1,3)
B (1,4) (2,4) (3,4)

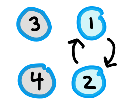




k^6 $\quad$ k^4 $\quad$ k^2

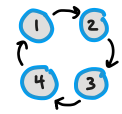
odd

6

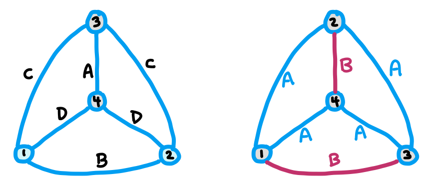


A (3,4)
B (1,2)
C (1,3) (2,3)
D (1,4) (2,4)

6



A (1,2) (2,3) (3,4) (1,4)
B (1,3) (2,4)

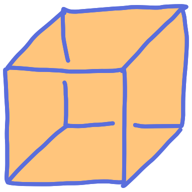


k^4 $\quad$ k^2

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}\left(k^6 + 9k^4 + 14k^2\right)$$

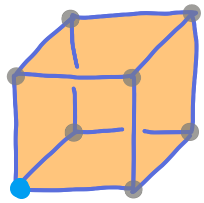
k	k^2	k^4	k^6	$14k^2$	$9k^4$	k^6	Σ	#
1	1	1	1	14	9	1	24	1
2	4	16	64	56	144	64	264	11 ☑

Symmetries of the cube

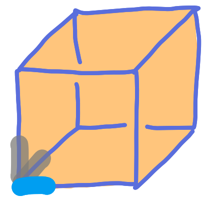


G = group of symmetries of cube in $\mathbb{R}^3$

(we ignore flips through $\mathbb{R}^4$)



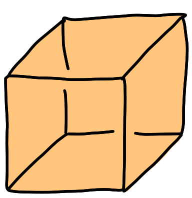
① Choose a corner
8 choices



② choose an edge meeting that corner
3 choices

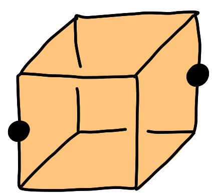
$|G| = 8 \cdot 3 = 24$

Can we find these 24 symmetries?



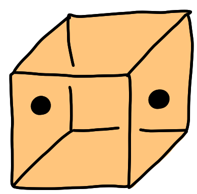
Identity 1

1



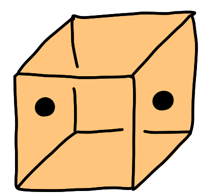
½ turn

6



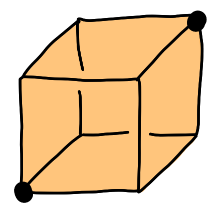
½ turn

3



¼ turn either way

6

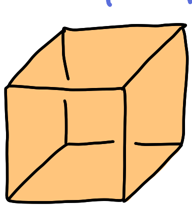


⅓ turn either way

8

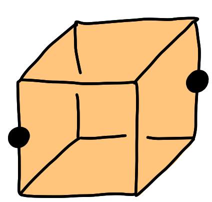
This $G \approx S_4$. Imagine 4 diagonal sticks inside the cube.
Easier: Label opposite corners the same, using $\{1,2,3,4\}$
Every permutation is possible.

How many ways can we k-color the faces of a cube, up to symmetry?



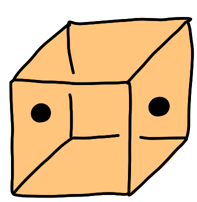
Identity 1

k^6



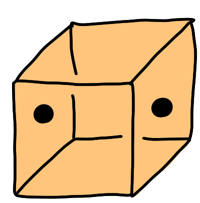
½ turn

$6k^3$



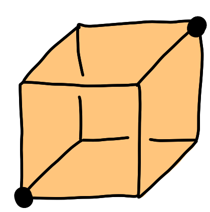
½ turn

$3k^4$



¼ turn either way

$6k^3$



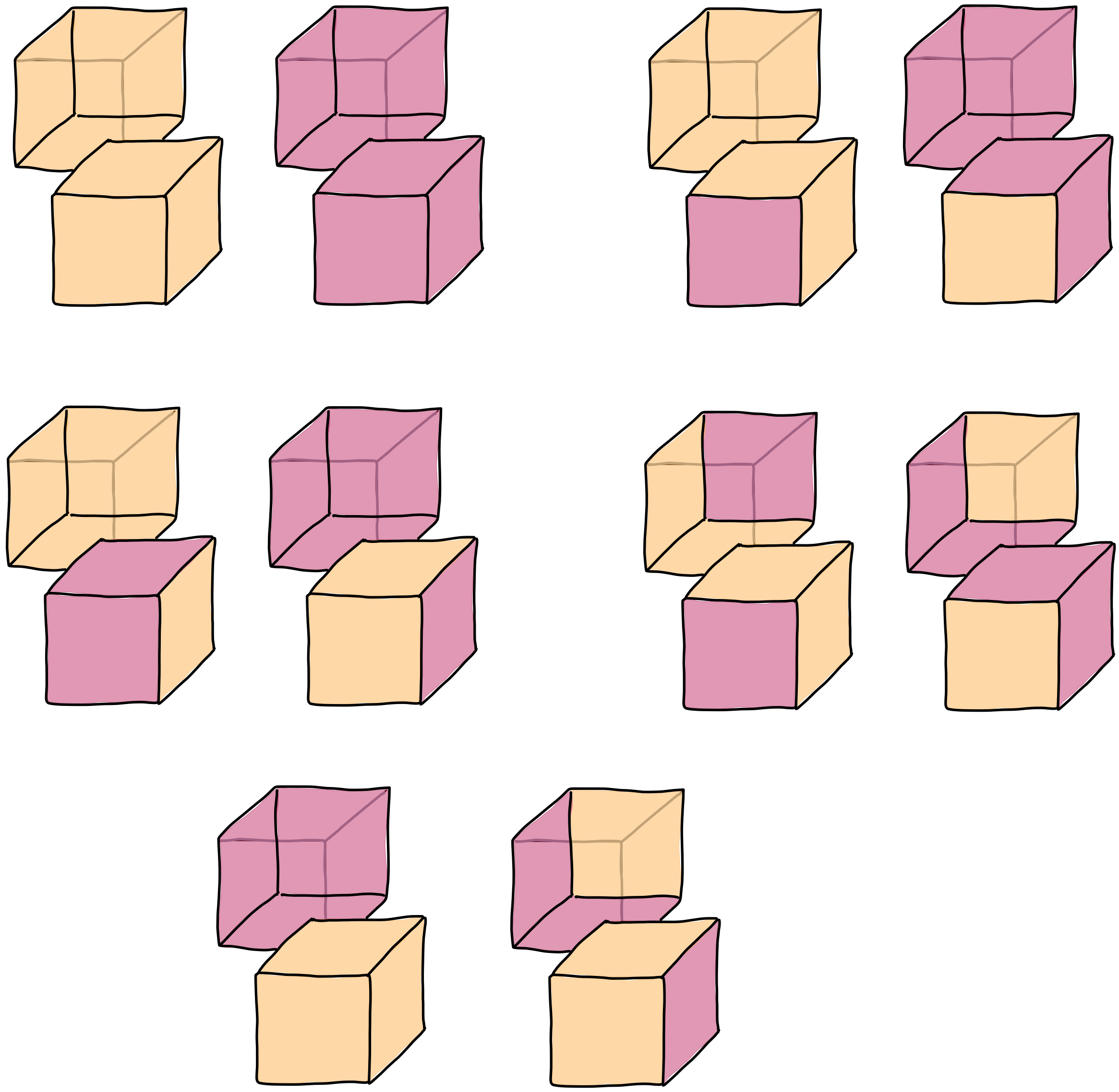
⅓ turn either way

$8k^2$

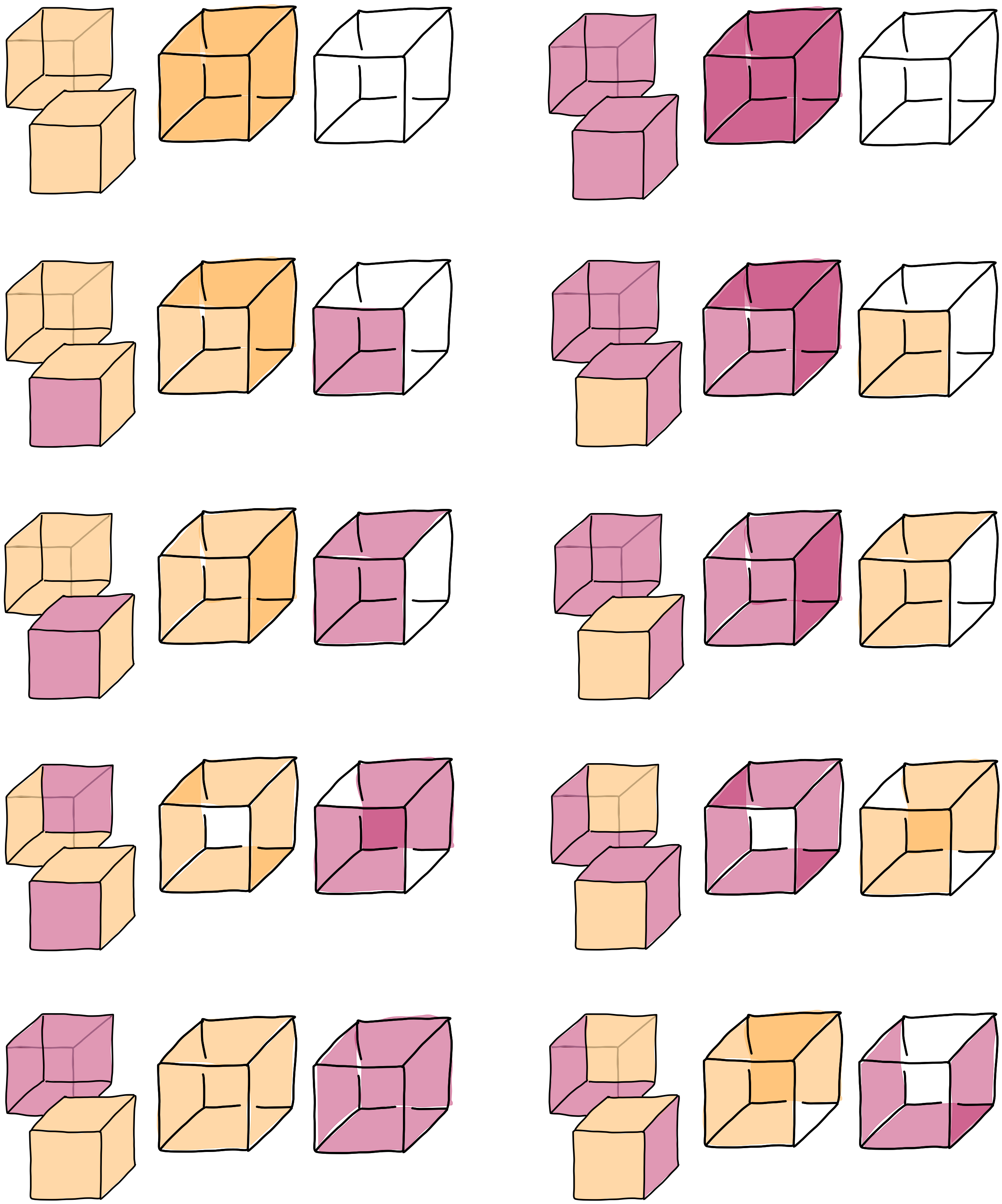
$$\frac{1}{24}\left(k^6 + 3k^4 + 12k^3 + 8k^2\right)$$

$$k=2 \Rightarrow \frac{1}{24}(64 + \underbrace{3\cdot 16}_{48} + \underbrace{12\cdot 8}_{96} + \underbrace{8\cdot 4}_{32}) = 240/24 = 10$$

Check $k = 2$: 10



After class: Try another way to draw these.
Two wire frames per pattern, to separate the faces of each color.

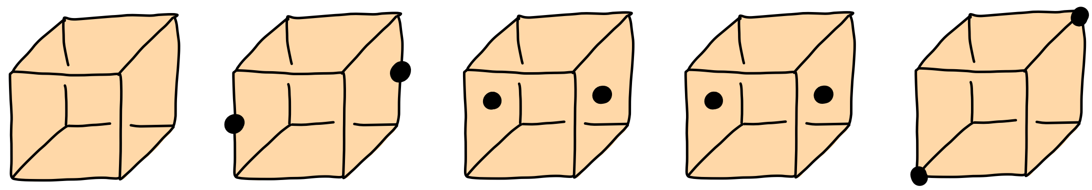


Check that action of S_4 induces every symmetry of cube

Four pairs of opposite corners, marked by ○ ● ● ●

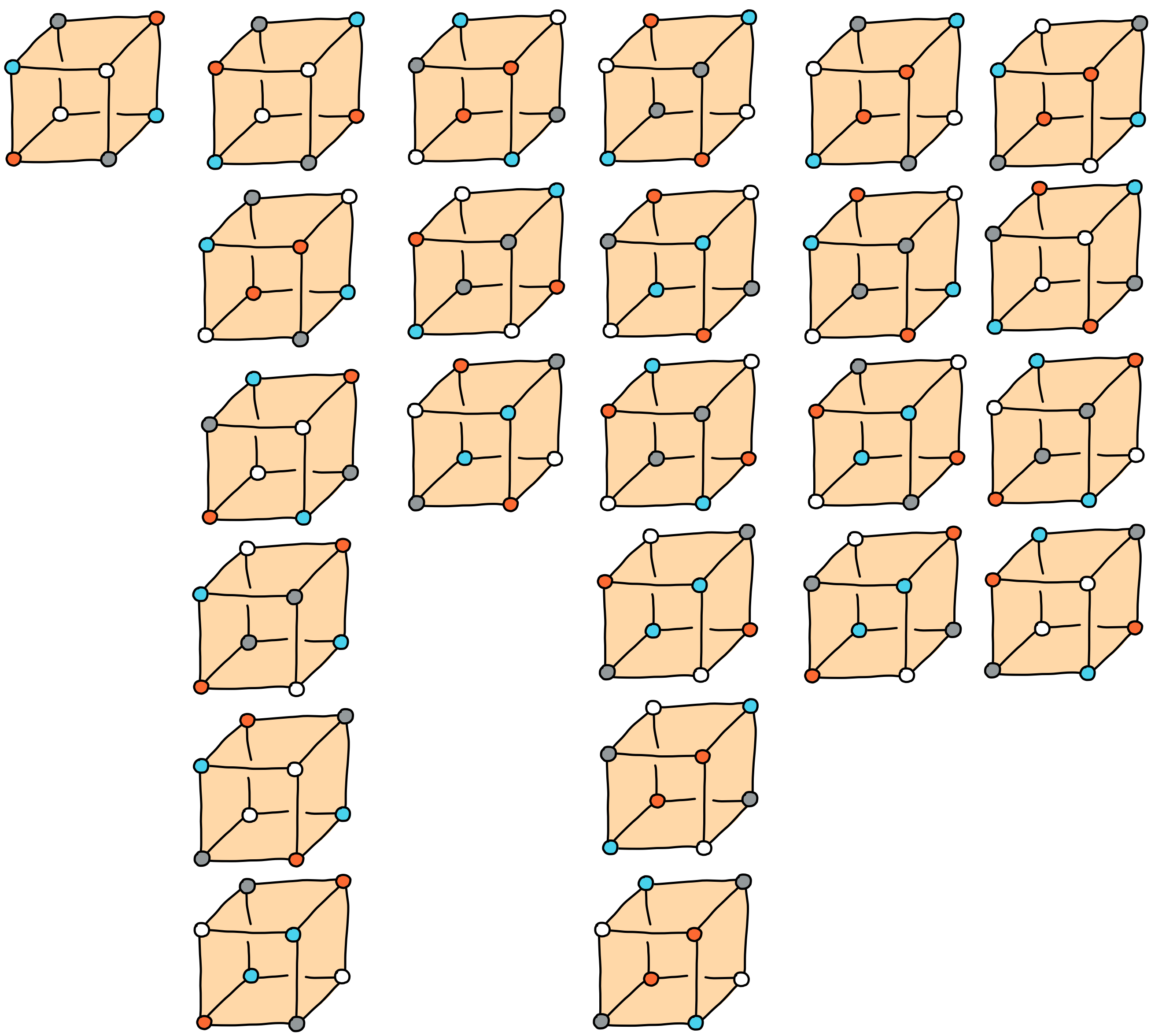
S_4 permutes these pairs

Every permutation corresponds to some rotation in space:

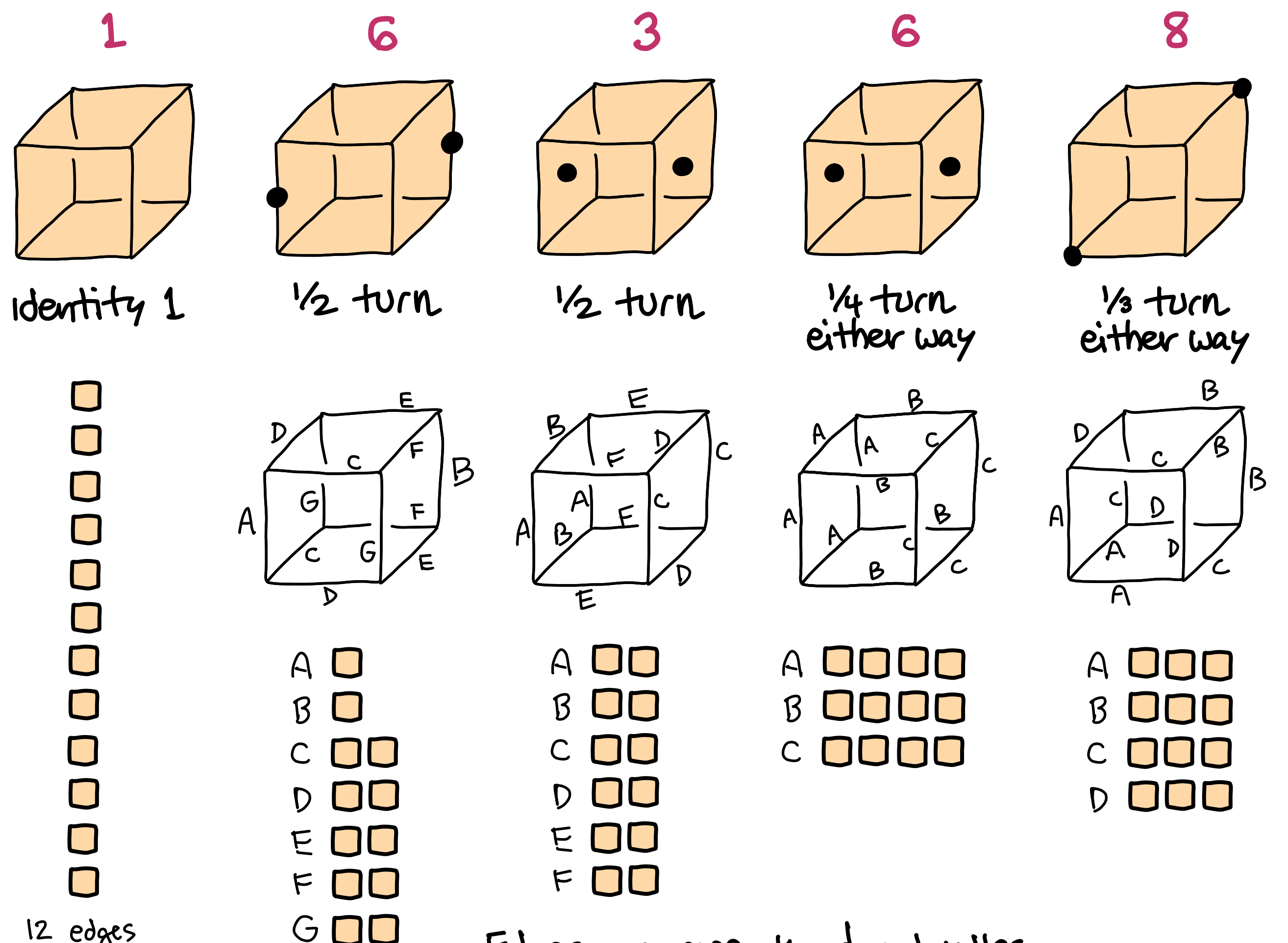


identity 1 | ½ turn | ½ turn | ¼ turn either way | ⅓ turn either way

1 | 6 | 3 | 6 | 8



How many ways can we choose k edges of a cube, up to symmetry?

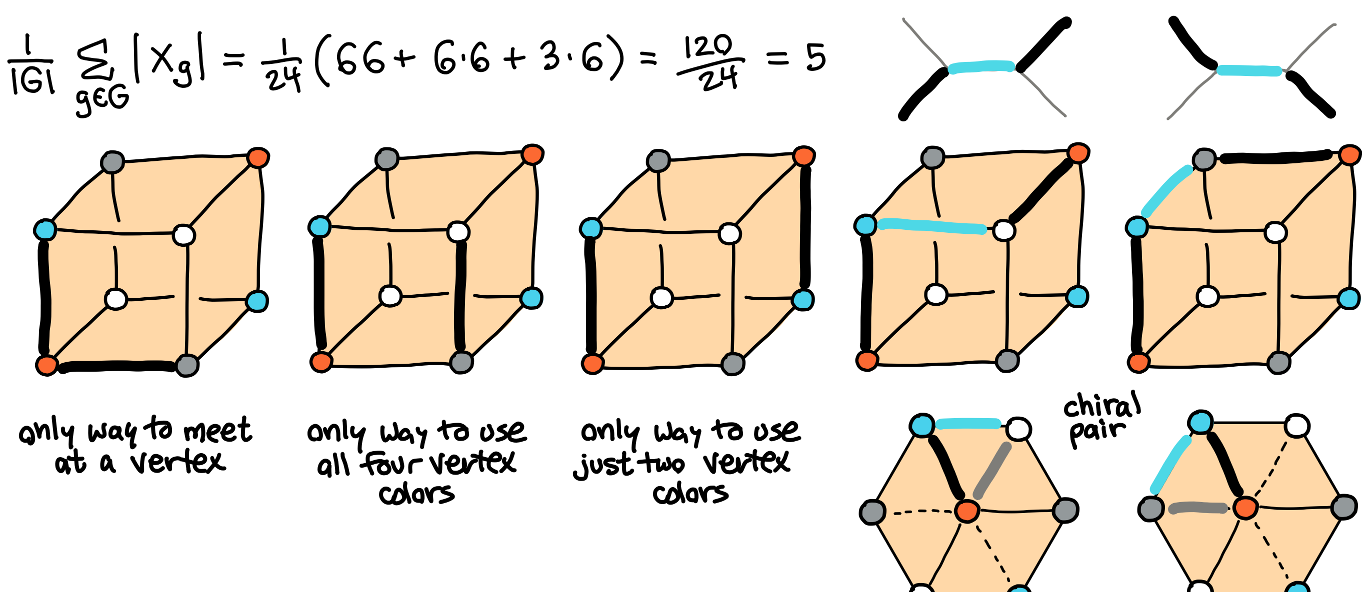


Edges come prepackaged in bundles
We need to make k buying entire bundles

K=2

$\binom{12}{2}=66$ 6 6 0 0

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}(66+6\cdot 6+3\cdot 6) = \frac{120}{24} = 5$$



March 25

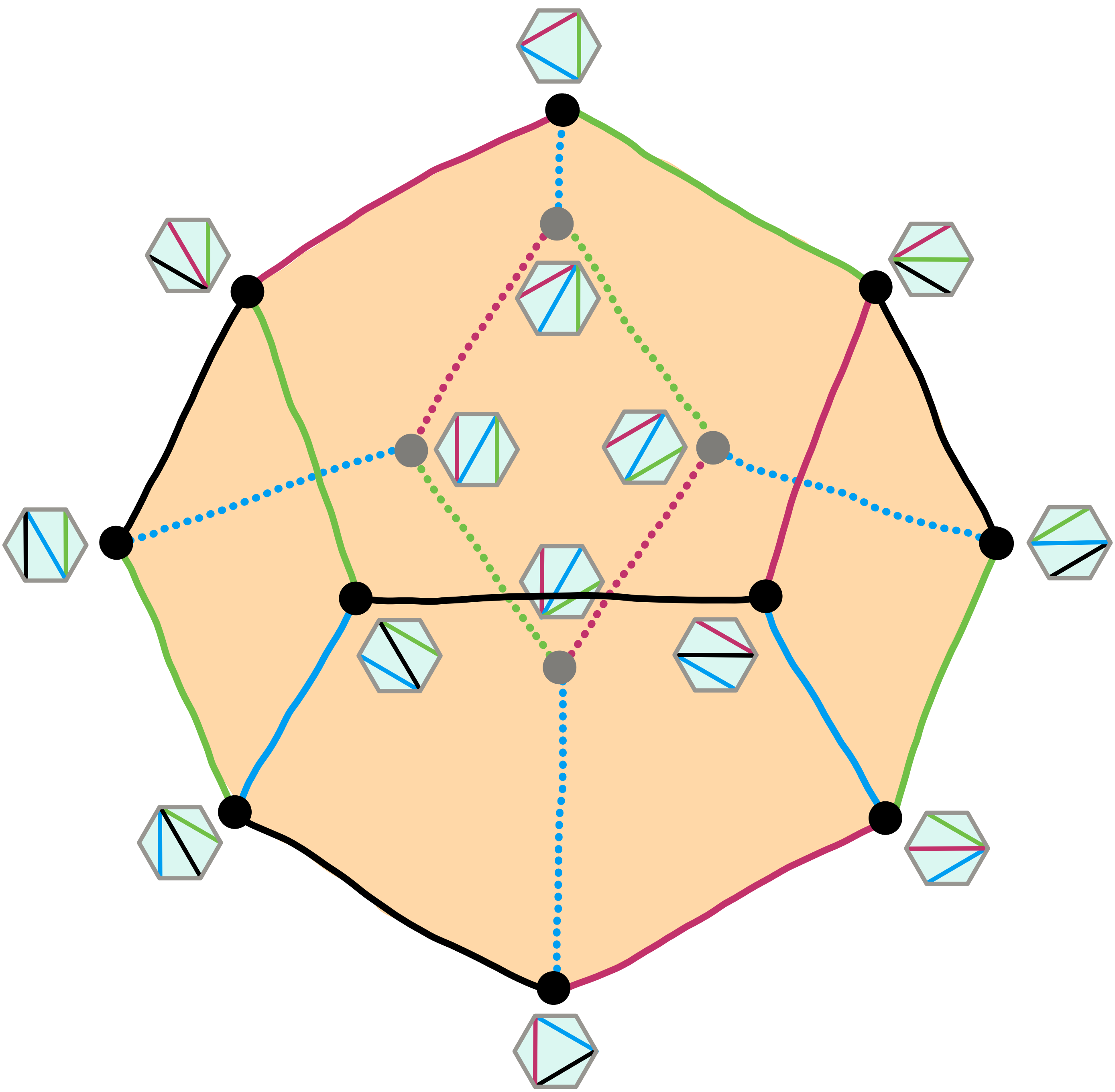
The Associahedron in $\mathbb{R}^3$

Euler characteristic of boundary

$\chi = v - e + f = 14 - 21 + 9 = 2$

like any sphere

c	f	e	v
1	9	21	14



$T(n,k)$ = number of dissections of an n-gon by k cuts

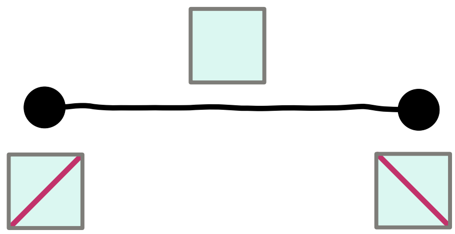
n-gon \ k cuts	0	1	2	3	4	5	6
3	1						
4	1	2					
5	1	5	5				
6	1	9	21	14			
7	1	14	56	84	42		
8	1	20	120	300	330	132	
9	1	27	225	825	1485	1287	429

⇐ the associahedron in $\mathbb{R}^3$

= Catalan numbers

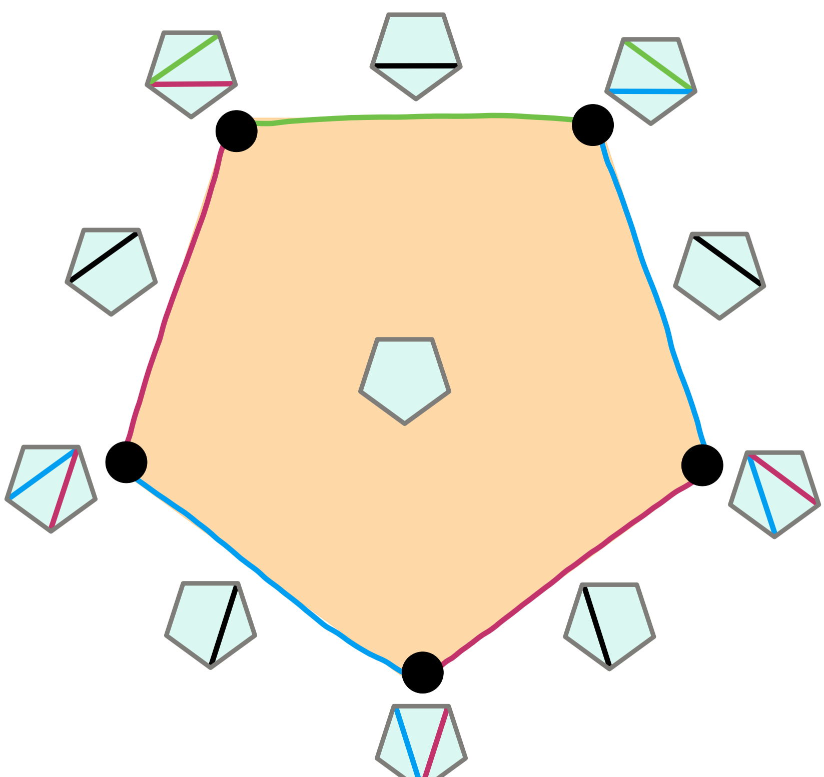
the associahedron in $\mathbb{R}^1$:

1	2

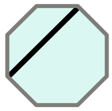


the associahedron in $\mathbb{R}^2$:

1	5	5



1 cut:

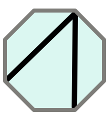


	①
3	
4	2
5	5
6	9
7	14
8	20
9	27

n	$\binom{n}{2}$	$-n$	①
4	6	4	2
5	10	5	5
6	15	6	9
7	21	7	14
8	28	8	20
9	36	9	27

$\binom{n}{2}$ pairs of vertices
$-n$ sides of n-gon
= # interior edges

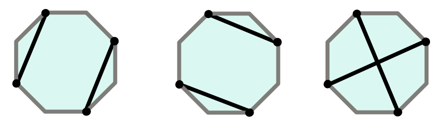
2 cuts:



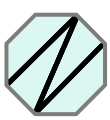
	②
3	
4	
5	5
6	21
7	56
8	120
9	225

n	①	$\binom{①}{2}$	$-\binom{n}{4}$	②
5	5	10	5	5
6	9	36	15	21
7	14	91	35	56
8	20	190	70	120
9	27	351	126	225

pairs of interior edges
– crossing pairs

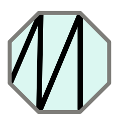


3 cuts:



	③
3	
4	
5	
6	14
7	84
8	300
9	825

4 cuts:



	④
3	
4	
5	
6	
7	42
8	330
9	1485

Can be done ad hoc.
Gets harder...

Many approaches

Formula:

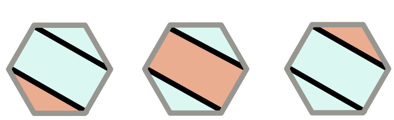
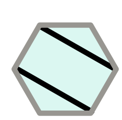
$T(n,k) =$ number of dissections of an n-gon by k cuts

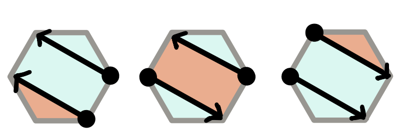
$$= \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$$

1890 Cayley
... 2000 Przytycki, Sikora

Meaning of each part:

$\boxed{\frac{1}{k+1}}$ We overcount, then divide. k cuts ⇒ k+1 regions.
Count k cuts with a marked region.

 counts  3 times

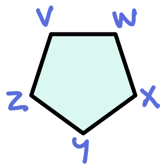


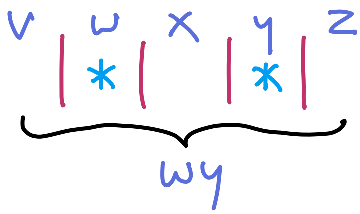
Orient each cut to keep marked region on the left.
Each cut now has a "start" •

$\boxed{\binom{n+k-1}{k}}$

This looks like a "bars & stars" monomial count,

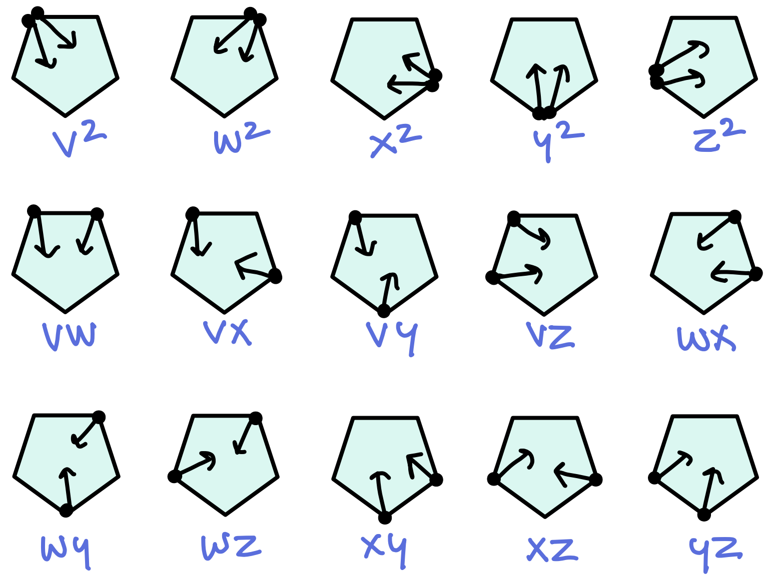
The k cuts can start anywhere, including several from the same vertex.





$$\frac{n-1 \quad k}{| \quad *} \qquad \binom{n+k-1}{k}$$

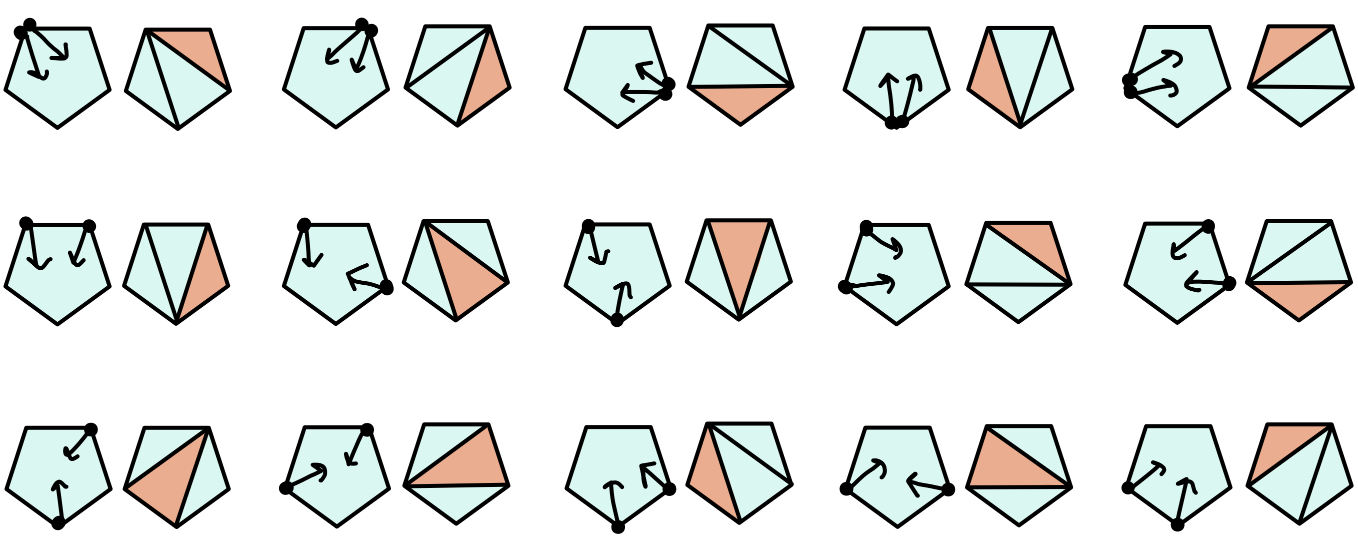
Choose the k *'s



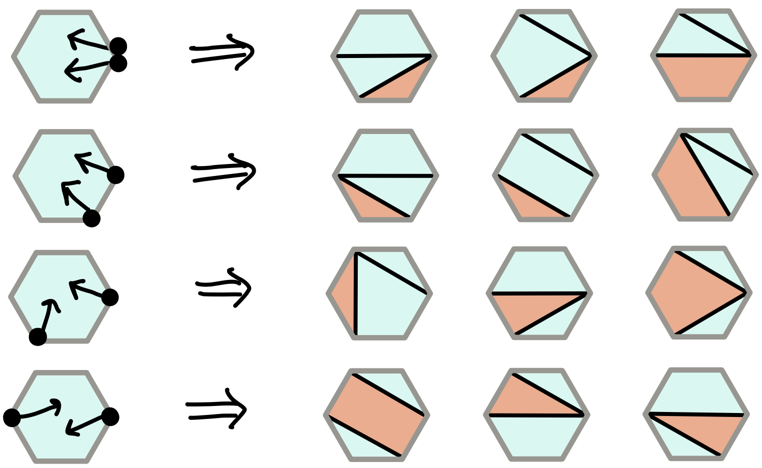
$\boxed{\binom{n-3}{k}}$

Counts ways to finish diagram, so cut directions are compatible with a choice of marked region.

$n=5$, $k=2$ $\Rightarrow$ $\binom{2}{2}=1$, unique way for our example.



$\boxed{\binom{n-3}{k}}$ $\quad n=6,\ k=2 \Rightarrow \binom{3}{2}=3$



Anyone up for a real life bonus question?

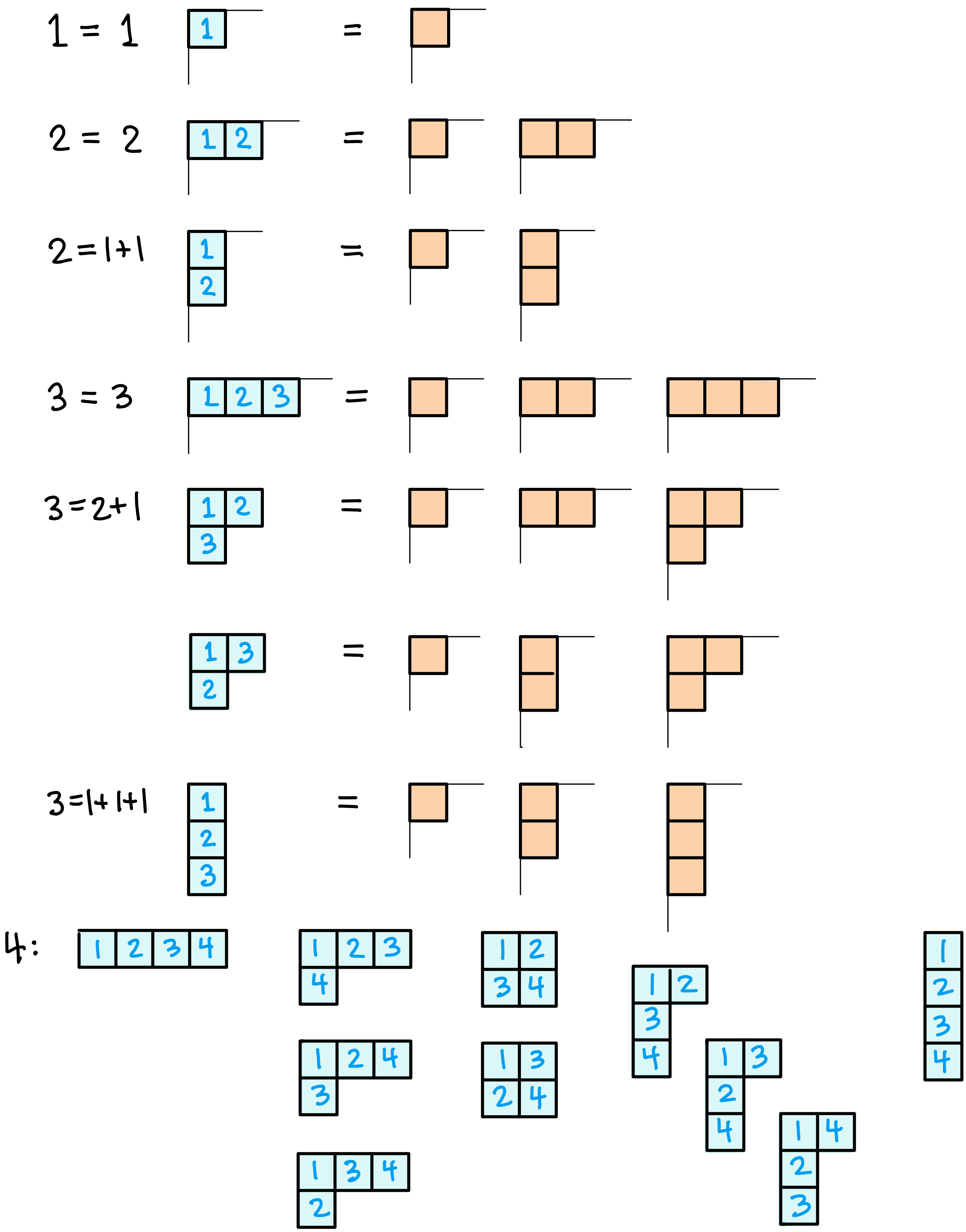
Challenge: Without looking at 2000 Przytycki, Sikora, find your own proof of this last step.

Play with this. I believe there could be a simpler argument.

Apparently unrelated topic (of course they're related!)

Young tableaux

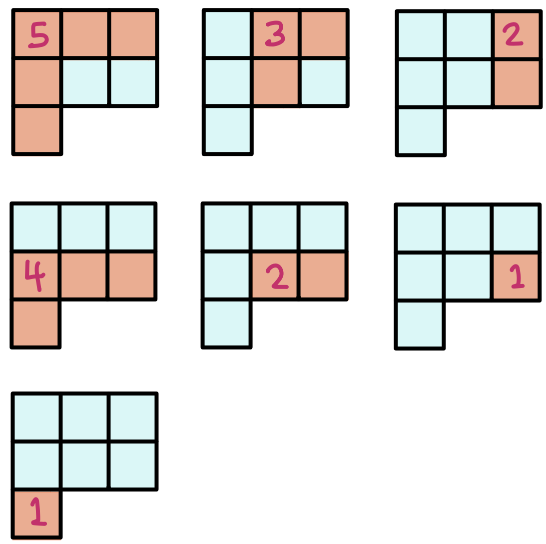
How many ways can we grow a staircase shape, step by step?



Hook length formula

For each cell, record the length of the "hook" down or over.

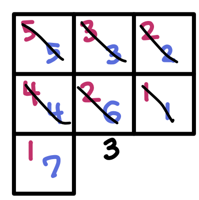
5	3	2
4	2	1
1		



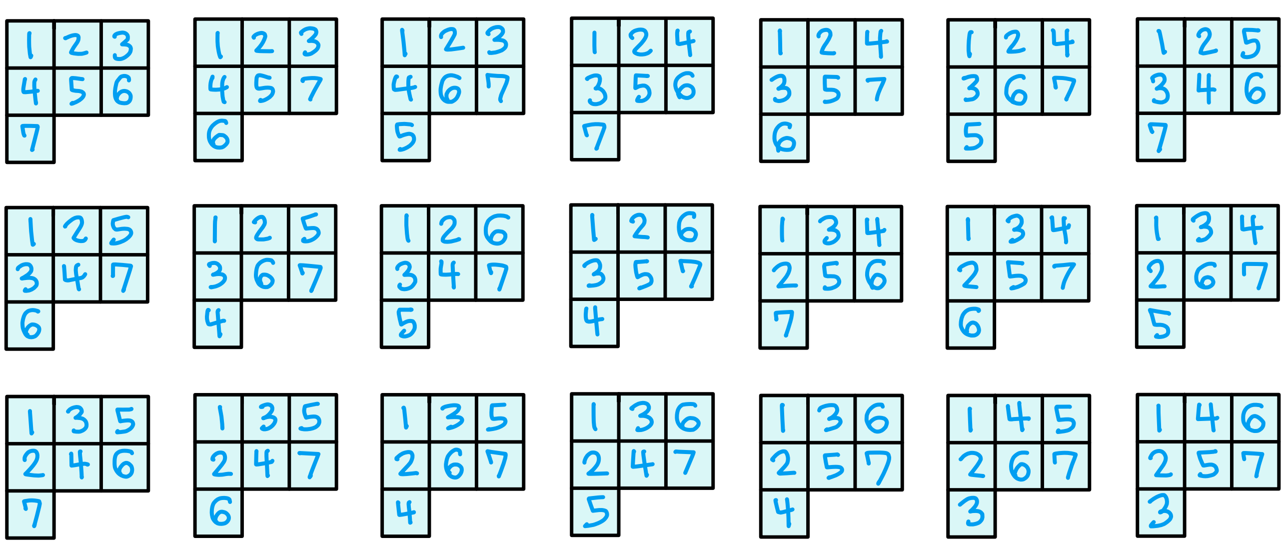
For n cells, divide n! by the product of the hook lengths.

$$\frac{7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{5}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{4}\cdot\cancel{2}\cdot 1\cdot 1} = 21$$

or



= 21



March 30

Recap, finish formula from last week.

Formula:

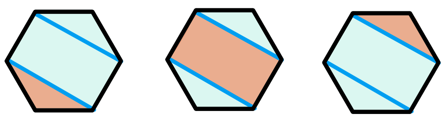
$T(n,k)$ = number of dissections of an n-gon by k cuts

$$= \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$$

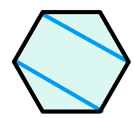
1890 Cayley
... 2000 Przytycki, Sikora

Meaning of each part:

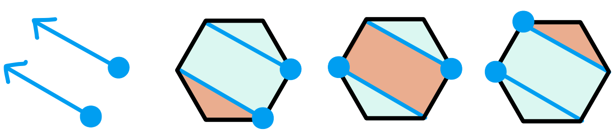
$\boxed{\frac{1}{k+1}}$ We overcount, then divide. k cuts $\Rightarrow$ $k+1$ regions.
Count k cuts with a marked region.



counts



3 times

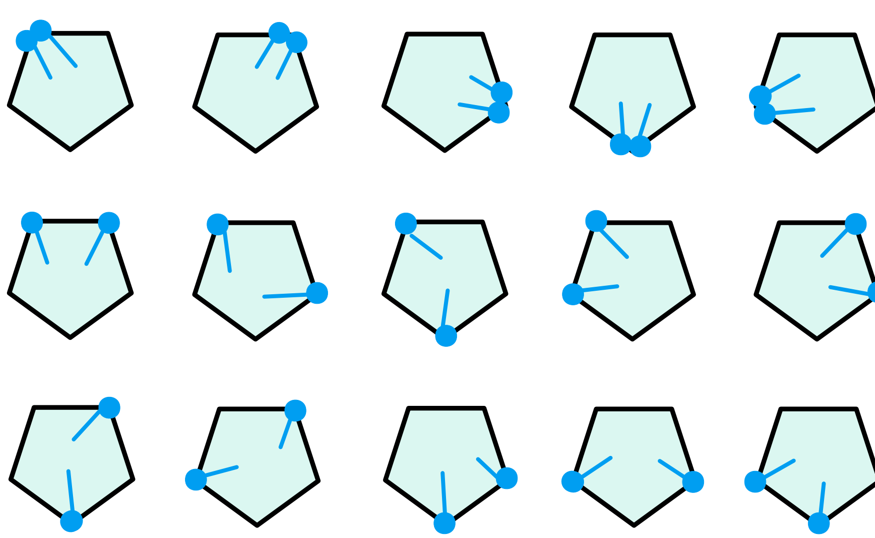


Orient each cut to keep marked region on the left.
Each cut now has a "start"

$\boxed{\binom{n+k-1}{k}}$ # monomials of degree k in n variables

= # ways to start k cuts on n corners

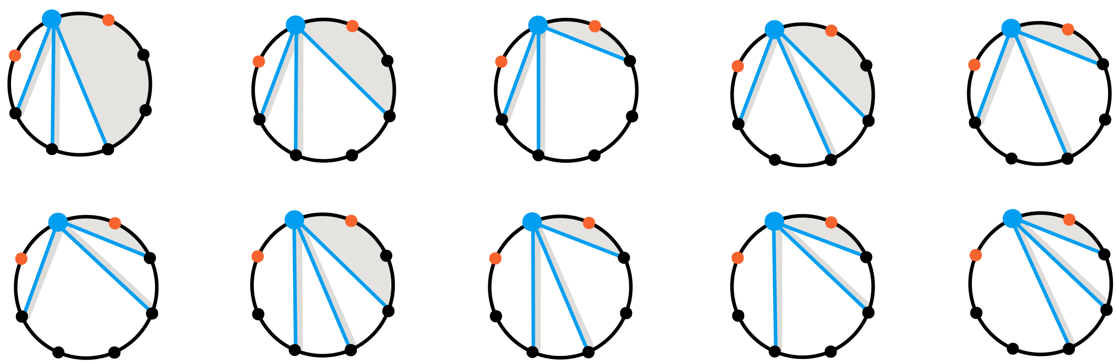
$n=5$, $k=2$ $\qquad \binom{n+k-1}{k} = \binom{6}{2} = 15$



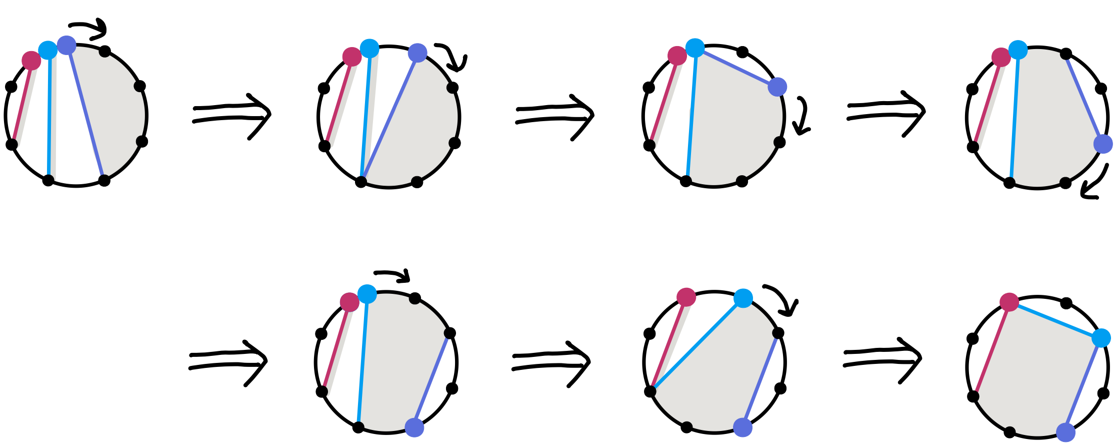
$\boxed{\binom{n-3}{k}}$ Counts ways to finish diagram, so cut directions are compatible with a choice of marked region.

Easy to see if all cuts start at same corner: There are $n-3$ eligible corners, we pick k of them.

$n=8, \quad k=3 \qquad \binom{n-3}{k} = \binom{5}{3} = 10$

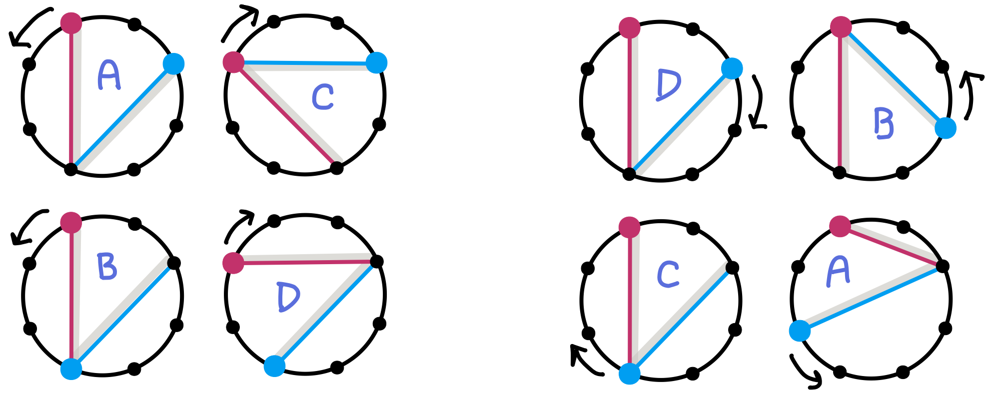


Modification of 2000 Przytycki, Sikora argument: Slide starting positions around like abacus beads. Transfer above configurations by reversible steps to any set of starting positions.



Usually we just rotate a cut to move its start. When two cuts collide, unique way to resolve conflict so there is still a consistent marked region.

Two kinds of transitions :

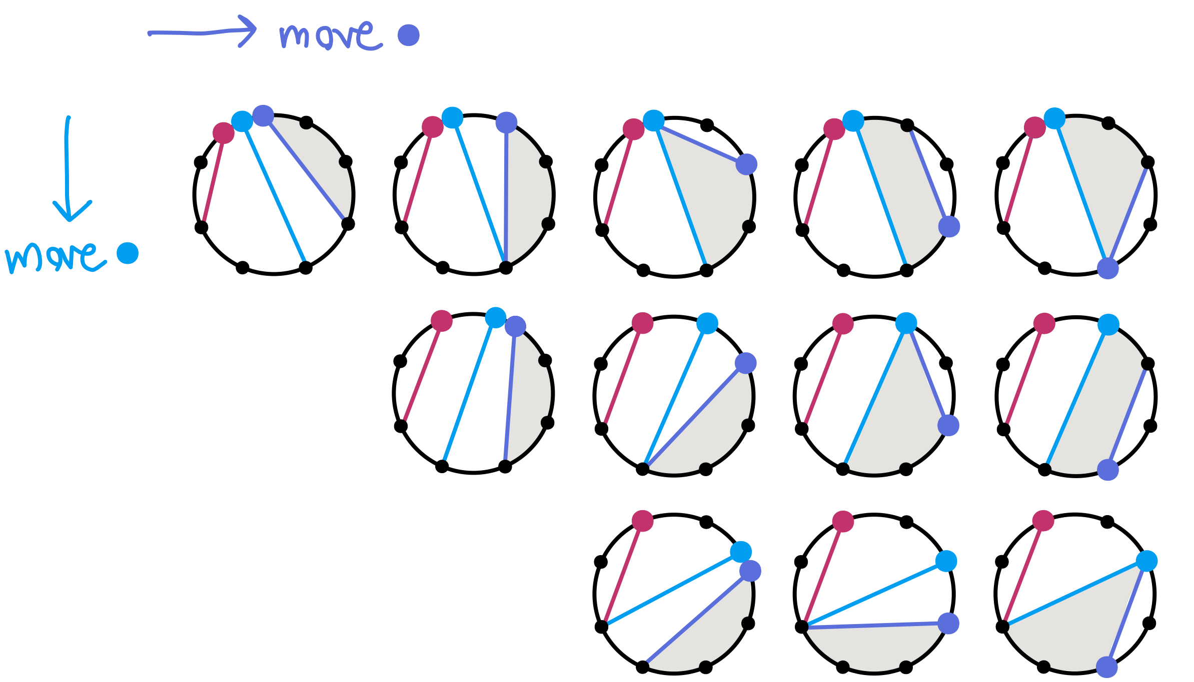


Other cases don't arise :

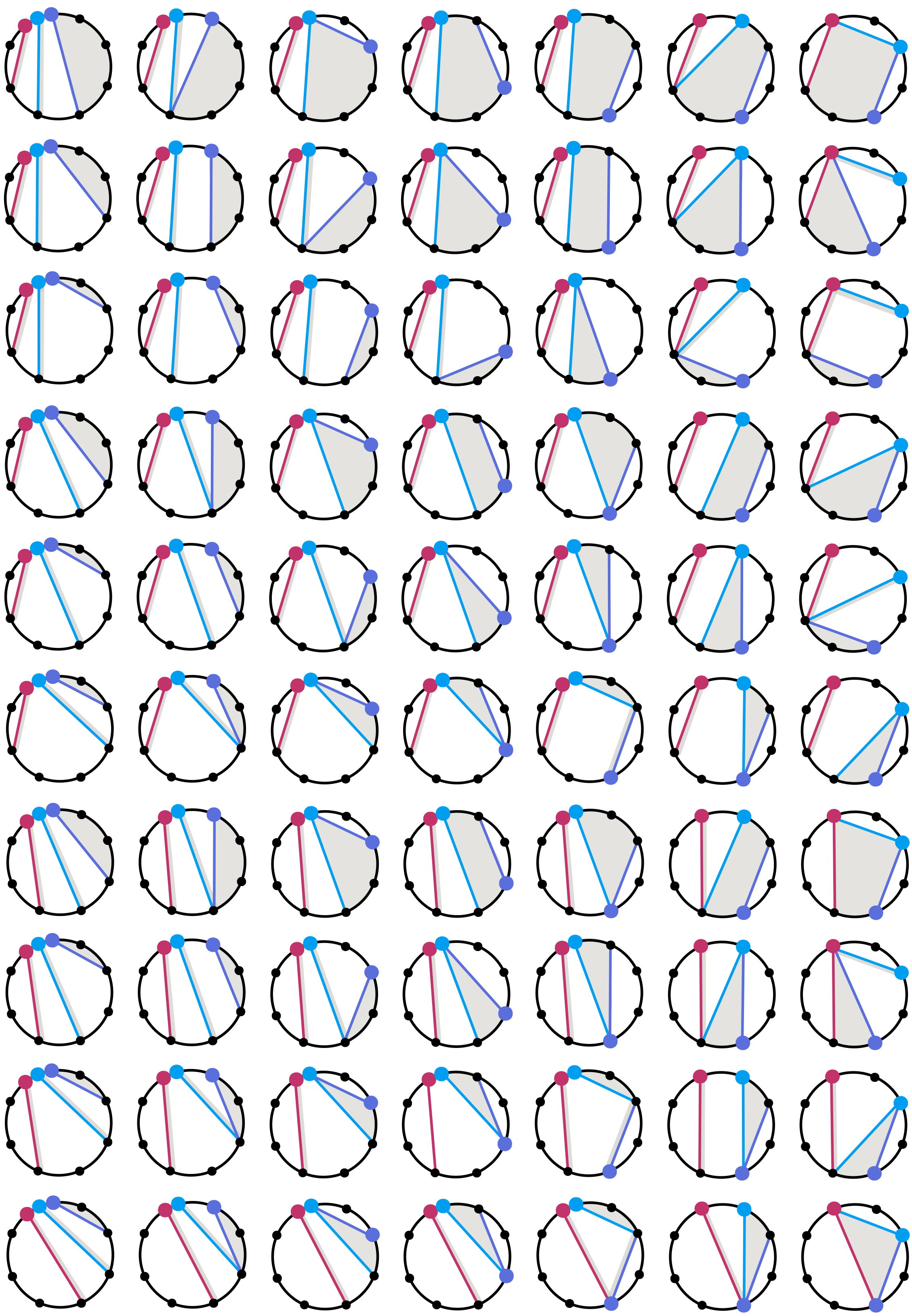
We don't let starts pass through each other

not consistent

The order in which we move starts doesn't matter. Not that we need to care. We get a 1:1 correspondence by retracing our steps.



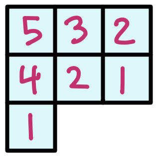
Example correspondence for $n=8$, $k=3$

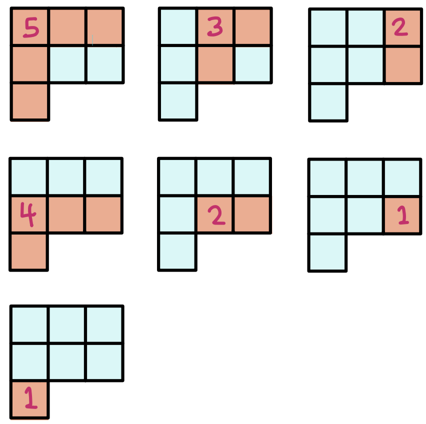


Young tableaux

Hook length formula

For each cell, record the length of the "hook" down or over.

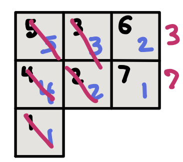




For n cells, divide n! by the product of the hook lengths.

$$\frac{7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{5}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{4}\cdot\cancel{2}\cdot 1\cdot 1} = 21$$

or



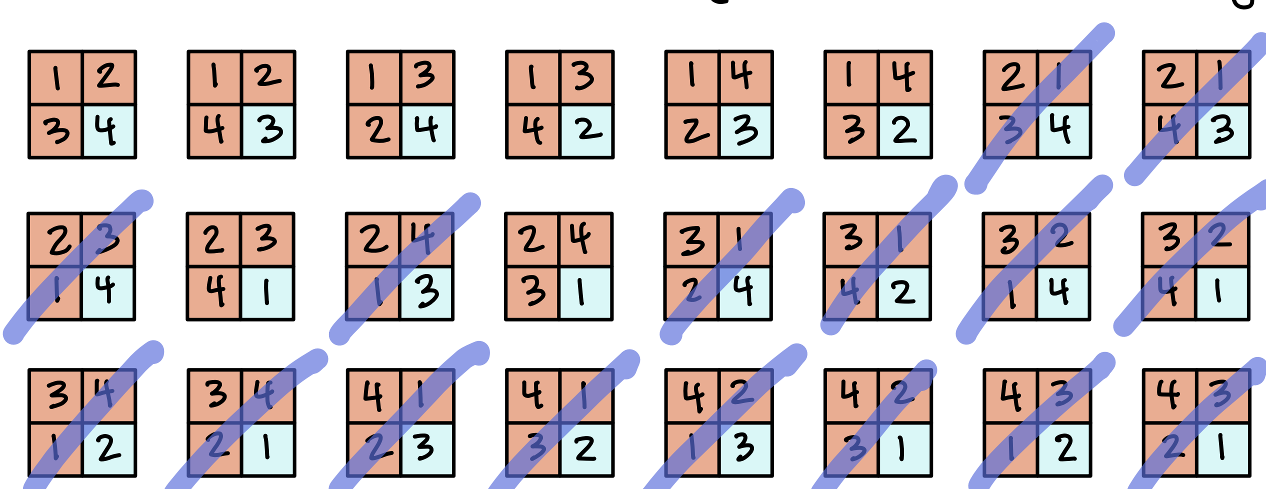
$= 21$

Still no proof that makes this obvious.

Knuth's heuristic argument:

Fill in tableau at random, and look at one hook.

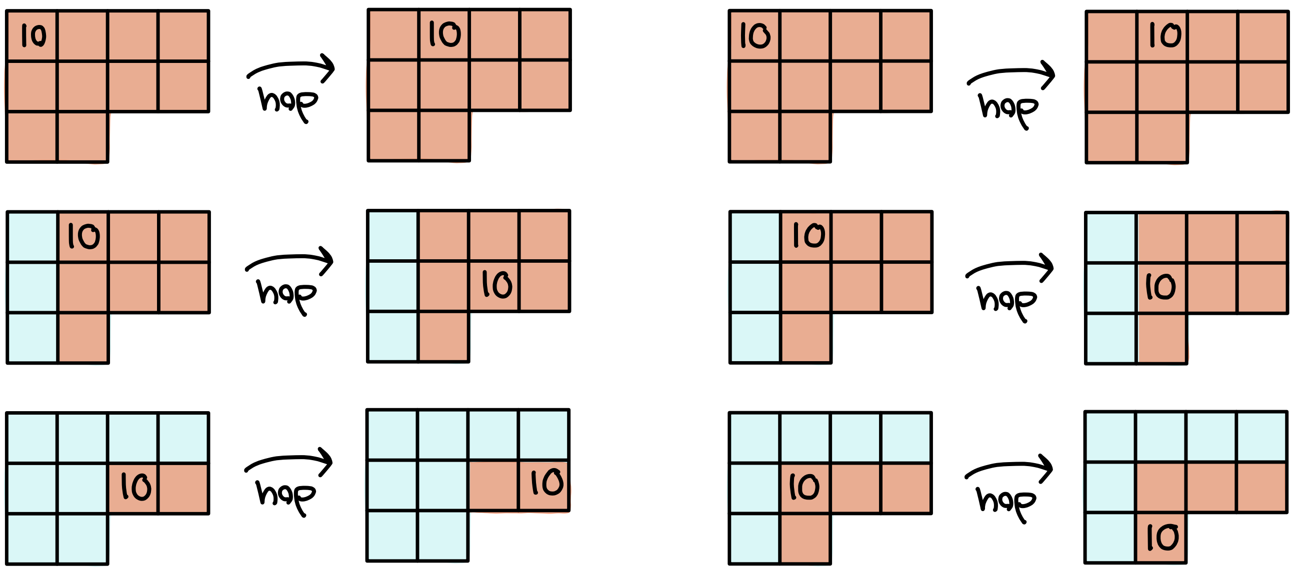
Chances of smallest element being at corner = 1/hook length



Problem: These probabilities aren't independent, for diffent hooks.

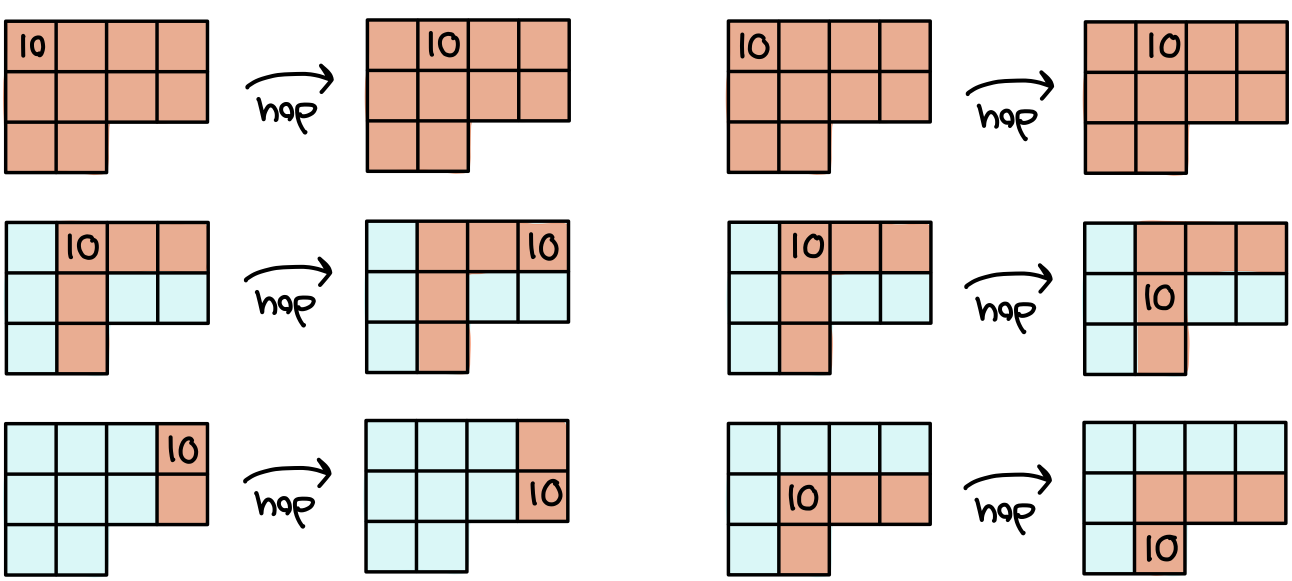
My idea in college (while taking a course with Herb Wilf):

Generate a Young tableau at random,
Start with n in upper left corner
Hop down/over uniformly at random till stuck.
Now iterate. Position $n-1$, then $n-2$, then...

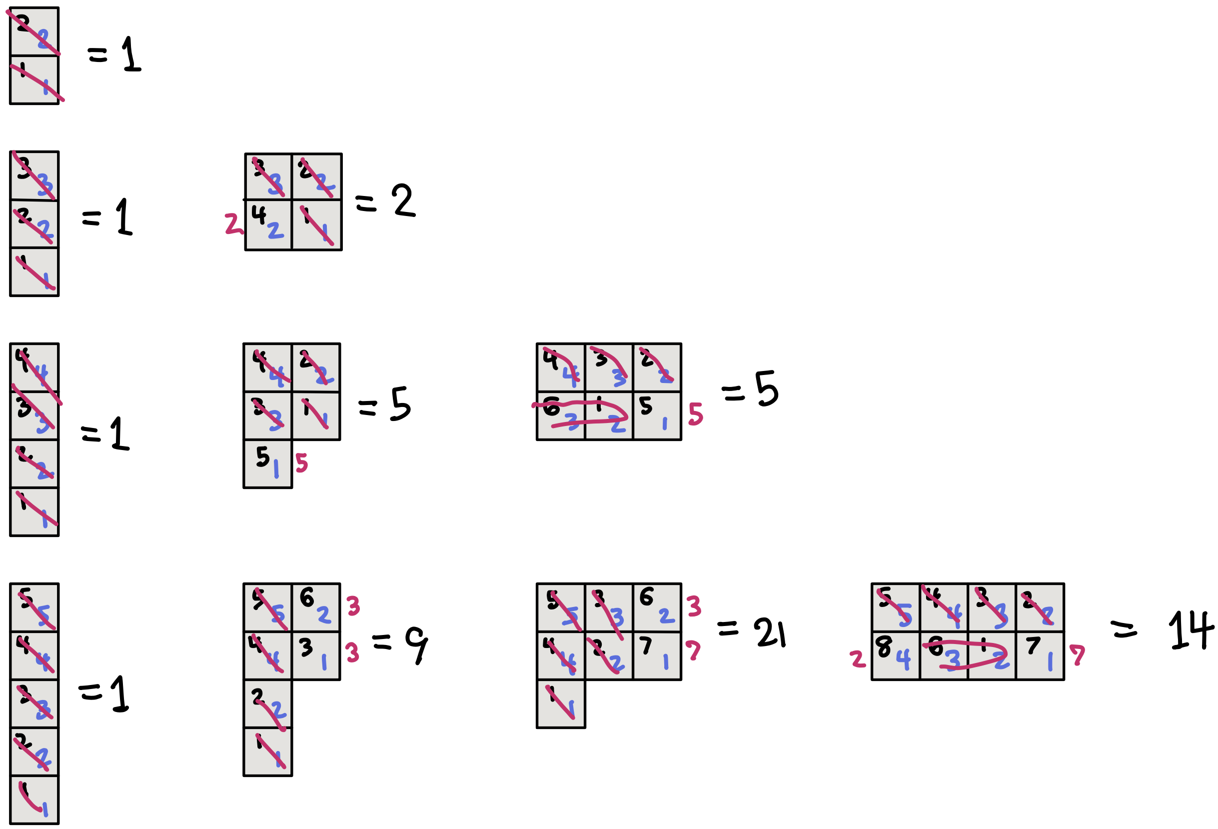


This doesn't quite work. Um, hook lengths? I still kick myself.

1979 Greene, Nijenhuis, Wilf came up with a better process:
After the first step, jump within hooks. Leads to proof of formula.



Special case: Two equal rows, one column

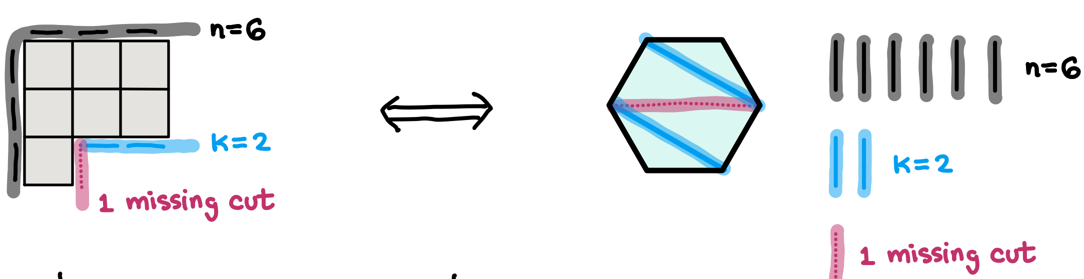


Compare:

$T(n,k)$ = number of dissections of an n-gon by k cuts

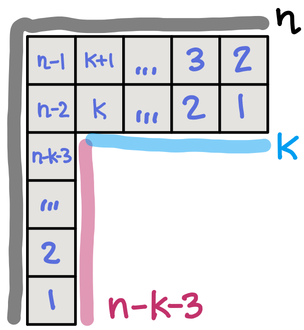
n-gon \ k cuts	0	1	2	3	4	5	6
3	1						
4	1	2					
5	1	5	5				
6	1	9	21	14			
7	1	14	56	84	42		
8	1	20	120	300	330	132	
9	1	27	225	825	1485	1287	429

Catalan numbers



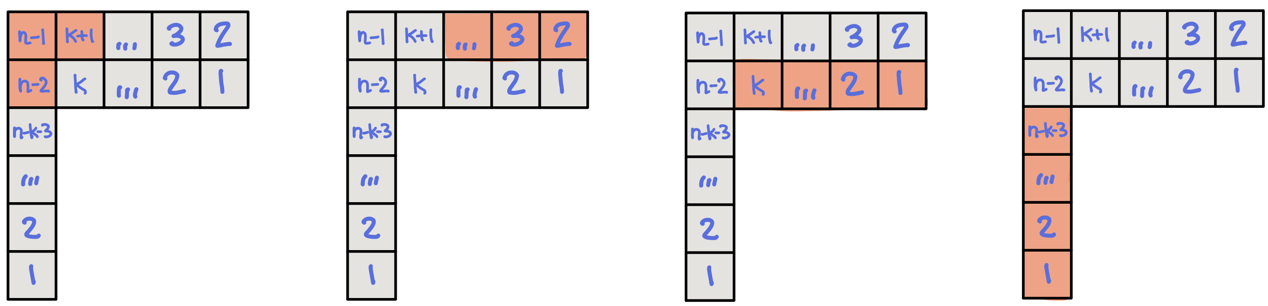
Formulas agree, on overlap.
These sets are in 1:1 correspondence.

Formulas agree, on overlap:



hook lengths

$$\underbrace{2k + (n-k-3) + 2}_{n+k-1} \text{ cells}$$



$(n-1)(n-2)(k+1)$ $\quad k!$ $\quad k!$ $\quad (n-k+3)!$

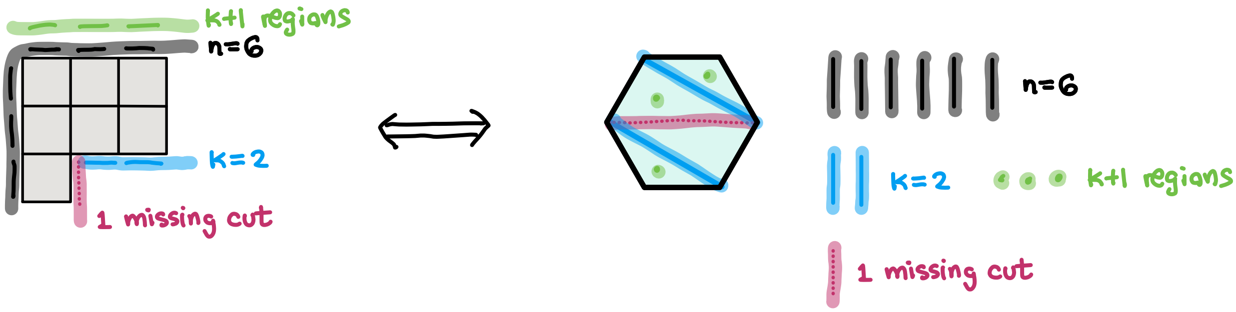
$$\frac{(n+k-1)\cdots n\,(n-1)(n-2)\,(n-3)\cdots(n-k+4)\,(n-k+3)\cdots 3\cdot 2\cdot 1}{(k+1)\;k!\;(n-1)(n-2)\;k!\;(n-k+3)!}$$

$$= \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$$

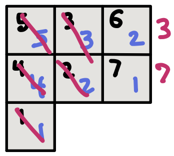
- Good that formulas agree
- Better to find 1:1 correspondence between sets
- Even better if correspondence:
  - has low complexity
  - preserves a neighbor graph ...
  - preserves a polytope

Here, we could learn more about Young tableaux from what we know about polygon dissections.

April 1 Stanley correspondence between polygon dissections and Young tableaux

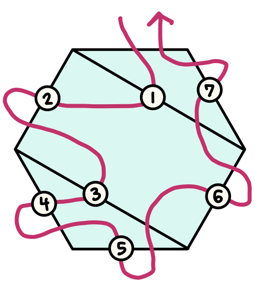


Formulas agree:

 $= 21$

$$\frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k} = \frac{1}{3}\binom{3}{2}\binom{7}{2} = 21$$

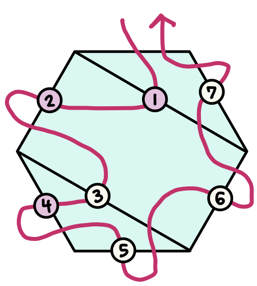
polygon dissections ⇒ Young tableaux:



1 2 3 4 5 6 7

Enter and exit top side

Mark every other wall by depth-first search

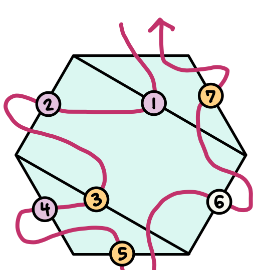


1	2	4

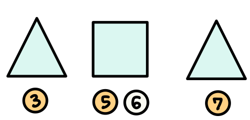
1 2 4

Select first exit from each chamber

This will become first row of Young tableau

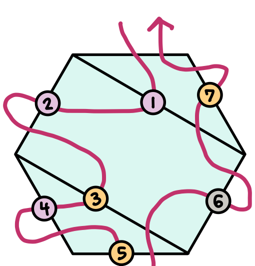


1	2	4
3	5	7



Group remaining markers to record chamber sizes

Select group starts for second row

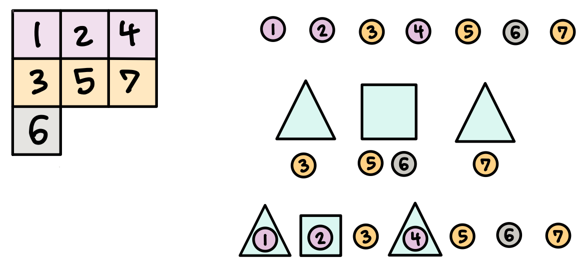


1	2	4
3	5	7
6		

6

Use remaining markers for tail of Young tableau

Young tableaux ⇒ polygon dissections:

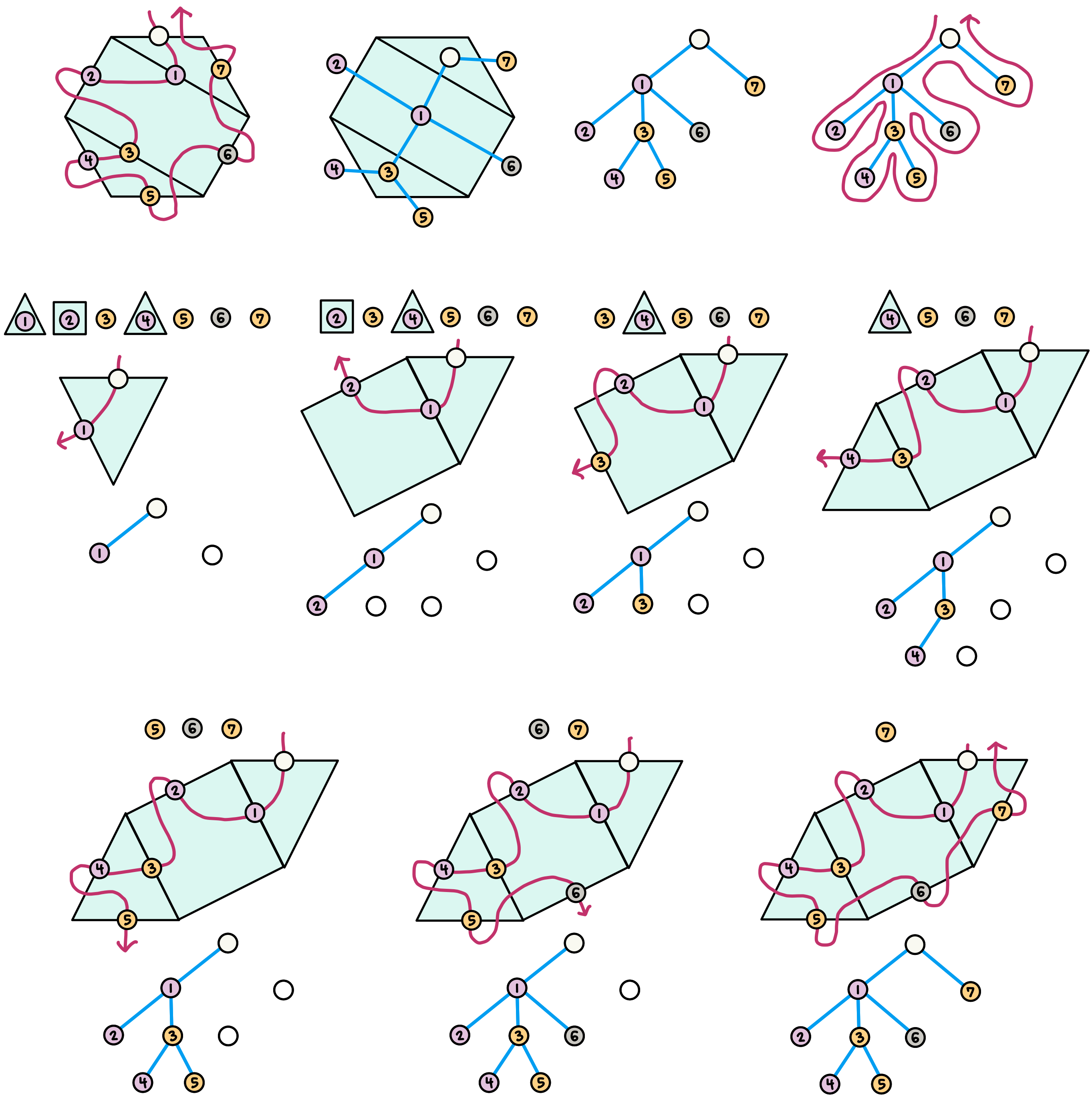


Read out numbers as markers

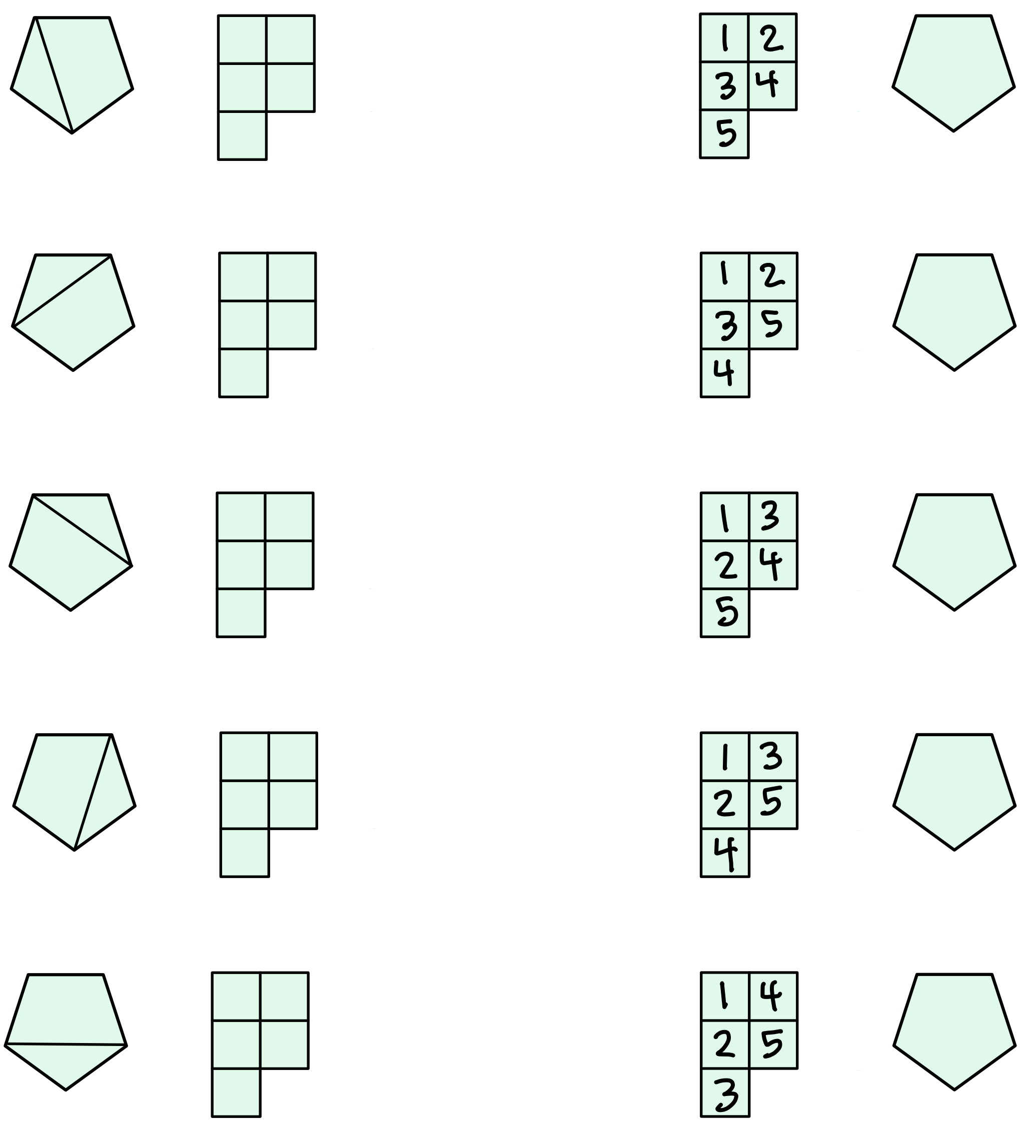
Recover chamber sizes

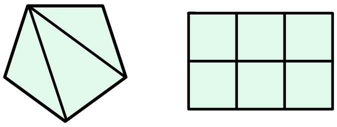
Move sizes to first exit markers

We can think of this as a dissection or a tree.

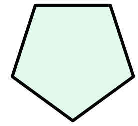


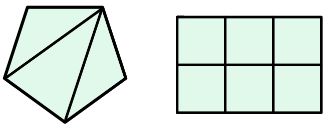

1 2
3 4
1 3
2 4


1 2
3 4
5
1 2
3 5
4
1 3
2 4
5
1 3
2 5
4
1 4
2 5
3

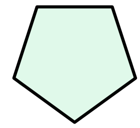


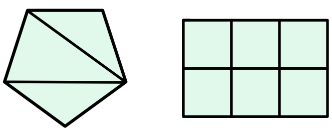
1	2	3
4	5	6



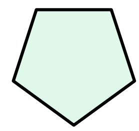


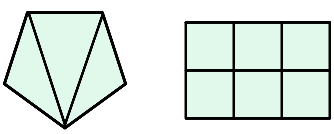
1	2	4
3	5	6



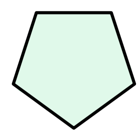


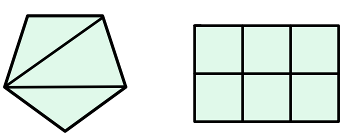
1	2	5
3	4	6



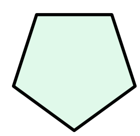


1	3	4
2	5	6

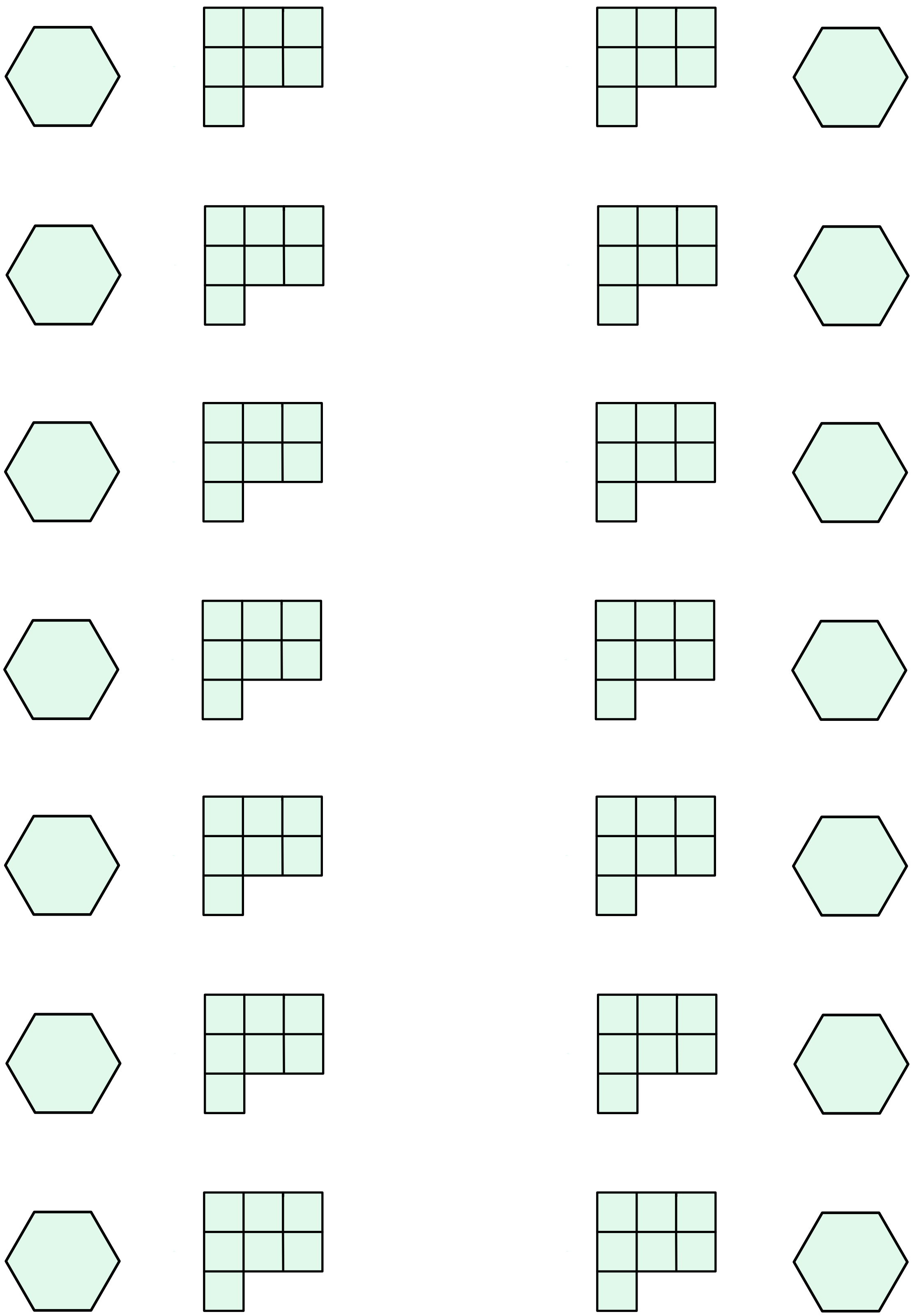




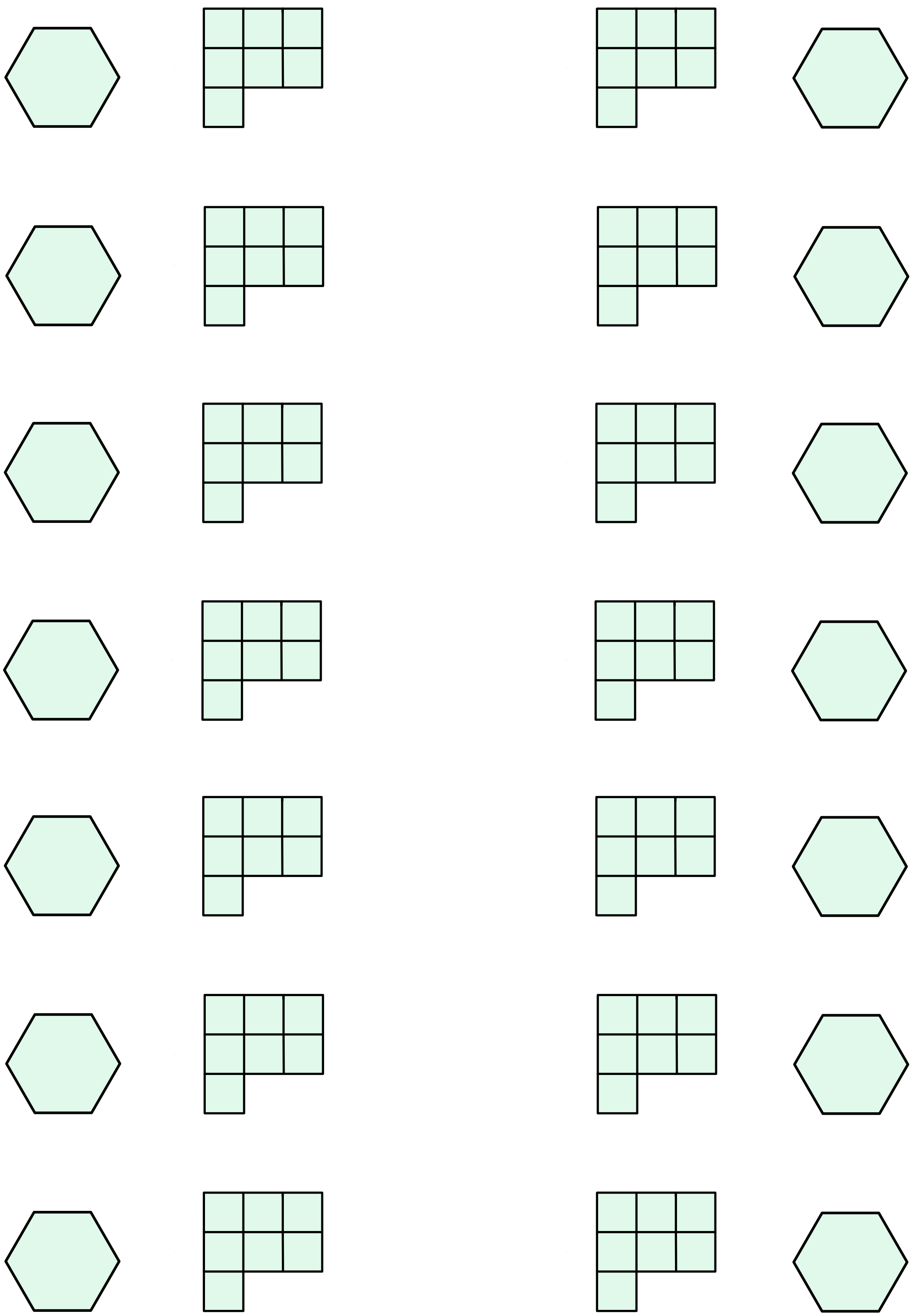
1	3	5
2	4	6



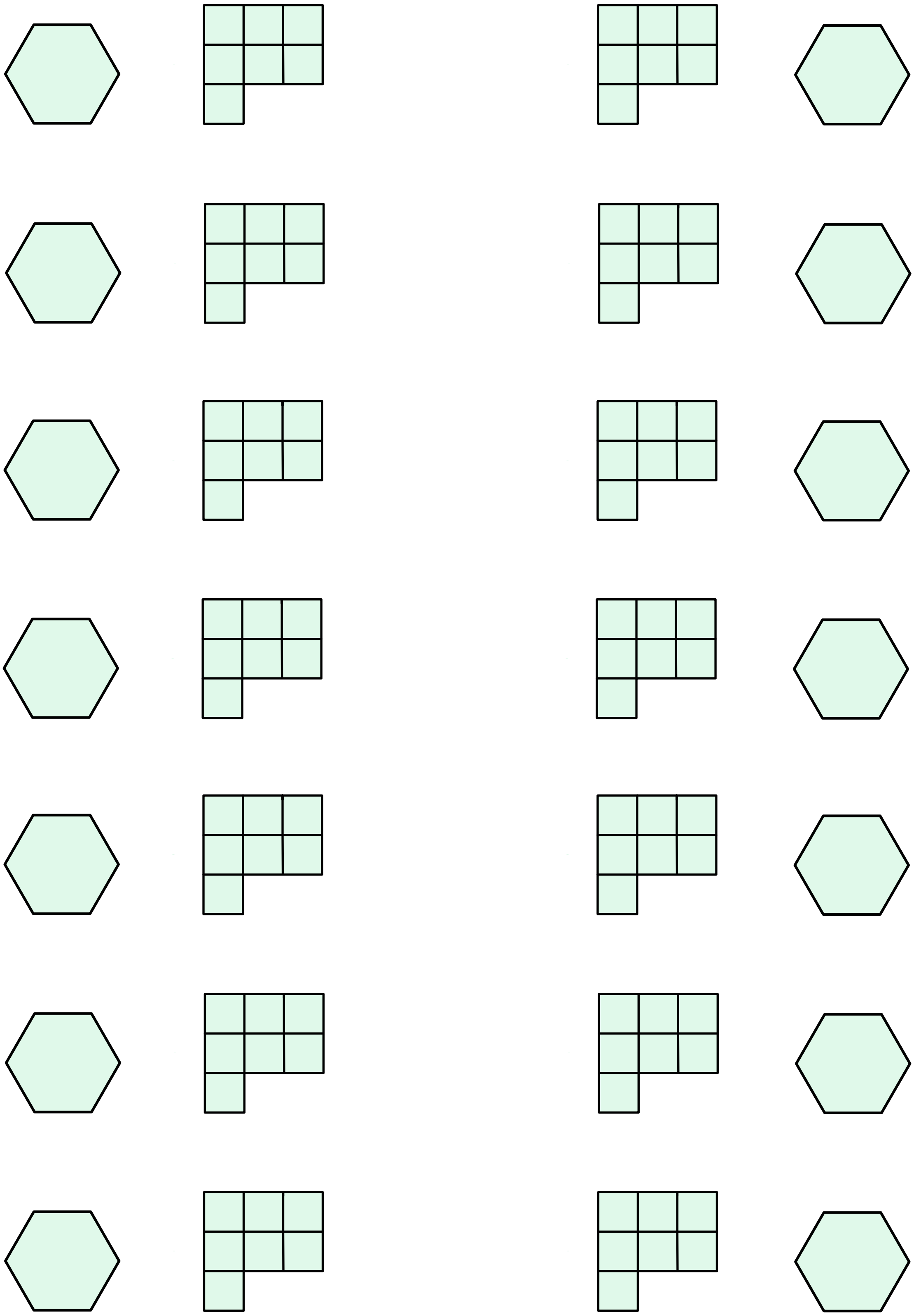
$n=6 \quad k=2 \quad 21$ cases

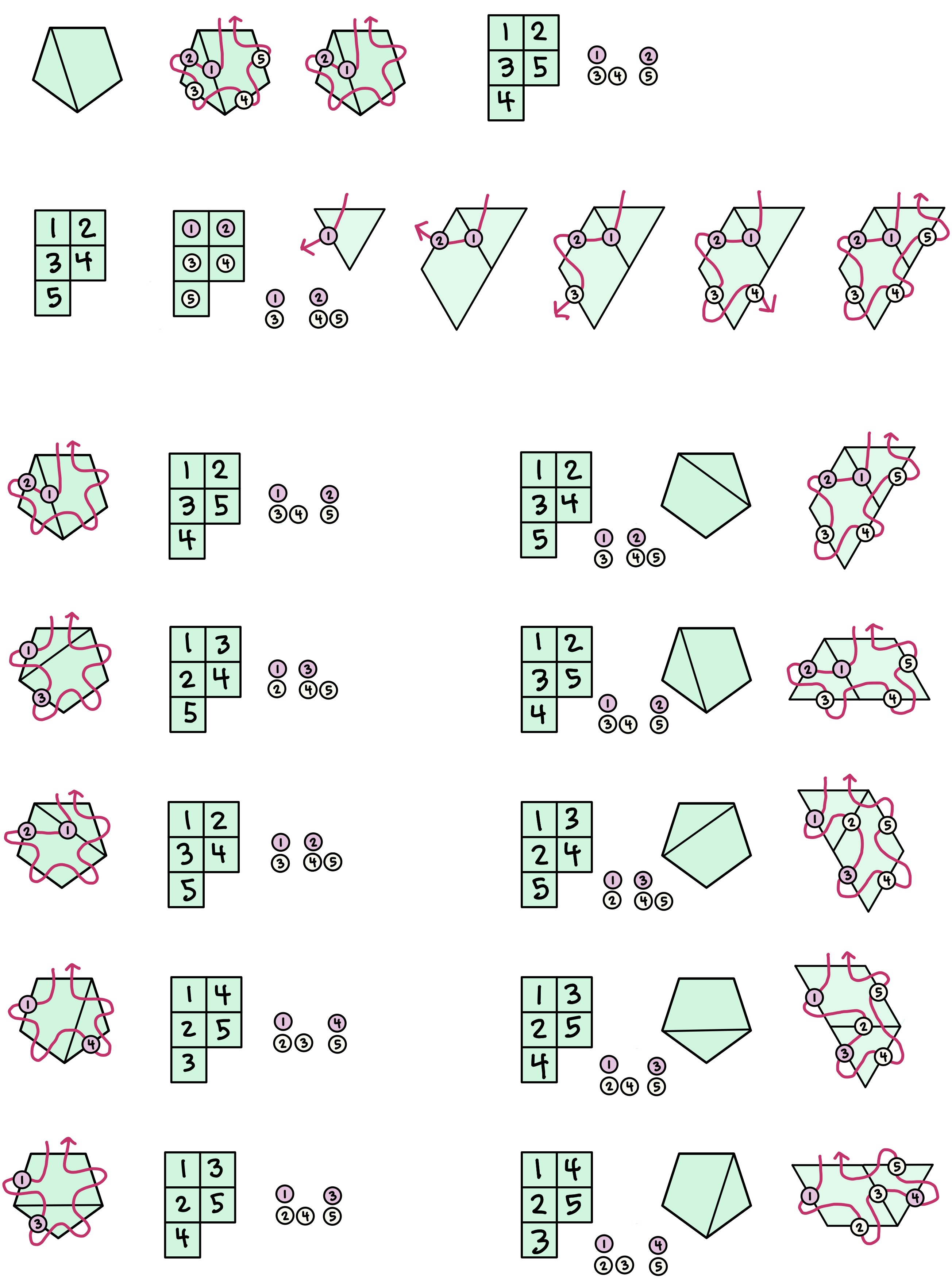


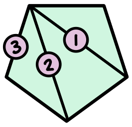
$n=6 \quad k=2 \qquad 21$ cases



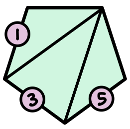
$n=6$ $k=2$ 21 cases



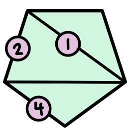




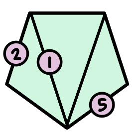
1	2	3
4	5	6



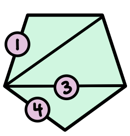
1	3	5
2	4	6



1	2	4
3	5	6

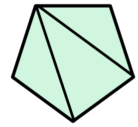


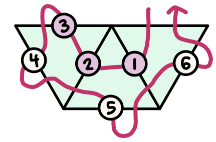
1	2	5
3	4	6



1	3	4
2	5	6

1	2	3
4	5	6





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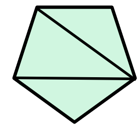

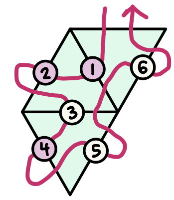

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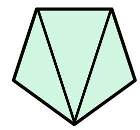

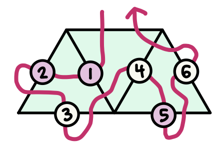

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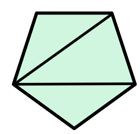

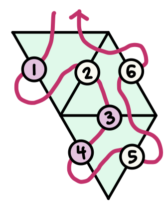

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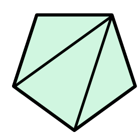
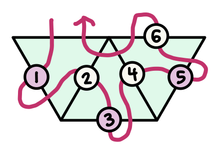
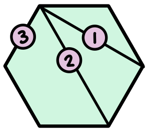




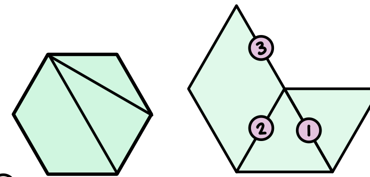
n=6 k=2 21 cases

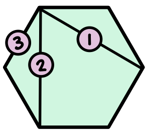


1	2	3
4	5	6
7		

1	2	3
4	5	6
7		

1 2 3
4 5 67





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1	2	3
4	5	7
6		

1 2 3
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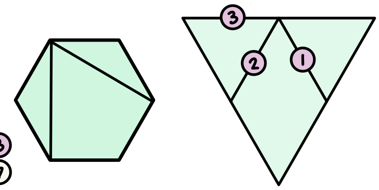

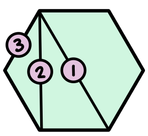

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5		

1 2 3
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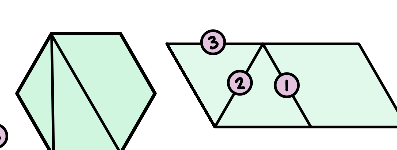

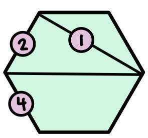

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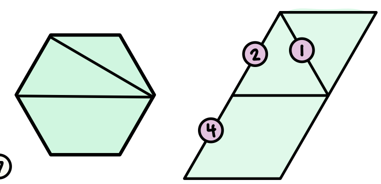
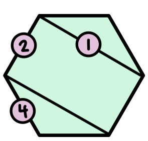
1	2	4
3	5	6
7		

1 2 4
3 5 67





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6		

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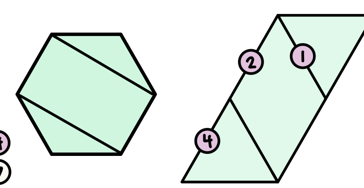
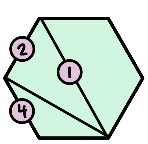
3	5	7
6		

1 2 4
3 56 7





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3	6	7
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1 2 4
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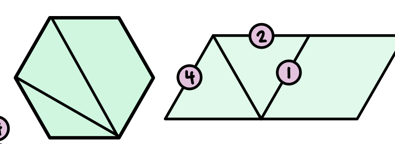

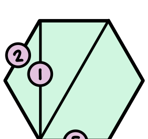

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7		

1	2	5
3	4	6
7		

1 2 5
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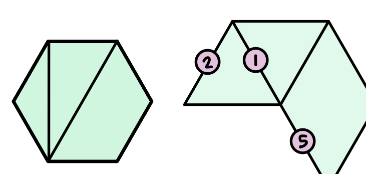
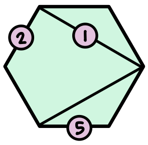


n=6 k=2 21 cases



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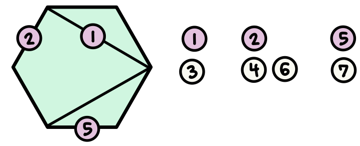

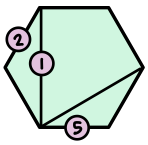

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4		

1	2	5
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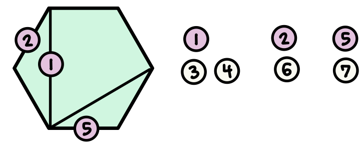
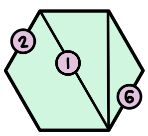
4		





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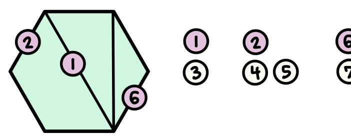

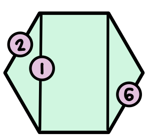

1	2	6
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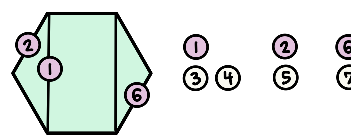

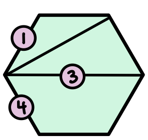

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1	3	4
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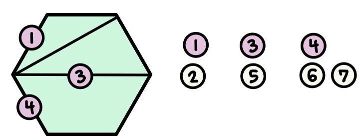

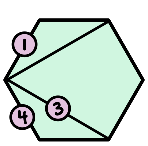

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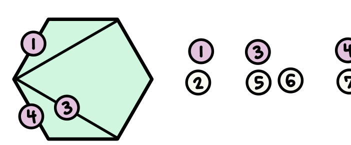

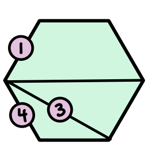

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2	6	7
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2	6	7
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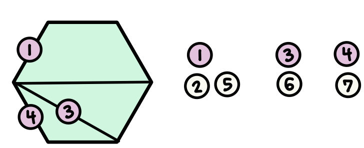
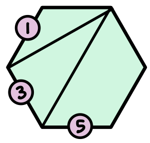


n=6 k=2 21 cases



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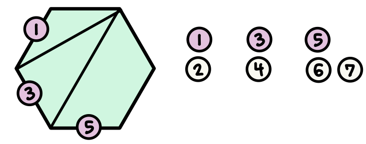

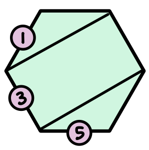

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2	4	7
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1	3	5
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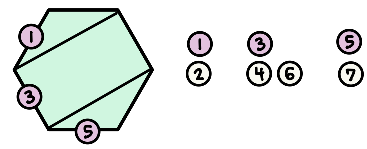

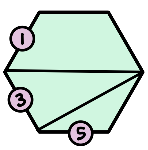

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1	3	5
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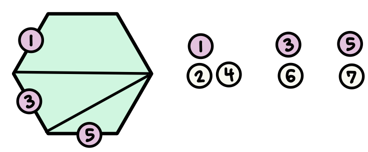

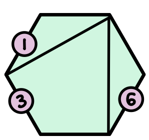

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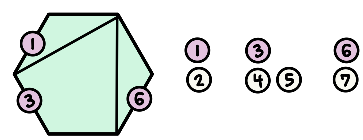

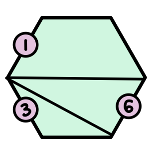

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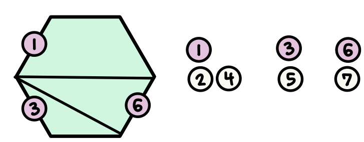

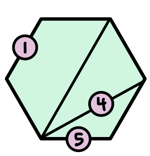

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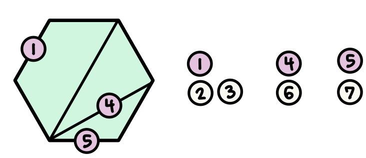

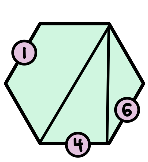

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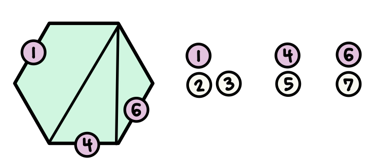
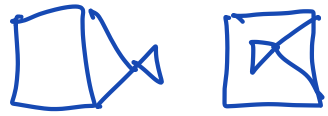
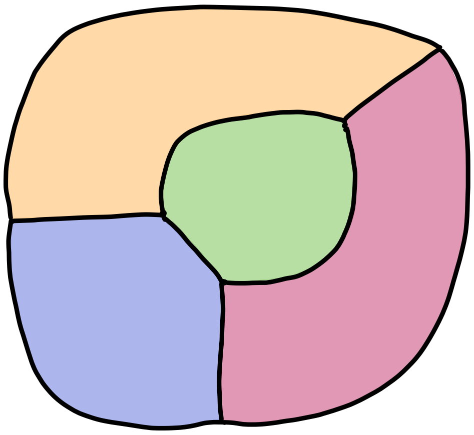
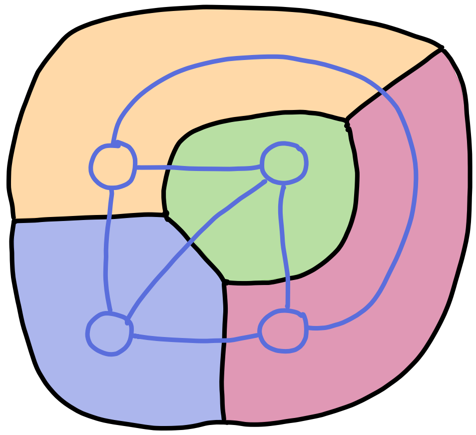


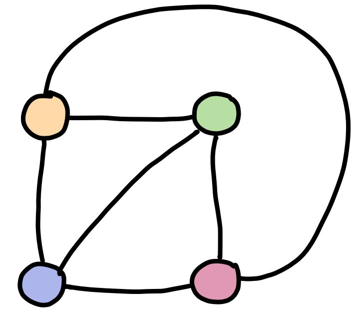
April 8 Graph Colorings



plane
planar







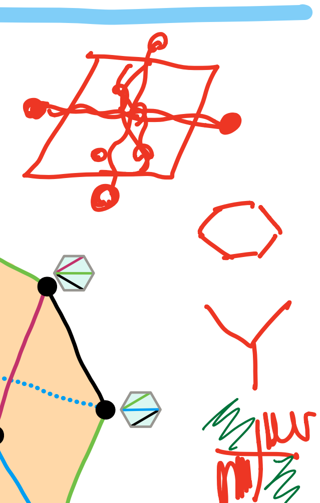
$\binom{4}{2} = 6$

A planar map can require 4 colors

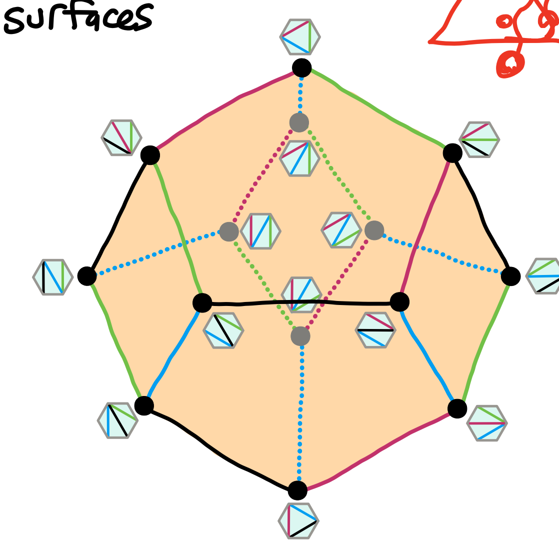
6 colors – easy to prove
5 colors – harder } we'll do
4 colors – very difficult, still no proof easily understood

Euler characteristic $\boxed{\chi = v - e + f}$

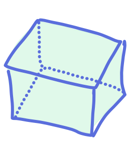
Invariant of simplicial, cellular surfaces



$\chi =$ # vertices
− # edges
+ # faces

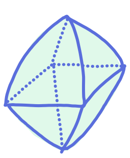


v e f

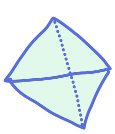


$\chi = 2 = 8 - 12 + 6$

dual



$\chi = 2 = 6 - 12 + 8$

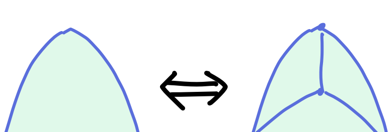


$\chi = 2 = 4 - 6 + 4$

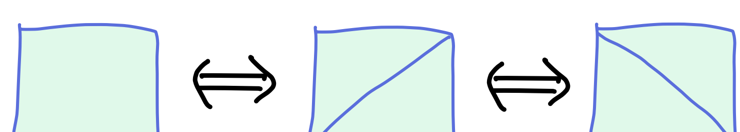
Associahedron

$\chi = 2 = 14 - 21 + 9$

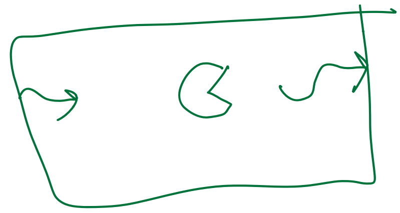
$\chi = 2$ for any topological sphere (genus 0)



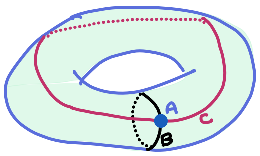
v e f
Δ 1−3+2 = 0

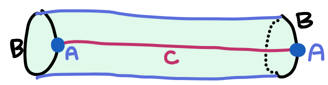


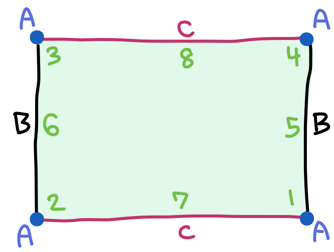
v e f
Δ 1−1 = 0

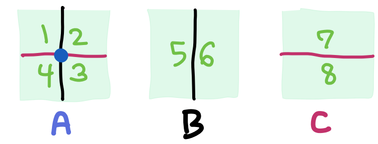


Torus (genus 1)

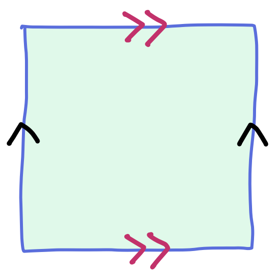




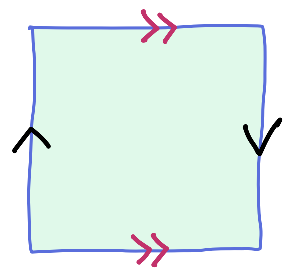




$$\chi = 0 = \overset{v}{1} - \overset{e}{2} + \overset{f}{1}$$



Torus



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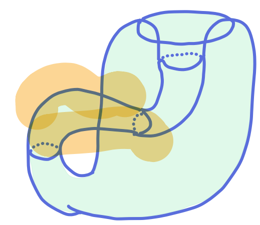
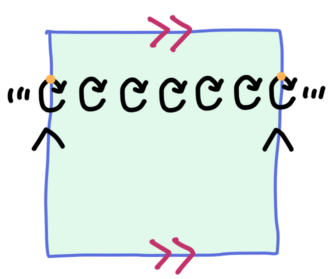
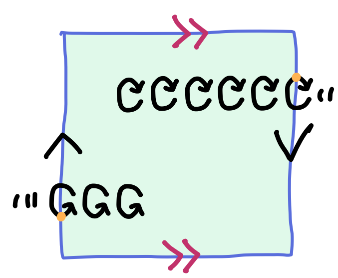


Klein bottle

(same χ)

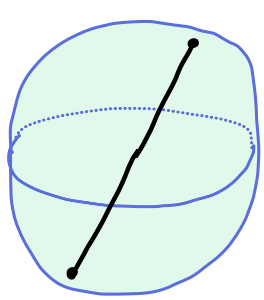


orientable

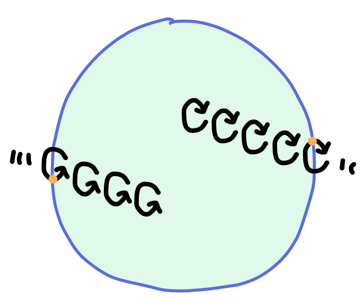


orientation-reversing path
$\Rightarrow$ not orientable

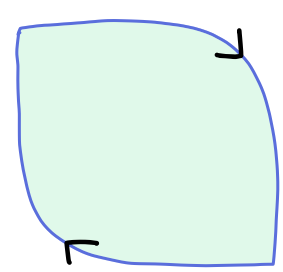
Projective plane



positions of a stick
if we can't tell ends apart

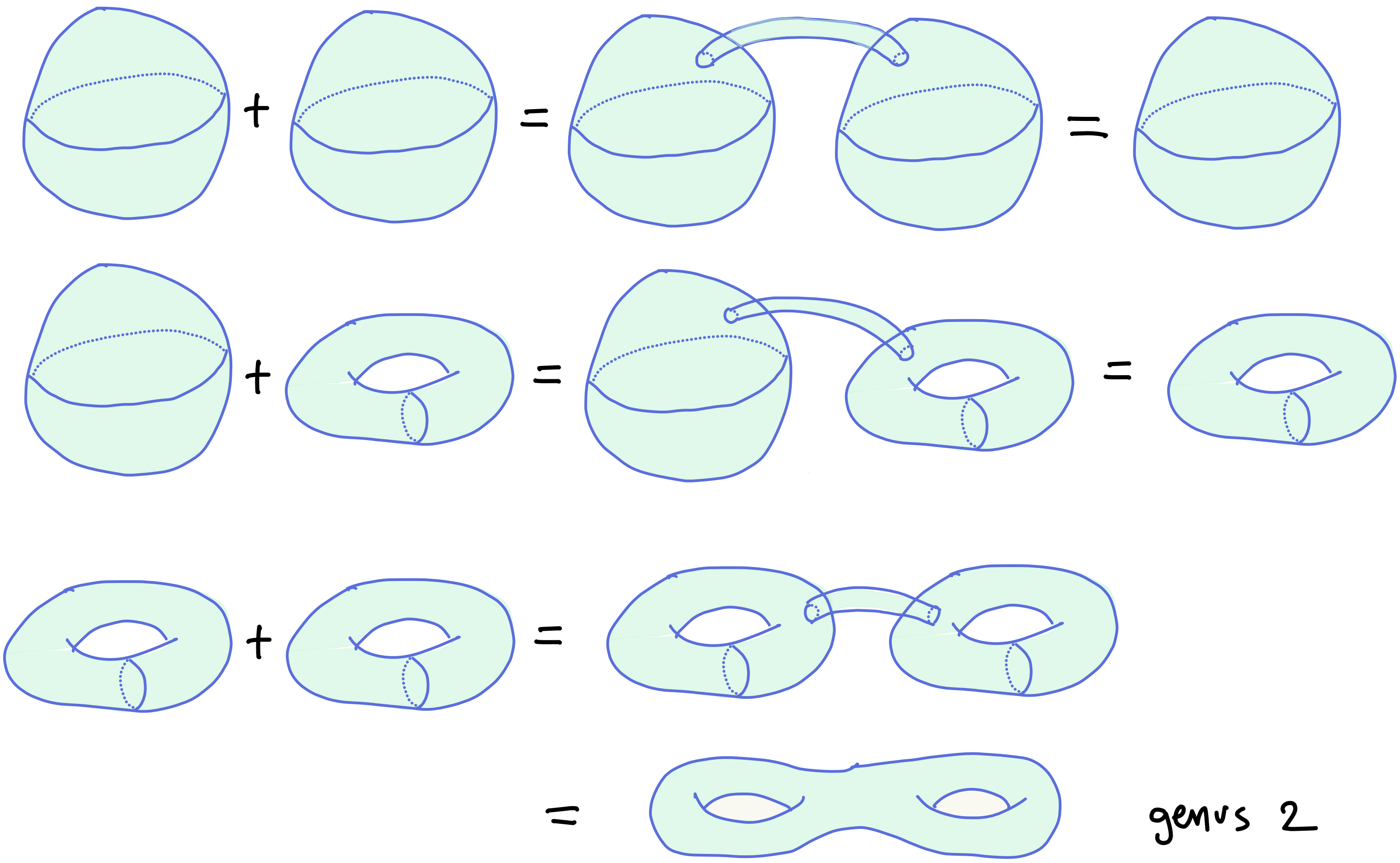


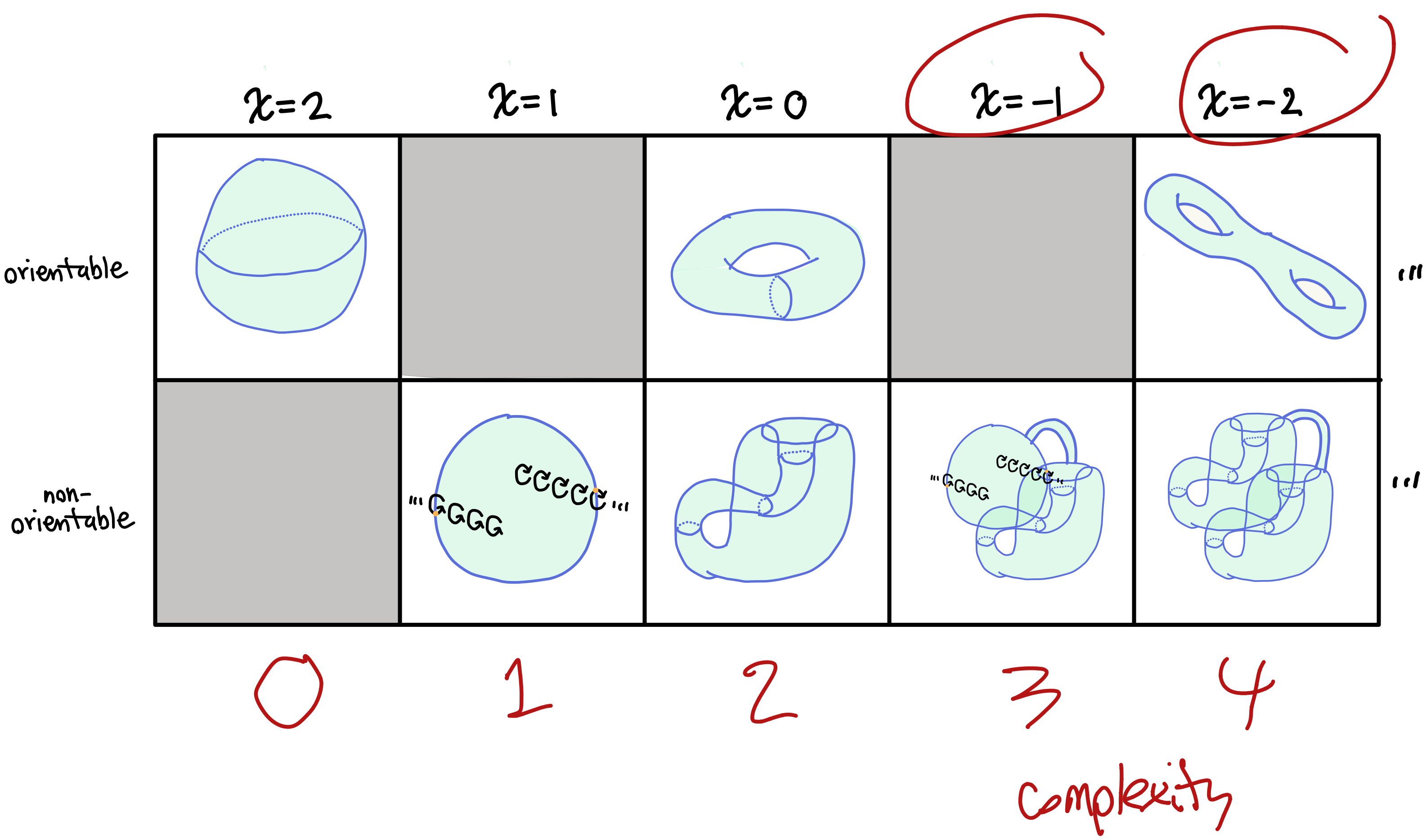
orientation-reversing path
$\Rightarrow$ not orientable

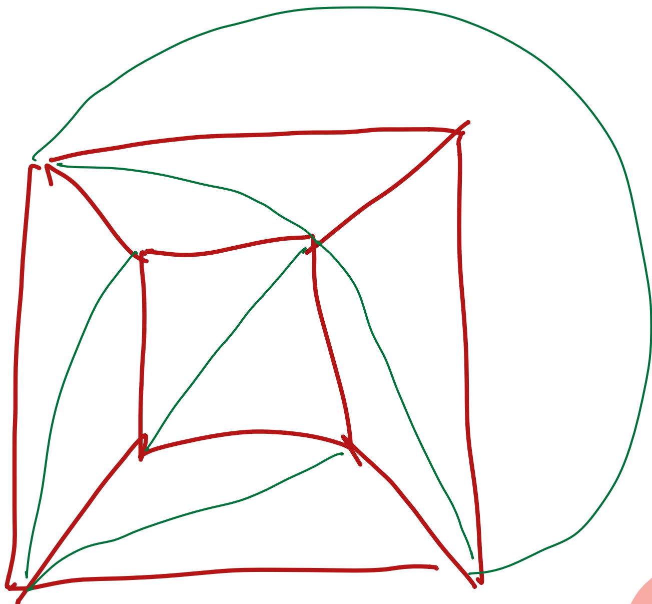


$$\chi = 1 = \overset{v}{1} - \overset{e}{1} + \overset{f}{1}$$

Surgery: Two surfaces can be "added" by connecting with a tube





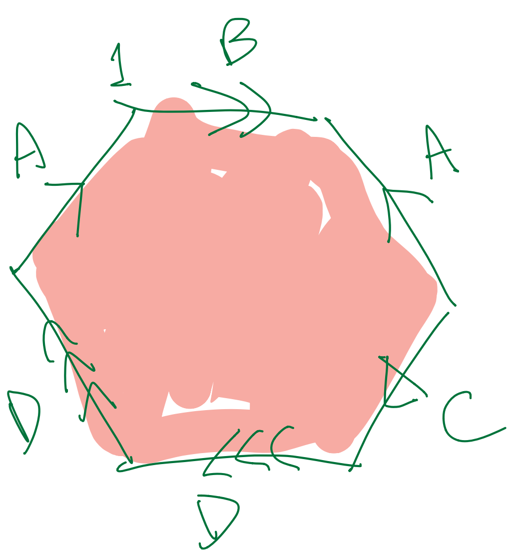


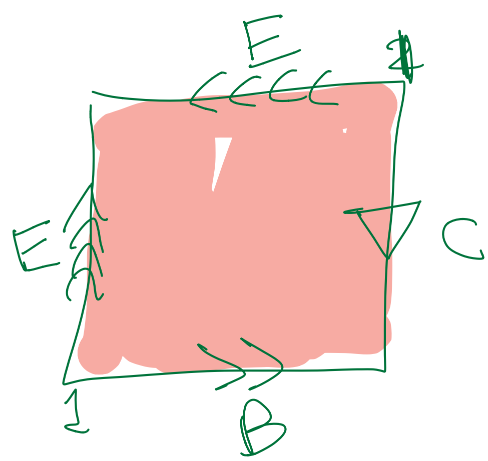
$\chi = 2$

$V - e + F = 2$

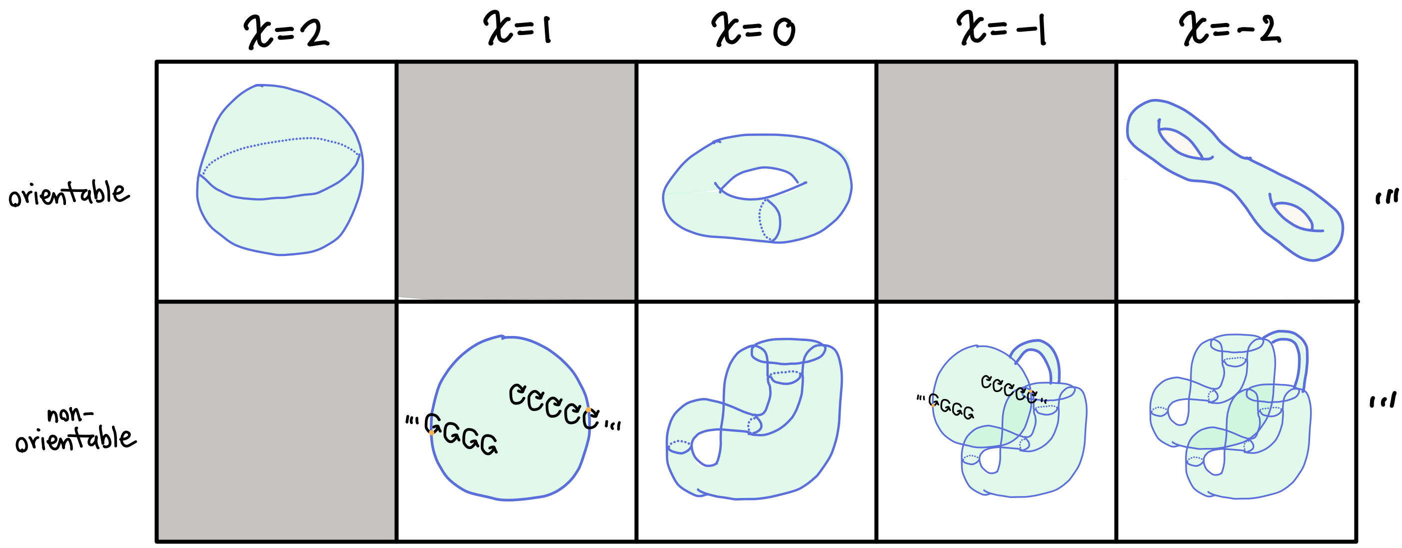
e, F

$2e = 3F$

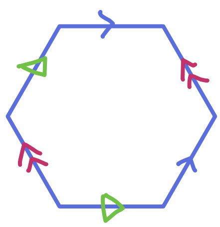




April 13 Graph Colorings

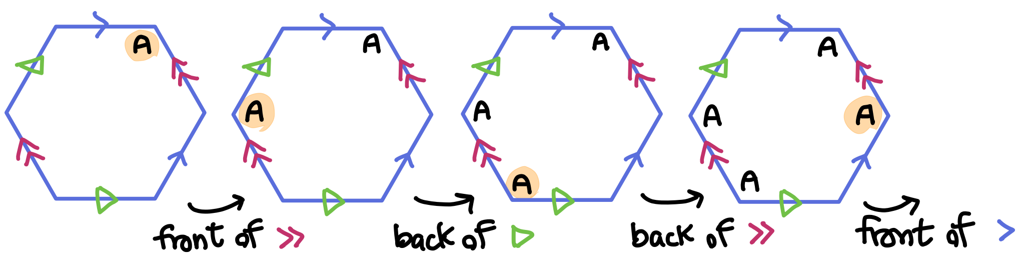


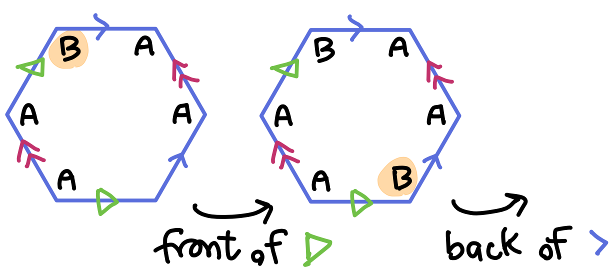
Identifying surfaces from their gluing diagrams



$$\chi = \underbrace{v}_{?} - \underbrace{e}_{3} + \underbrace{f}_{1}$$

chase identifications to enumerate vertices



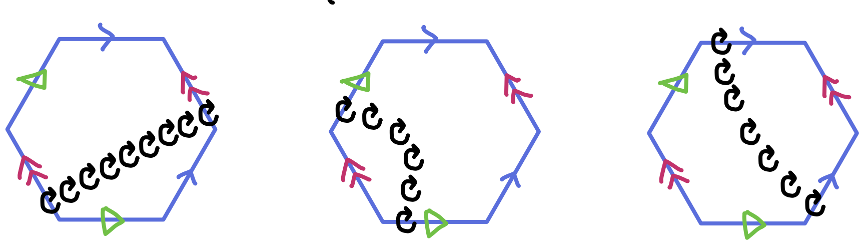


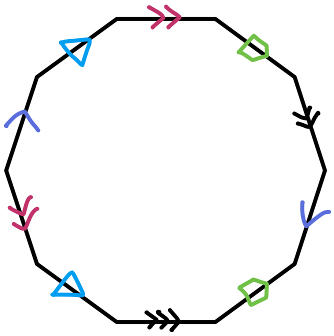
... so $V = 2$ A, B

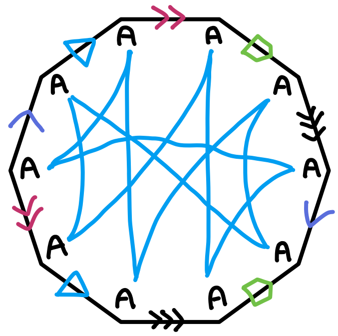
$\chi = 2 - 3 + 1 = 0$

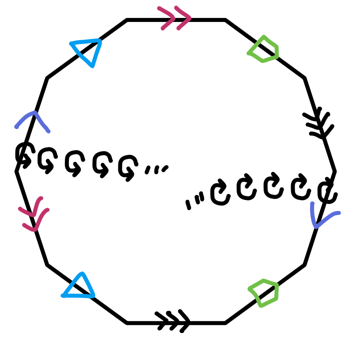
torus or Klein bottle

Is it orientable? No loops that reverse orientation. $\Rightarrow$ torus





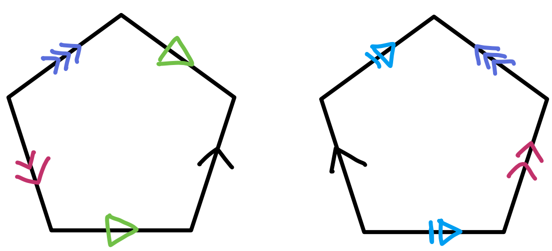


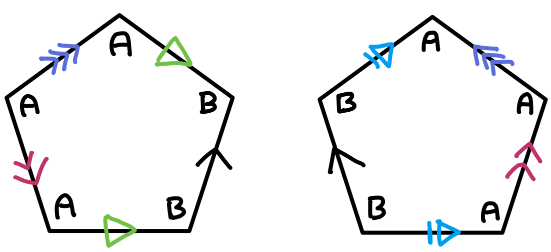


$$\chi = 1 - 5 + \underbrace{1 = -3}_{\text{This alone tells us can't be orientable}}$$

orientation-reversing loop

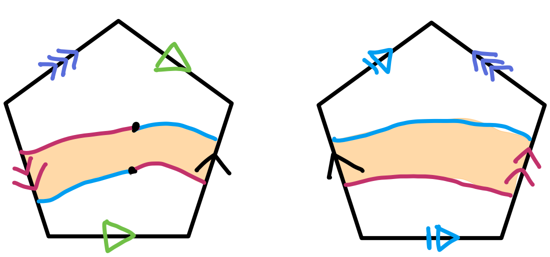
We can also glue multiple pieces:





$$\chi = 2 - 5 + 2 = -1$$

Again must be non-orientable



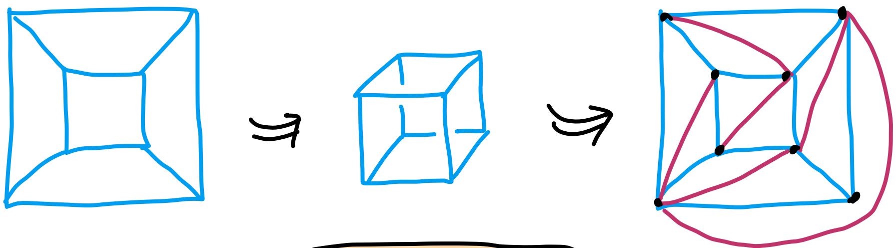
We can find Möbius strip inside surface

Same idea as ... ...

Apply Euler characteristic $\chi = v - e + f$ to planar graphs:

$$\boxed{\text{Every planar graph has a vertex of degree} \leq 5}$$

① View a planar graph as drawn on a sphere $(\chi = 2)$
Make extra cuts so graph is triangulation of the sphere



$$\begin{array}{c} \quad v \quad e \quad f \\ \chi = 8 - 18 + 12 = 2 \end{array}$$

$$\boxed{v - e + f = 2}$$

② Every triangle has 3 sides. This counts every edge twice.

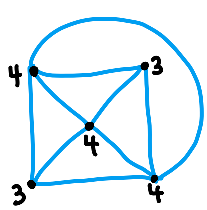
$$\boxed{2e = 3f}$$

③ Let d be the average degree of a vertex.

Each edge has two ends, so dv counts each edge twice.

$$\boxed{dv = 2e} \Rightarrow e = dv/2, \; f = dv/3$$

Check these equations in an example:



v	e	f	d
5	9	6	18/5

$$\tfrac{1}{5}(4+4+4+3+3) = \tfrac{18}{5}$$

$$\underset{5}{v} - \underset{9}{e} + \underset{6}{f} = 2 \;\; ☑$$

$$\underset{2\cdot 9}{2e} = \underset{3\cdot 6}{3f} \;\; ☑$$

$$\underset{\frac{18}{5}\cdot 5}{dv} = \underset{2\cdot 9}{2e} \;\; ☑$$

$$2 = v - e + f = v - dv/2 + dv/3$$

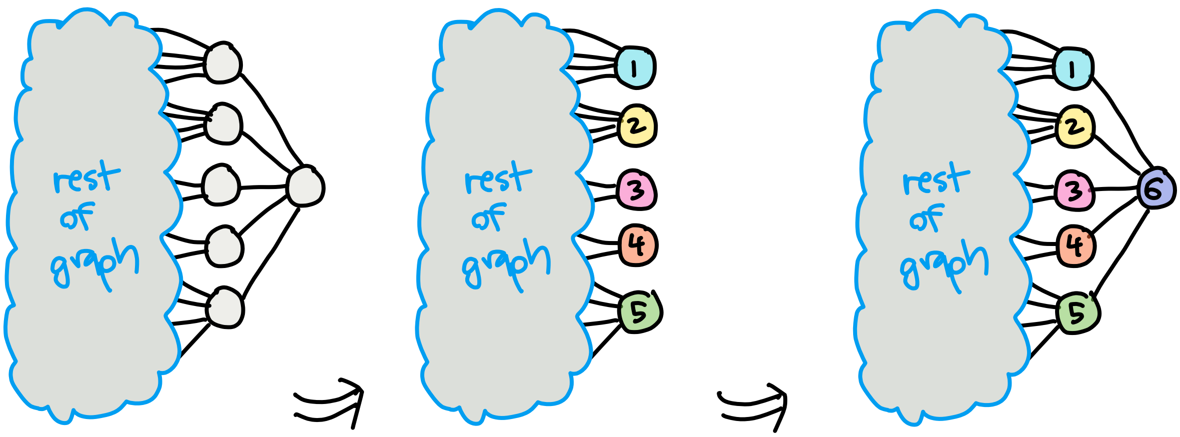
$$= v\,(\underbrace{1 - d/6}_{\text{must be positive}}) \Rightarrow \boxed{d < 6}$$

Every planar graph can be colored using 6 colors

Inductive proof / recursive algorithm:

Find a vertex of degree ≤ 5
Delete it, and 6-color the smaller graph left
Now add it back, and choose a color not used by its neighbors

{ 1 2 3 4 5 6 }

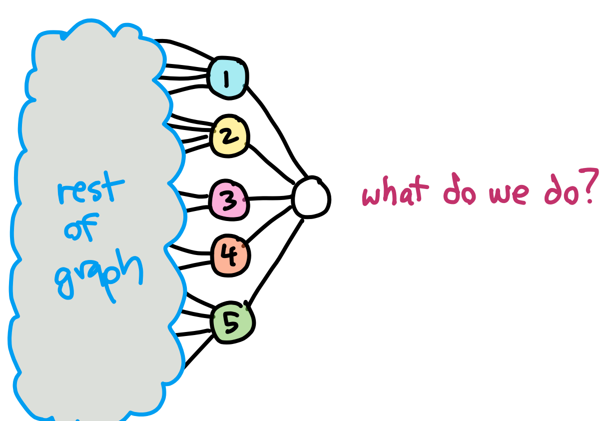


...

Every planar graph can be colored using 5 colors

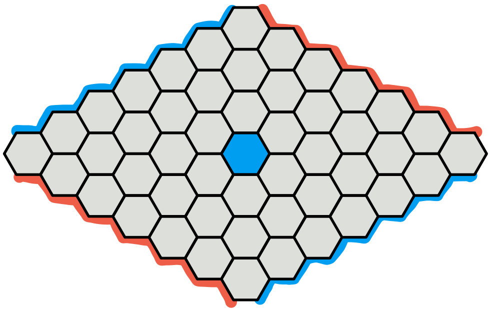
{ 1 2 3 4 5 }

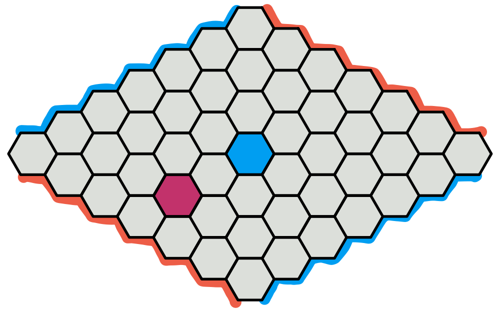
Same proof, but we need an idea to handle this case:
(we're out of colors)

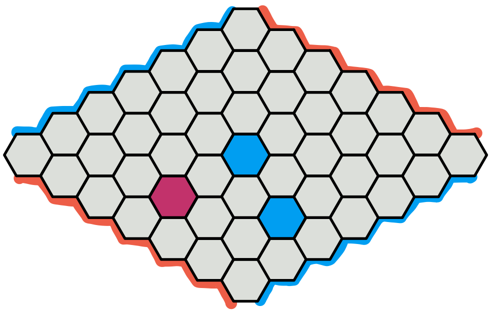


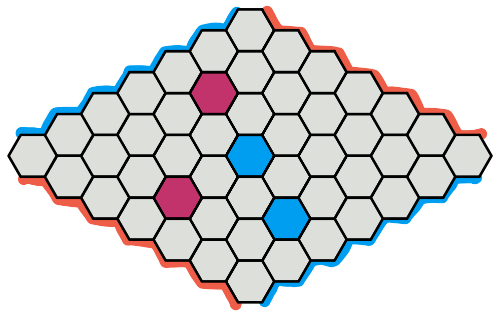
Surface topology enters again

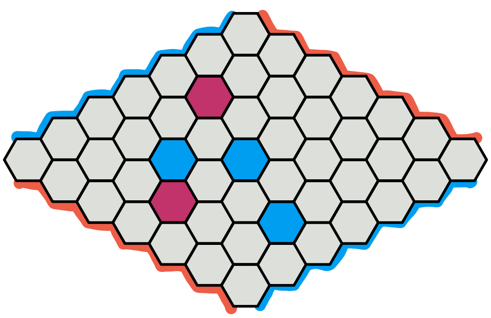
Same idea as proof some player wins Nash/Hex

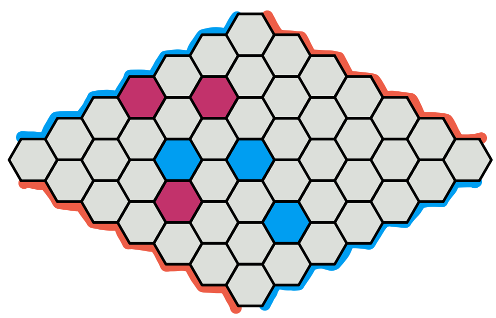


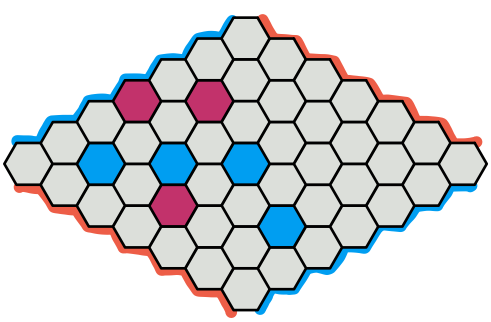




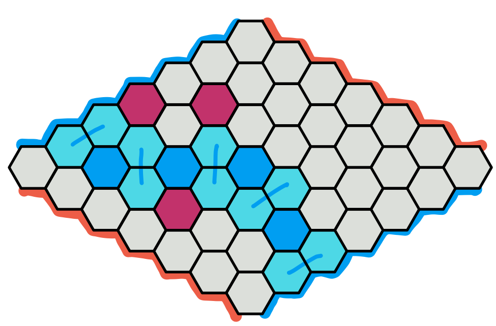








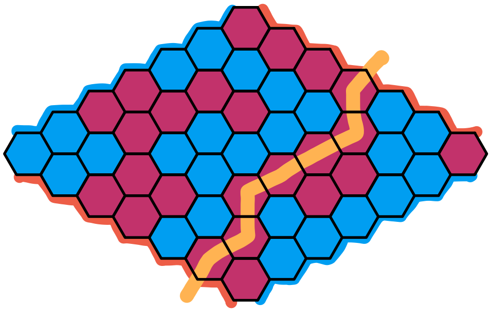
... Red resigns



Blue has a forced win

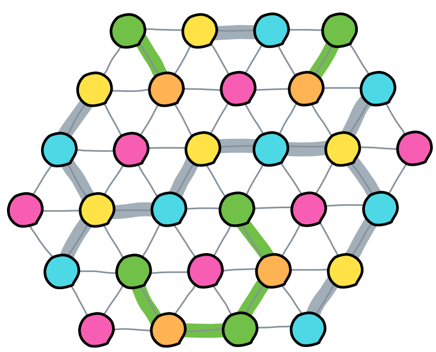
(14x14 is much harder)

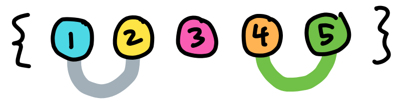
Topology: If we color every cell, Red or Blue has a win



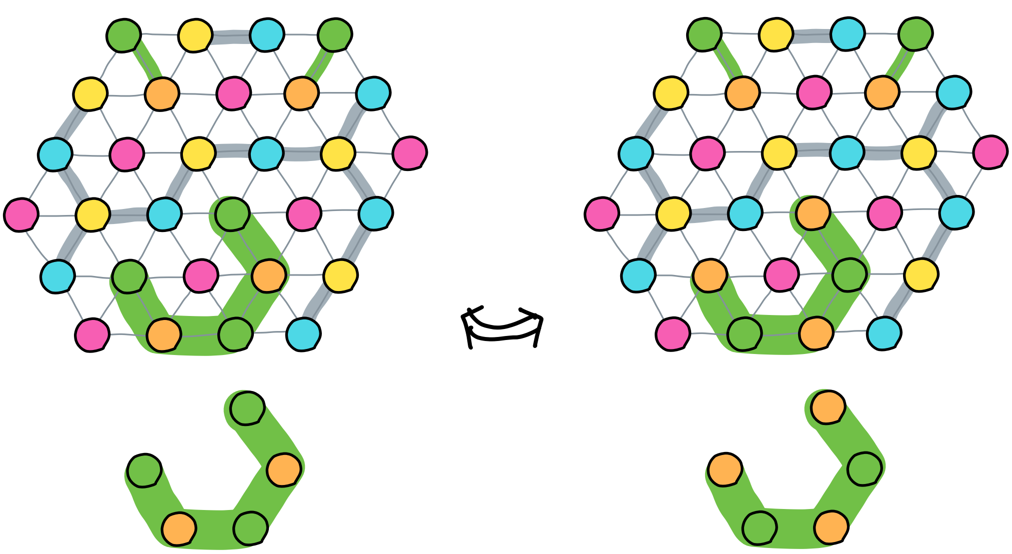
A bridge Red to Red blocks any possible bridge Blue to Blue.

Same with chains of colors in a planar graph.

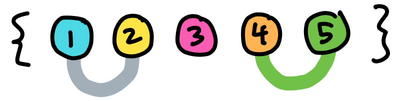


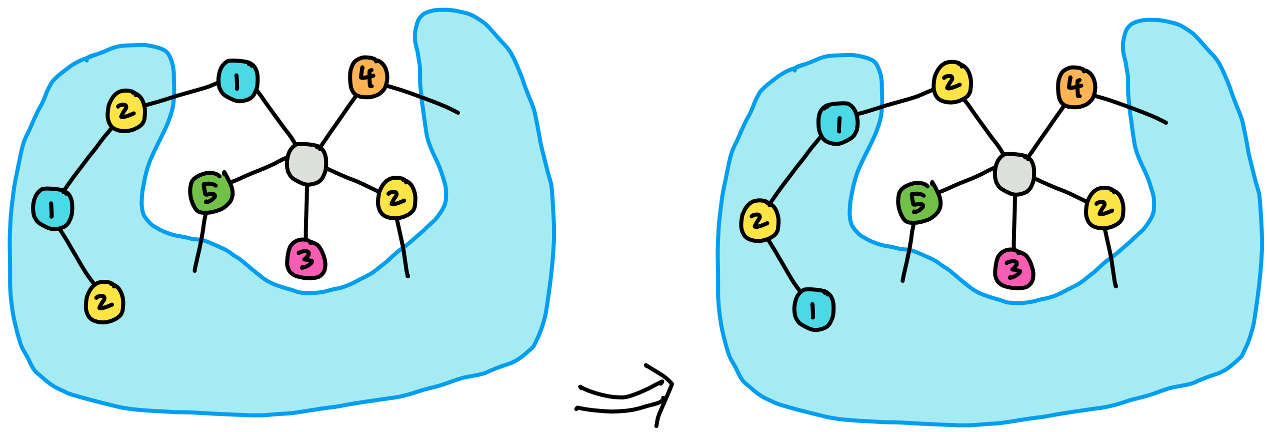


Chains block each other

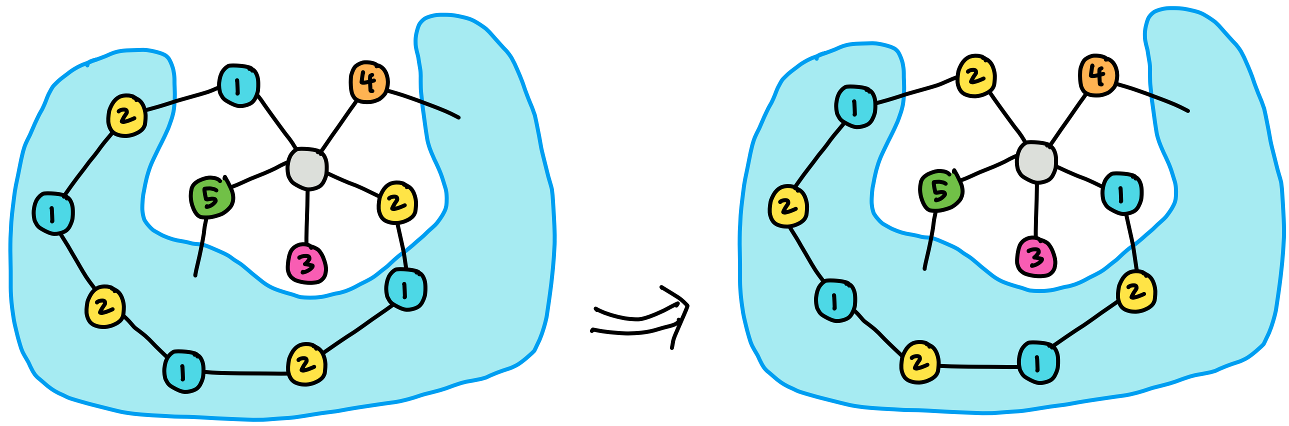


We can swap the colors in a connected chain, and no one else cares

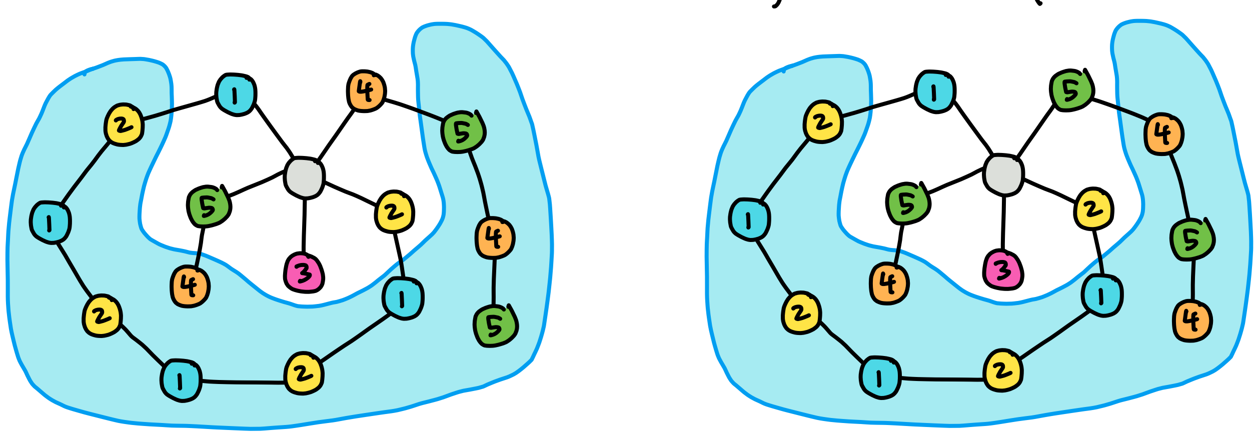




Swap 1 2 to make room



Doesn't work if they form a loop



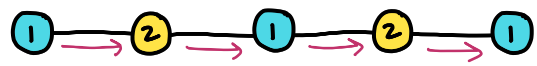
If so, loop separates 4 5 so we can toggle a 4 5 strand.

Chromatic polynomial

Let $f(n)$ = # ways to color a graph G using up to n colors

$f(n)$ is a polynomial in n

G is a vine on K vertices:

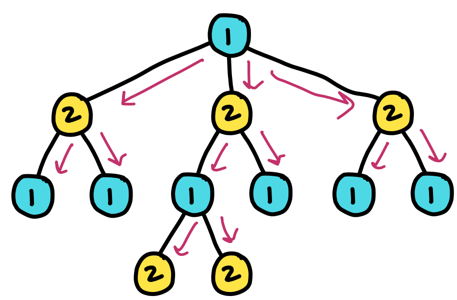


First vertex can be any color: n choices

Each following vertex can't repeat previous color: $(n-1)$ choices

$$f_K(n) = n(n-1)^{K-1}$$

G is a tree on K vertices:

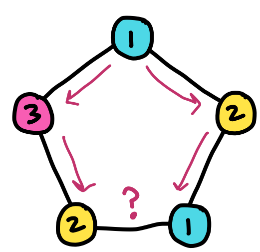


Same argument

Grow tree one vertex at a time

$$f_K(n) = n(n-1)^{K-1}$$

G is a cycle on K vertices:

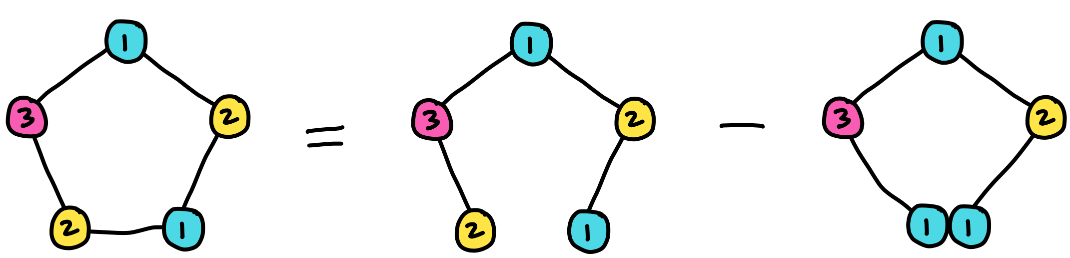


Ignore bottom edge

⇒ too many colorings

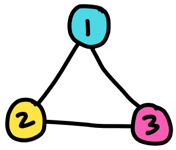
Subtract invalid colorings

= valid colorings if we collapse edge

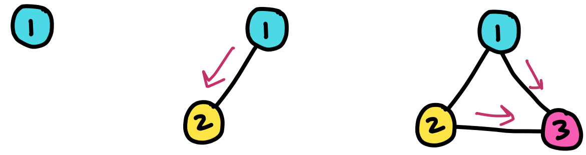


$$f_K(n) = n(n-1)^{K-1} - f_{K-1}(n)$$

Start induction:



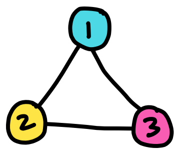
$=$



$n \quad (n-1) \quad (n-2)$

$$f_3(n) = n(n-1)(n-2)$$

Or recurse, see what happens



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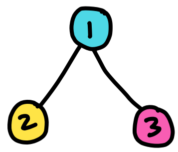

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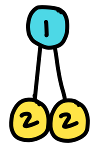
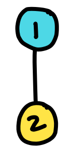
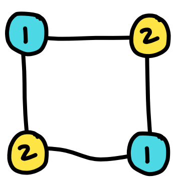


(same as )

$$f_3(n) = n(n-1)^2 - n(n-1)$$
$$= n(n-1)[(n-1)-1] = n(n-1)(n-2) \quad \boxed{\checkmark}$$



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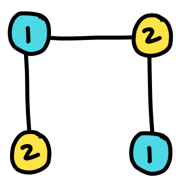

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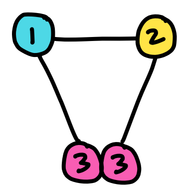
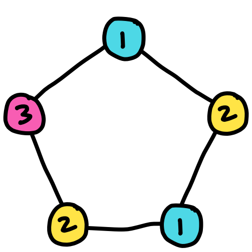
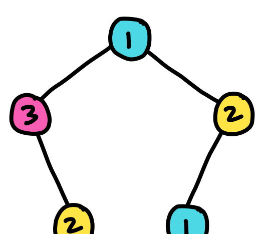


$$f_4(n) = n(n-1)^3 - n(n-1)(n-2)$$
$$= n(n-1)\left[(n-1)^2-(n-2)\right]$$



$=$



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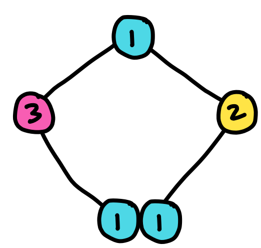
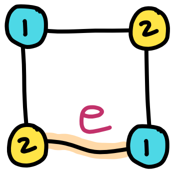


$$f_5(n) = n(n-1)^4 - n(n-1)\left[(n-1)^2-(n-2)\right]$$
$$= n(n-1)\left[(n-1)^3-(n-1)^2+\underbrace{(n-1)-1}_{\text{better way to show pattern}}\right]$$

General construction: Deletion / Contraction

Graph G, edge e



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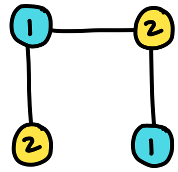

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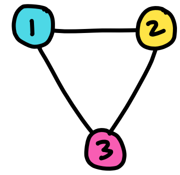
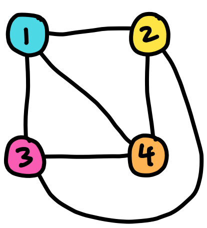


G $G\backslash e$ G/e

$$f_G(n) = f_{G\backslash e}(n) - f_{G/e}(n)$$

Example: Complete graph on 4 vertices



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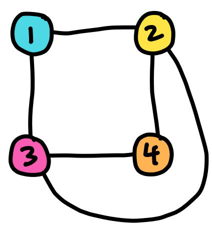

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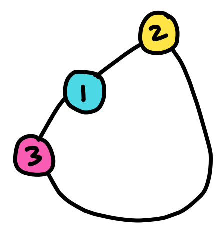

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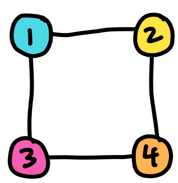

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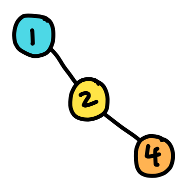

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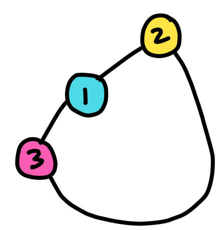
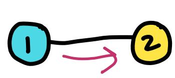
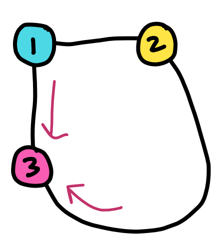


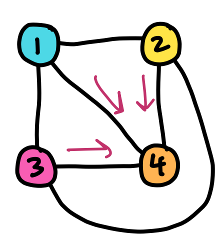
$$n(n-1)\left[(n-1)^2-(n-2)\right] - n(n-1)^2 - n(n-1)(n-2)$$

$$= n(n-1)\left[\underbrace{(n-1)^2-(n-2)-(n-1)-(n-2)}_{n^2-5n+6 \;=\; (n-2)(n-3)}\right]$$

$= n(n-1)(n-2)(n-3)$ as expected







n $(n-1)$ $(n-2)$ $(n-3)$

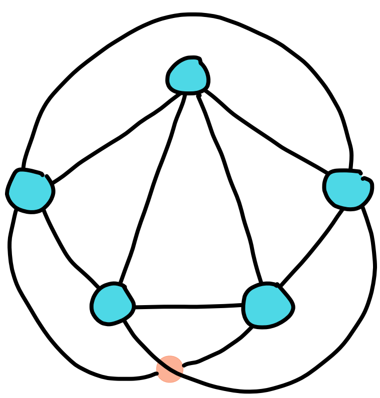
Graph Minors

A graph H is a minor of a graph G
$\Leftrightarrow$ H can be obtained from G by deletion and contraction.

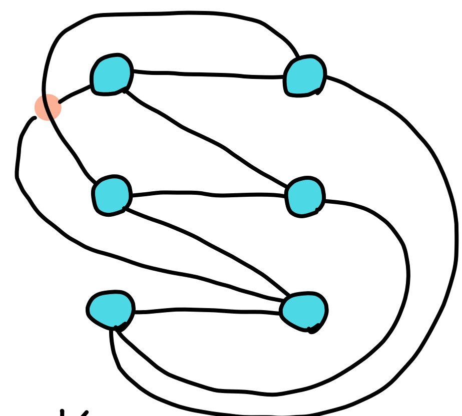
History:

Kuratowski's theorem (1930): A graph G is planar $\Leftrightarrow$
it does not contain a subdivision of K_5 or $K_{3,3}$

Wagner's theorem (1937): A graph G is planar $\Leftrightarrow$
it does not contain K_5 or $K_{3,3}$ as a minor



K_5



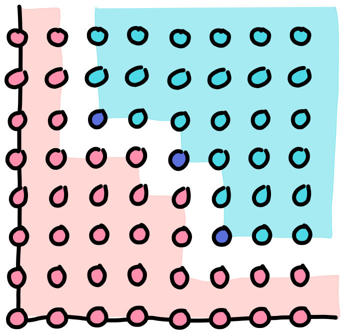
$K_{3,3}$

Interpretation: A property of graphs is minor closed
if it is preserved by taking minors.
A graph not having this property is minor minimal
if each of its minors has the property.

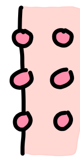
- Any minor of a planar graph is also planar.
- K_5 and $K_{3,3}$ are the minor minimal nonplanar graphs

So Kuratowski / Wagner theorem is a finiteness theorem.

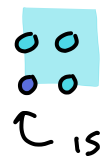
Hilbert basis theorem (1890)



Think of ↰ ↓ as deletion/contraction



is minor closed



is minor minimal

Graph minors are same idea:

Robertson-Seymour theorem (1983-2004, 500 pages)

Every minor closed property of graphs can be characterized by a finite set of forbidden minors.

⇕

Everyone has heard of Everest but K2 is a bigger deal.

⇕

Four color theorem

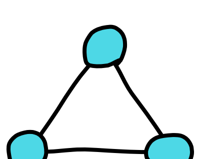
1976 Appel, Haken
1997 Robertson, Sanders, Seymour, Thomas
2005 Gonthier

Other examples:

- If G can be drawn on a $\underbrace{\text{surface}}_{\text{plane, sphere, torus,...}}$, then so can any minor

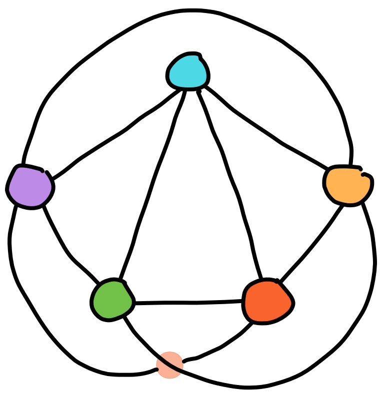
  forbidden minors complicated/unknown

- If G is a forest (union of trees ⟺ no cycles) then so is any minor.



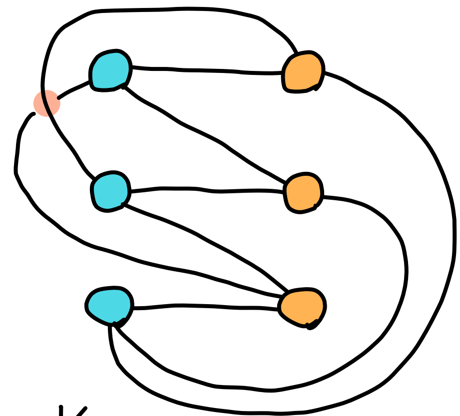
The triangle is unique forbidden minor

Four color theorem



K_5

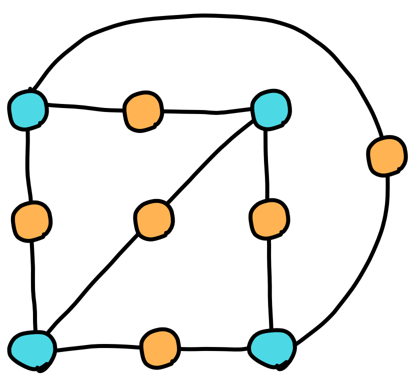
bad



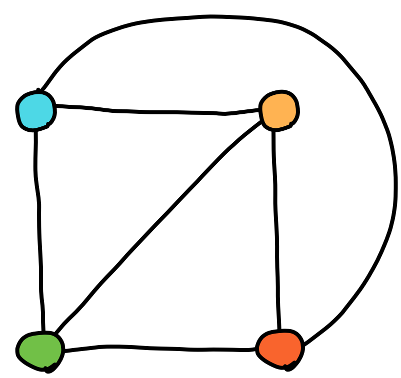
$K_{3,3}$

ok

Being n-colorable is not minor closed:



$\Longrightarrow$



2-colorable $\Longrightarrow$ 4-colorable

Hadwiger's conjecture $\Longrightarrow$ **Four color theorem**

If G is loopless and contains no K_{n+1} minor, then G is n-colorable.

Known for $1 \leq n \leq 5$

Matroid theory

Let $v_1, \ldots, v_k$ be vectors in the vector space W over the field K

Let $\mathcal{S}$ = the set of subsets $B \subset \{v_1, \ldots, v_k\}$ that are linearly independent

How can we spot a valid $\mathcal{S}$?

Easily seen rules not quite enough. Exchange axiom:

If A, B independent sets and $|A| > |B|$ then $\exists v \in A \setminus B$ so $B \cup \{v\}$ is independent

An abstract matroid (data for $\mathcal{S}$ satisfying these rules) is representable over the field K
$\Leftrightarrow$ we can find $v_1, \ldots, v_k$ vectors in the vector space W over K, with this structure $\mathcal{S}$

- Can a graph G be drawn on a surface S?
  Forbidden minors, depending on S

- Can a matroid $\mathcal{S}$ be represented over a field K?
  Forbidden minors, depending on K.

Definitions are more technical, but theory is entirely parallel.

Graphs are actually a special case of matroids:
A set of edges is independent $\Leftrightarrow$ they don't contain a cycle.

There is a chromatic polynomial for matroids.