Exam 1

Combinatorics, Dave Bayer, February 11-14, 2021

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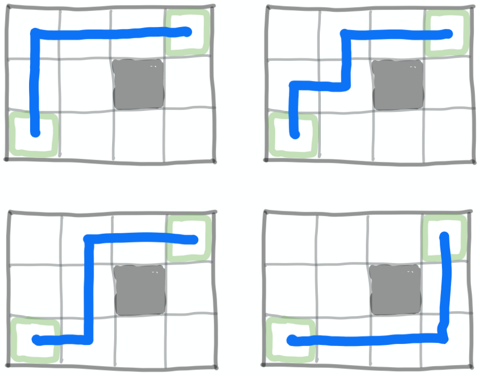
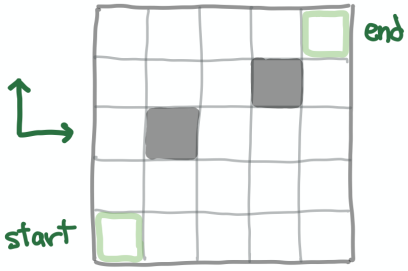
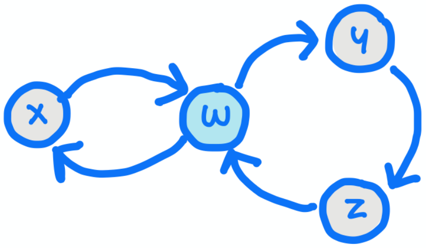
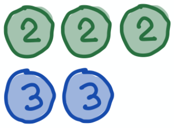


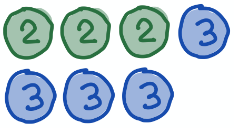


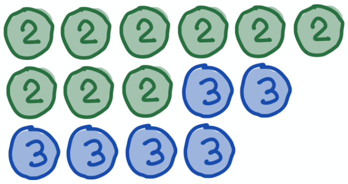
[2] Let $f(n)$ be the number of n step paths from w to itself on the directed graph below. What is $f(12)$?



[3] Let $f(n)$ be the number of ways of making change for n cents, using 2 cent and 3 cent coins. As shown below, $f(6) = 2$, $f(9) = 2$, and $f(12) = 3$. What is $f(18)$?







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1	2
3	4

1	2	3
4	5	6

1	2	4
3	5	6

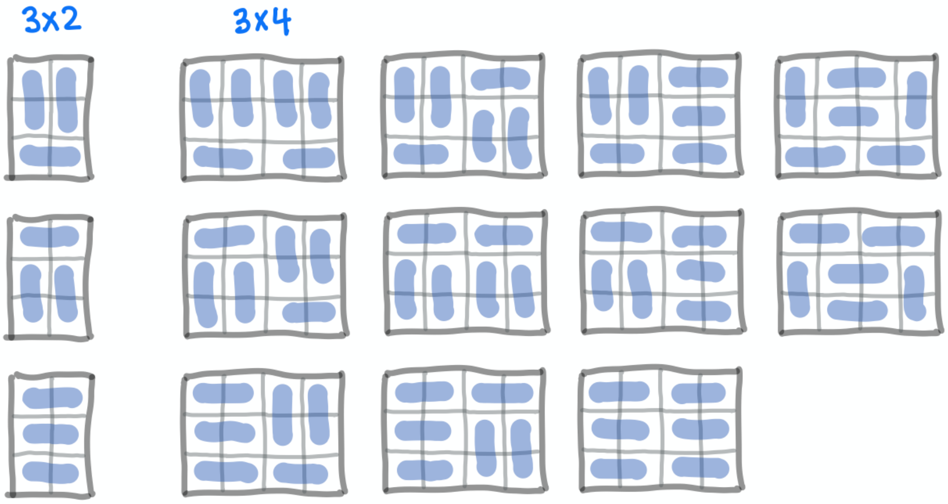
1	2	5
3	4	6

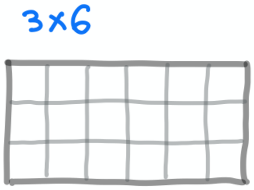
1	3
2	4

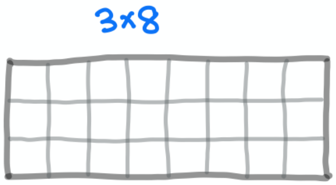
1	3	4
2	5	6

1	3	5
2	4	6

[5] Let $f(n)$ be the number of ways of arranging 1×2 bricks in a $3 \times 2n$ grid. As shown below, $f(1) = 3$ and $f(2) = 11$. Find $f(3)$ and $f(4)$. What can you say about $f(n)$?







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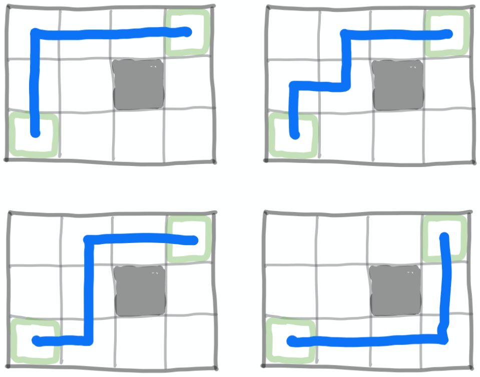

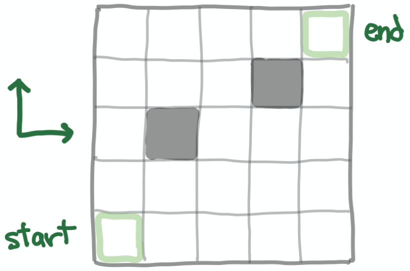

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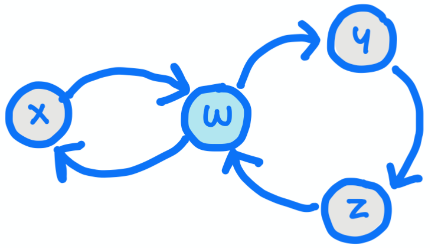

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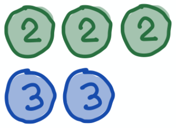

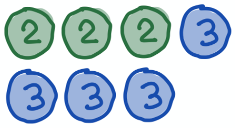

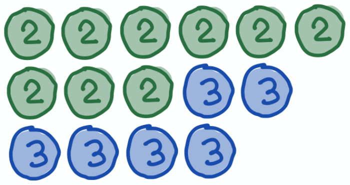

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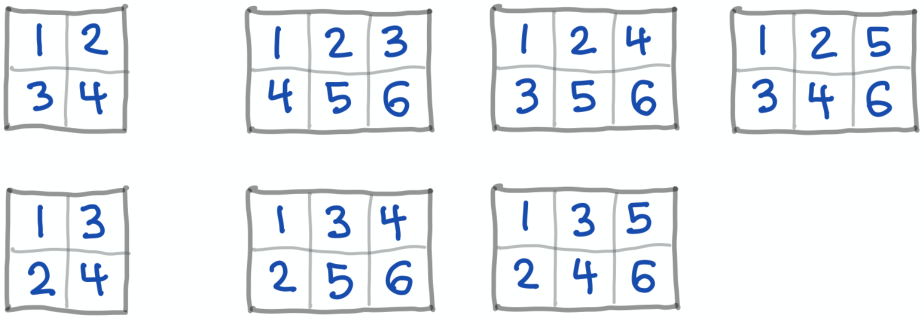

1	2
3	4

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3
2	4

1	3	4
2	5	6

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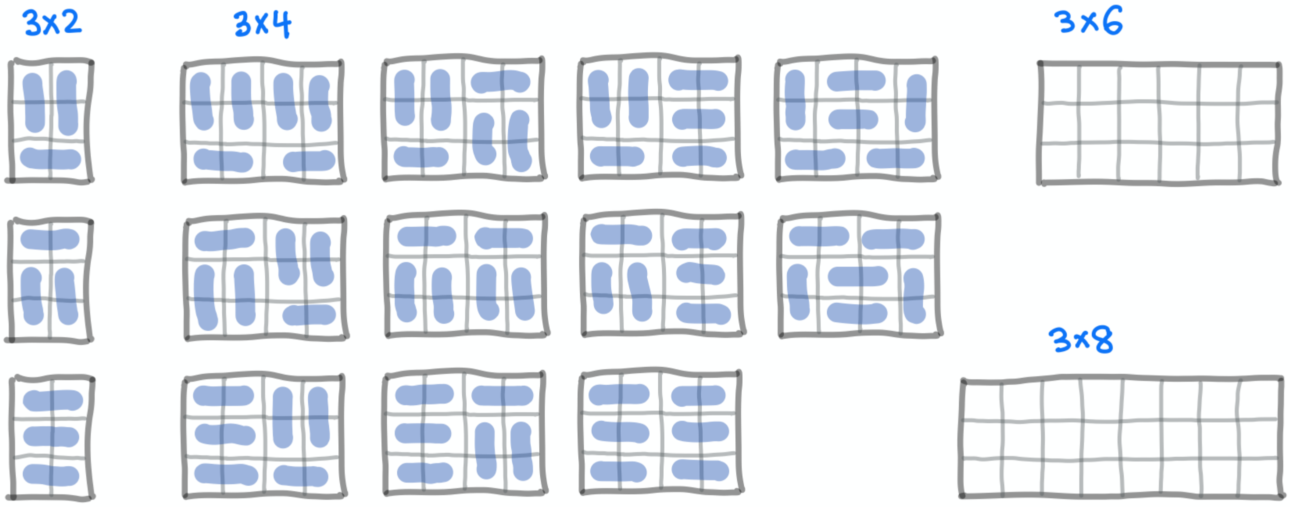

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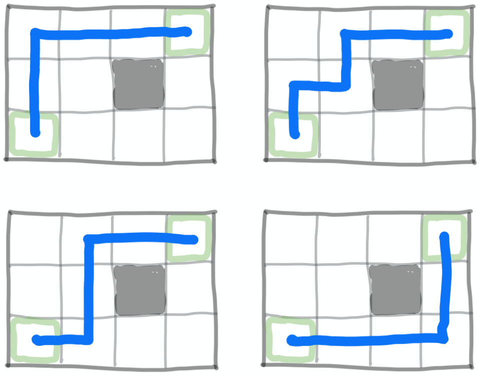
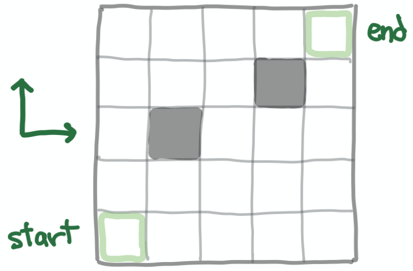
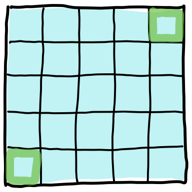




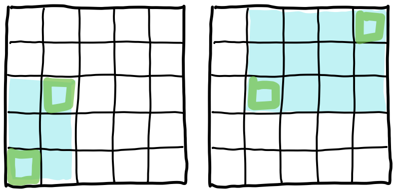
1	2	6	6	18
1	1	4	▨	12
1	▨	3	7	12
1	2	3	4	5
1	1	1	1	1

$=$



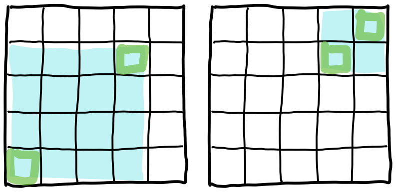
$$\binom{8}{4} = \frac{\cancel{8}\cdot 7\cdot \overset{2}{\cancel{6}}\cdot 5}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 70$$

$-$



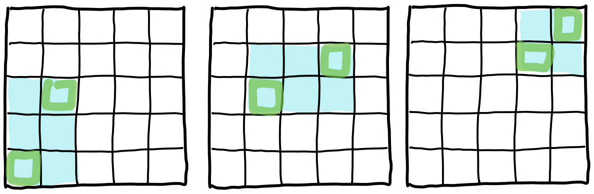
$$\binom{3}{1}\binom{5}{3} = 3\cdot 10 = 30$$

$-$



$$\binom{6}{3}\binom{2}{1} = 20\cdot 2 = 40$$

$+$



$$\binom{3}{1}\binom{3}{2}\binom{2}{1} = 18$$

$$70 - 30 - 40 + 18 = 18$$ ☑

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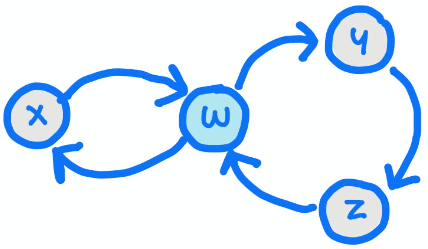
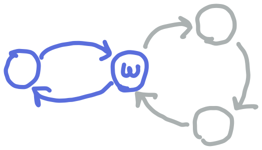
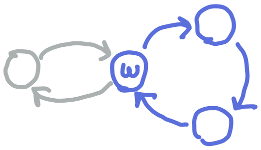




2 step loop 🟦🟦

$\boxed{f(12) = 12}$



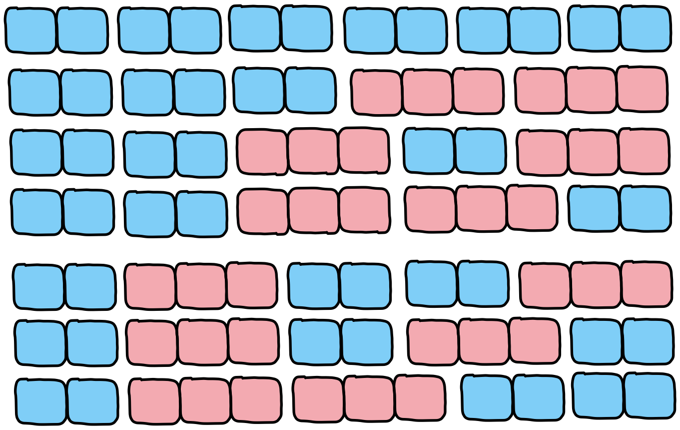
3 step loop 🟥🟥🟥

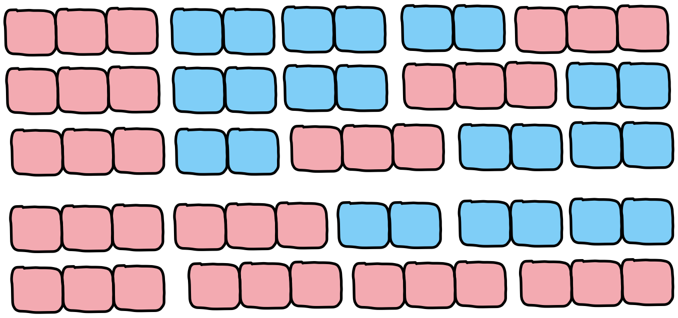
$$f(n) = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ f(n-2) + f(n-3), & n>0 \end{cases}$$

f(n) → 🟦🟦... f(n-2)

f(n) → 🟥🟥🟥... f(n-3)

n	0	1	2	3	4	5	6	7	8	9	10	11	12
f(n-2)			1		1	1	1	2	2	3	4	5	7
f(n-3)				1		1	1	1	2	2	3	4	5
f(n)	1		1	1	1	2	2	3	4	5	7	9	12





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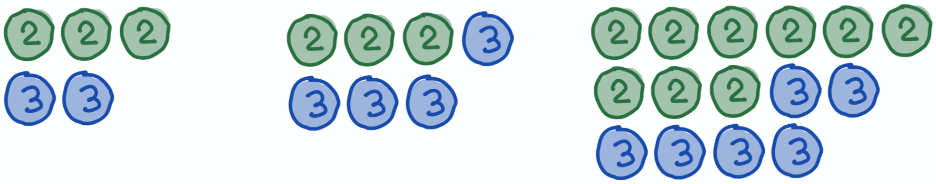

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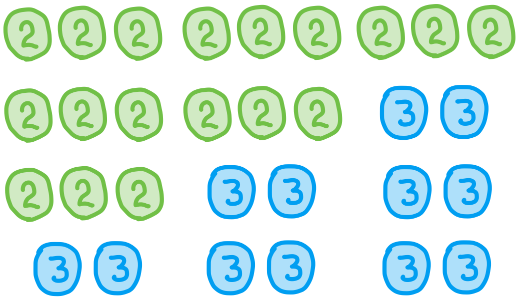
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1		1		1		1		1		1		1		1		1		1
3	1		1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4



$$\boxed{f(18) = 4}$$

$$g(t) = 1 +$$

$$g(t) = (1 + \underbrace{t^2 + t^4 + t^6 + \dots}_{{}^{1}/_{1-t^2}})(1 + \underbrace{t^3 + t^6 + t^9 + \dots}_{{}^{1}/_{1-t^3}})$$

$$\boxed{g(t) = \frac{1}{1-t^2-t^3+t^5}}$$

$$(1-t^2)(1-t^3) = 1 - t^2 - t^3 + t^5$$

check: $(1-t^2-t^3+t^5)\,g(t) = 1$

$g(t)$

1	1		1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4
$-t^2$			1		1	1	1	1	2	1	2	2	2	2	3	2	3	3	3
$-t^3$				1		1	1	1	1	2	1	2	2	2	2	3	2	3	3
$+t^5$						1		1	1	1	1	2	1	2	2	2	2	3	2
	1																		

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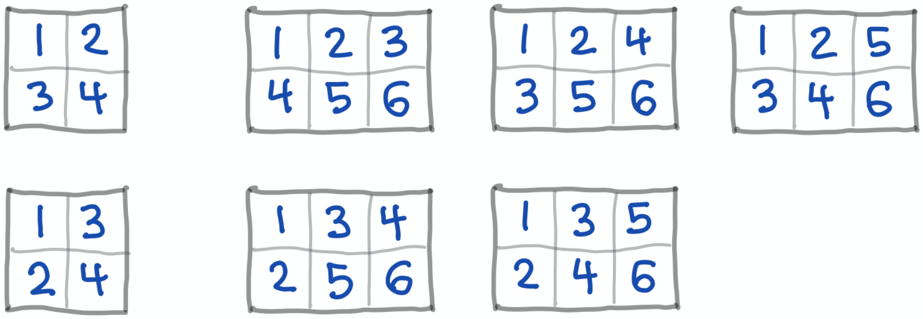
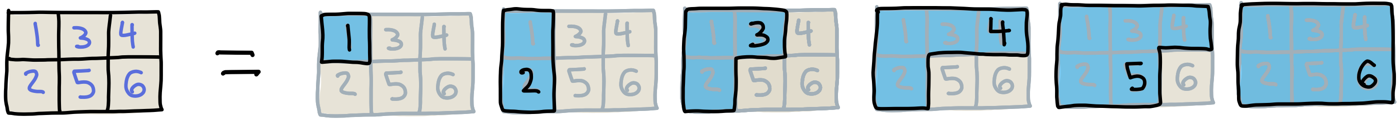
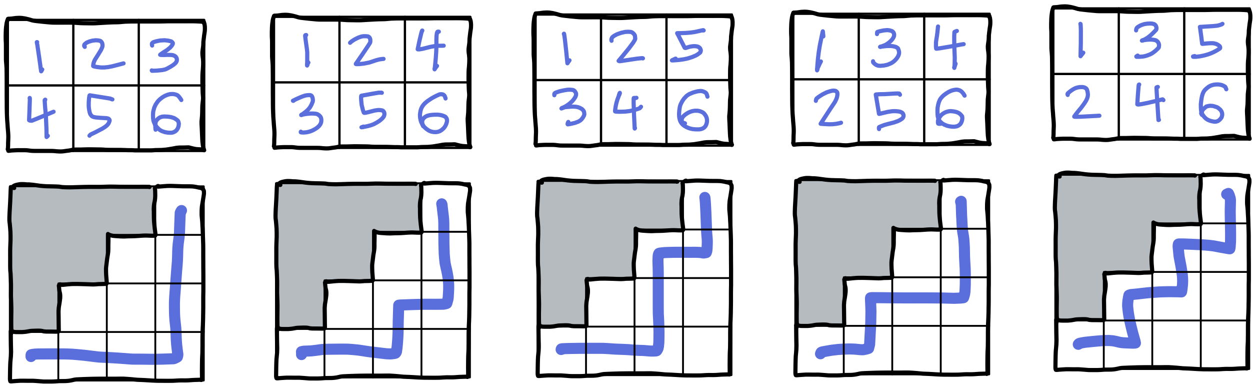


We can think of a Young tableau as instructions for growing a staircase:



The second row can't get ahead of the first row.
Think of first row as →, second row as ↑
This is same problem as not exceeding diagonal in a lattice walk.



We recognize the Catalan numbers.

n	1	2	3	4	5	6
f(n)	1	2	5	14	42	132

$$\boxed{f(5) = 42}$$

$$\boxed{f(n) = C_n}$$

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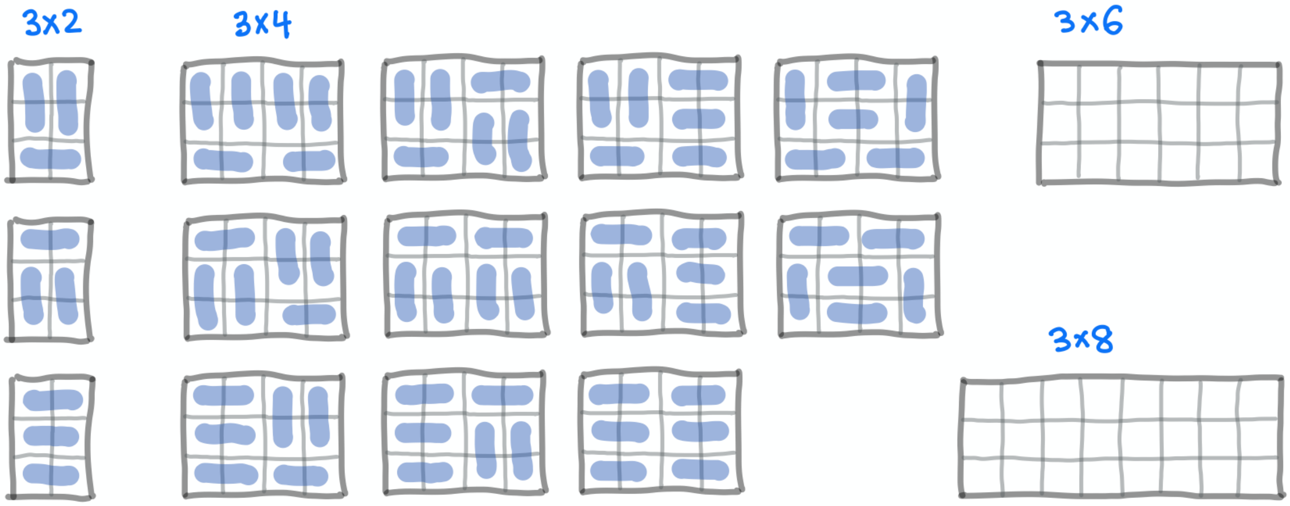

$\boxed{f(3) = 41}$ $\boxed{f(4) = 153}$

See class notes, February 16

Exam 2

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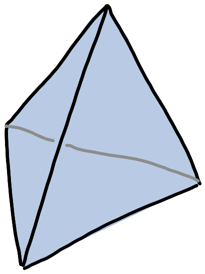
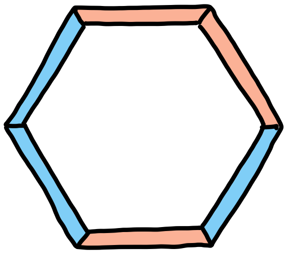
To receive full credit for correct answers, please show all work.

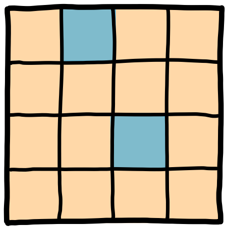
[1] How many ways can we choose three edges of a regular tetrahedron, up to rotational symmetry? Confirm your answer by finding all patterns up to symmetry.



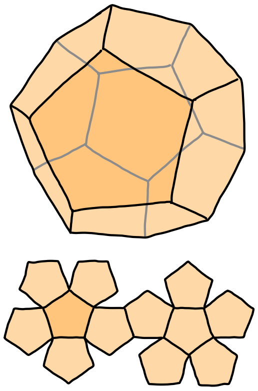
[2] How many ways can we k-color the six sides of a regular hexagon, up to rotational and flip symmetries? Confirm your answer for $k = 2$, by finding all patterns up to symmetry.



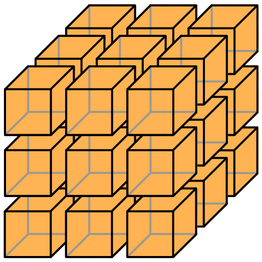
[3] How many ways can we choose two squares of a 4×4 board, up to rotational and flip symmetries? Confirm your answer by finding all patterns up to symmetry.



[4] How many ways can we choose 2 or 3 faces of a regular dodecahedron up to rotational symmetry? Confirm your answers by finding all patterns up to symmetry.



[5] How many ways can we choose two cubes from a $3 \times 3 \times 3$ array of 27 cubes, up to rotational symmetry? (This is not a *Rubik's Cube*. The symmetries are the 24 rotations we have studied of a solid cube.)



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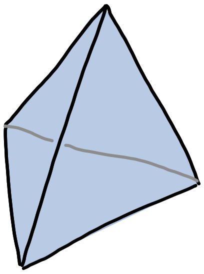

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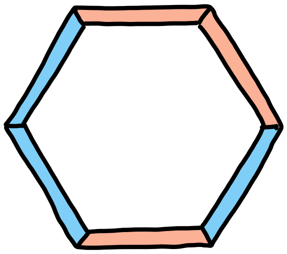

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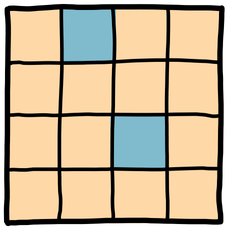

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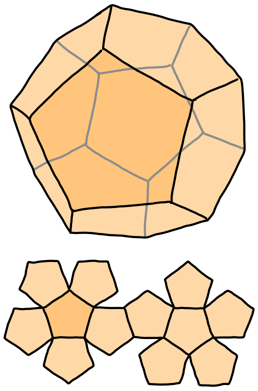

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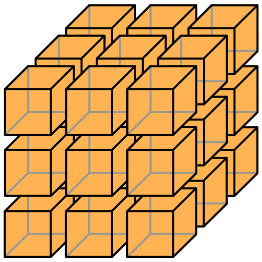

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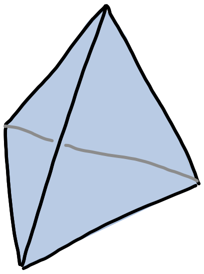
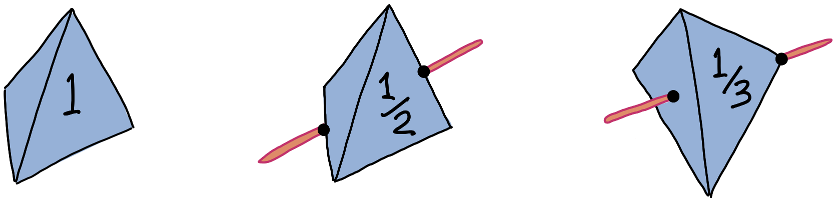
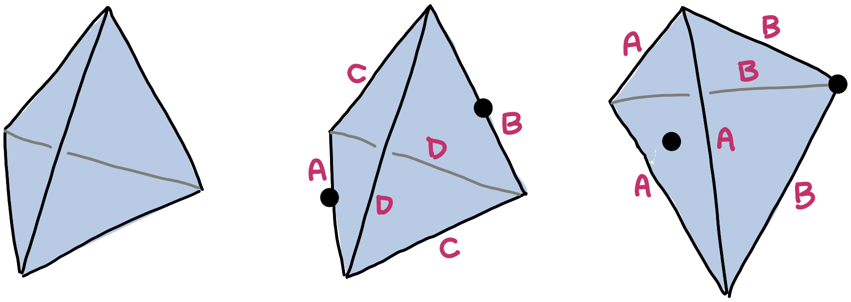




1 3 8



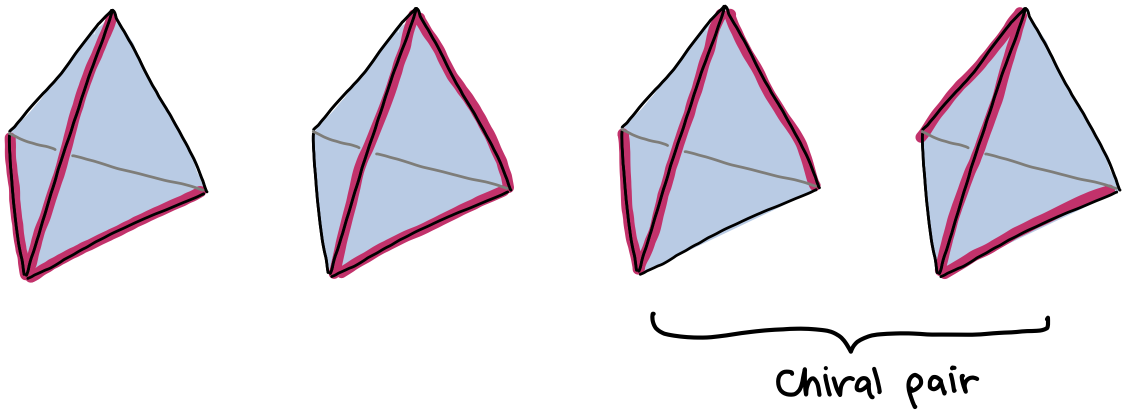
A B CC DD

AAA BBB

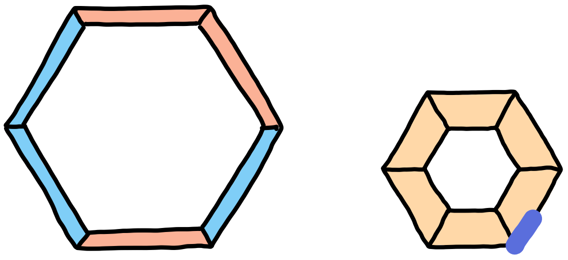
$\binom{6}{3}=20$ $2\cdot 2=4$ 2

$$\frac{1}{12}(20+\underset{12}{3\cdot4}+\underset{16}{8\cdot2}) = 48/12 = \boxed{4}$$

Check:



[2] How many ways can we k-color the six sides of a regular hexagon, up to rotational and flip symmetries? Confirm your answer for $k = 2$, by finding all patterns up to symmetry.

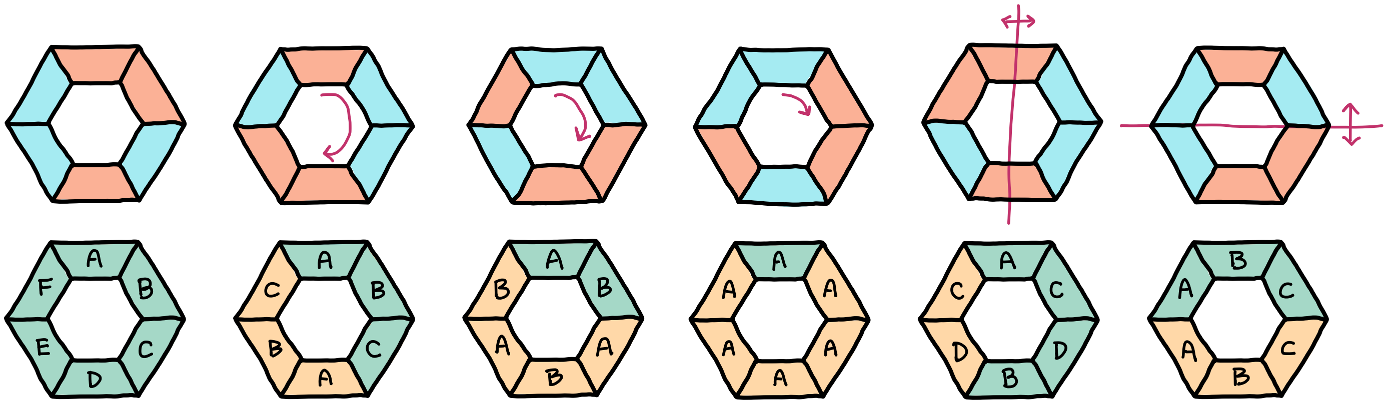


$|G| = 6$ vertices · 2 edges = 12 cases

6 rotations

6 flips

possibilities

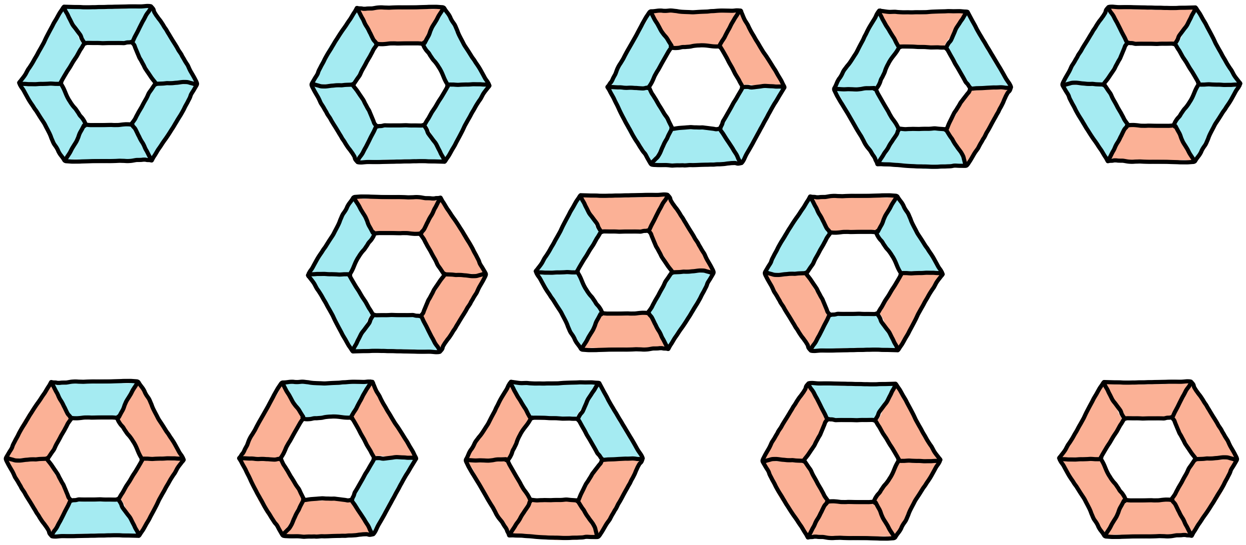


1	1	2	2	3	3
Identity	½ turn	⅓ turns	⅙ turns	side flips	vertex flips
k^6	k^3	k^2	k	k^4	k^3

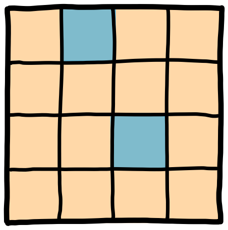
$$\boxed{\frac{1}{12}(k^6+3k^4+4k^3+2k^2+2k)}$$

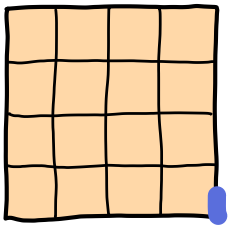
$$k=2: \quad \frac{1}{12}(64+\underset{48}{3\cdot16}+\underset{32}{4\cdot8}+\underset{8}{2\cdot4}+\underset{4}{2\cdot2}) = 156/12 = \boxed{13}$$

Check:

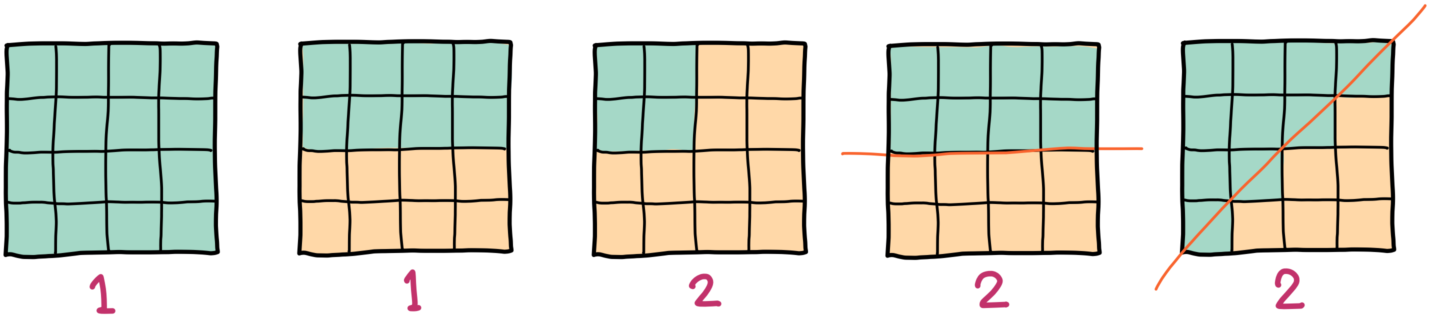


[3] How many ways can we choose two squares of a 4 × 4 board, up to rotational and flip symmetries? Confirm your answer by finding all patterns up to symmetry.





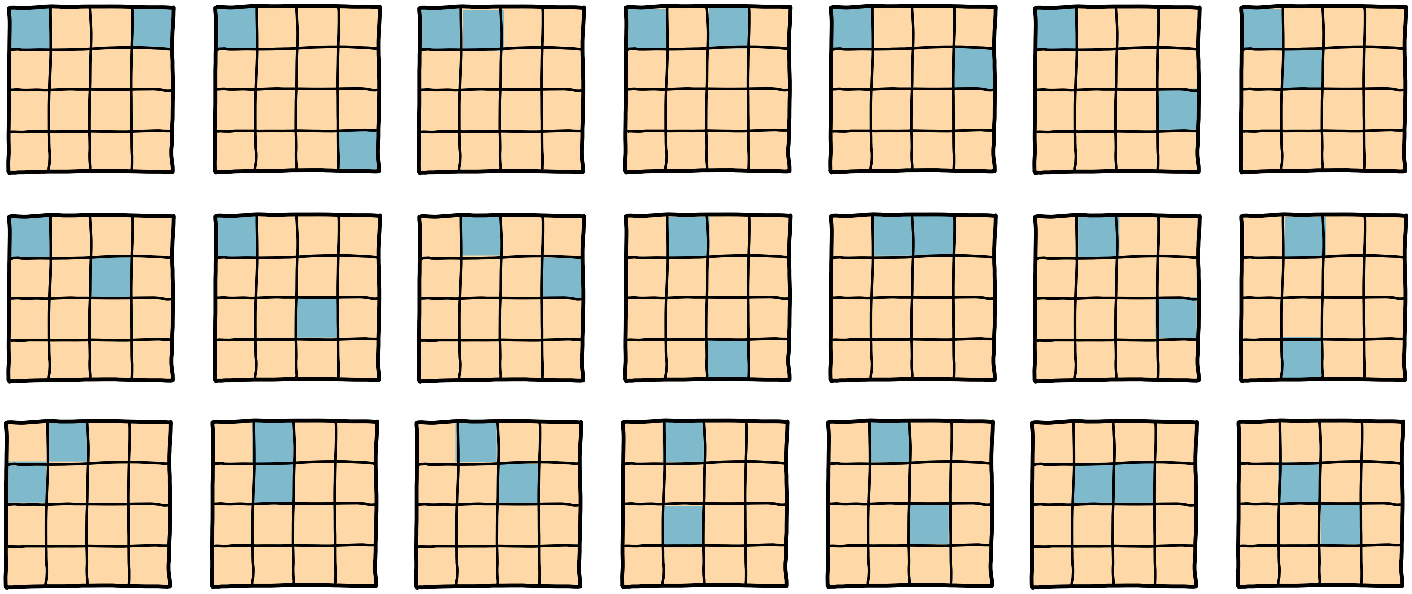
$|G| = 4 \text{ corners} \cdot 2 \text{ edges} = 8$



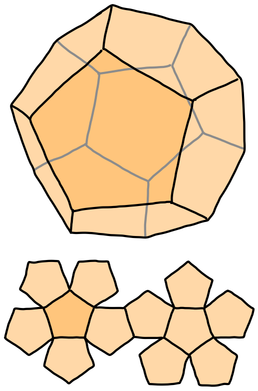
1	1	2	2	2
Identity	½ turn	¼ turns	side flips	vertex flips
$\binom{16}{2} = 8\cdot15 = 120$	8	0	8	$\binom{4}{2}+6=12$

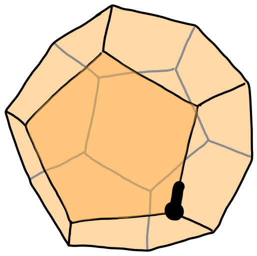
$$\frac{1}{8}\left(120+8+\underbrace{2\cdot8}_{16}+\underbrace{2\cdot12}_{24}\right) = 168/8 = \boxed{21}$$

Check:



[4] How many ways can we choose 2 or 3 faces of a regular dodecahedron up to rotational symmetry? Confirm your answers by finding all patterns up to symmetry.





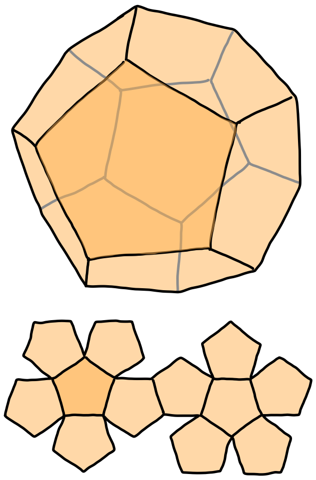
12 pentagon faces

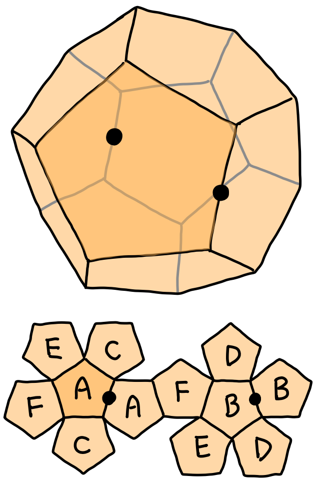
30 edges $5\cdot12/2$

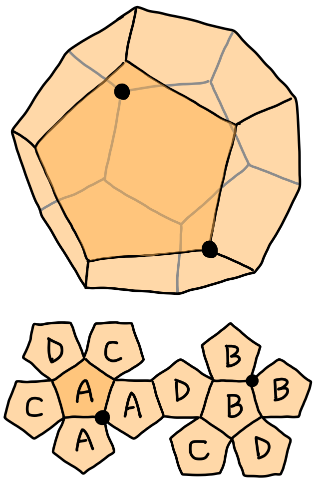
20 vertices $5\cdot12/3$

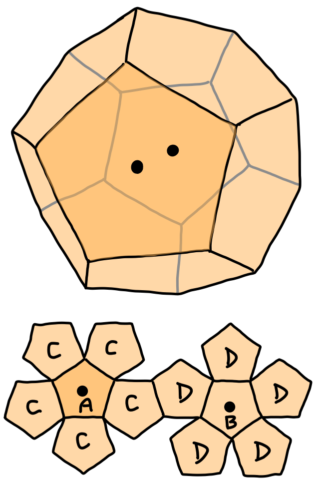
Choose vertex then edge

$|G| = 20\cdot3 = 60$







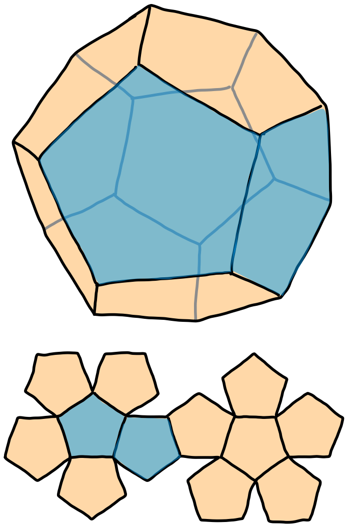


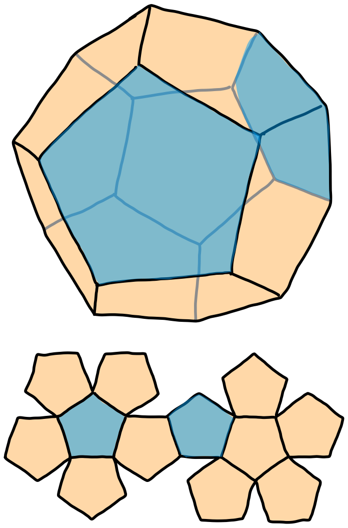
	1	15	20	24
	Identity	Edge ½ turns	Vertex ⅓ turns	Face turns
	$\frac{{}^{6}\cancel{12}\cdot 11}{\cancel{2}\cdot 1}$ $\frac{{}^{2}\cancel{12}\cdot 11\cdot 10}{\cancel{3}\cdot\cancel{2}\cdot 1}$	AA BB CC DD EE FF	AAA BBB CCC DDD	A CCCCC B DDDDD
k=2	$\binom{12}{2}=66$	6	0	1
k=3	$\binom{12}{3}=220$	0	4	0

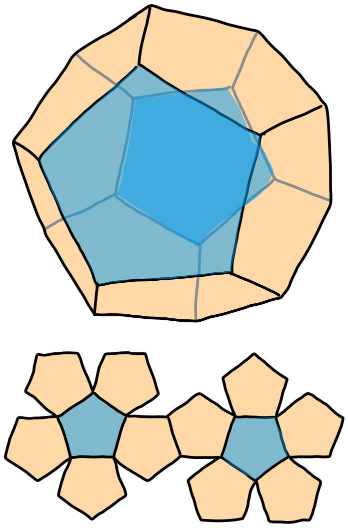
k=2 $\frac{1}{60}(66+15\cdot6+20\cdot0+24\cdot1) = 180/60 = \boxed{3}$

k=3 $\frac{1}{60}(220+15\cdot0+20\cdot4+24\cdot0) = 300/60 = \boxed{5}$

k=2 $\boxed{3}$







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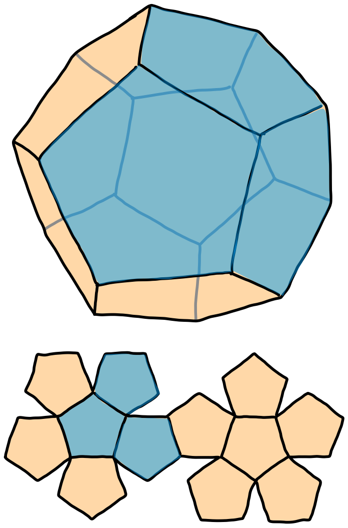
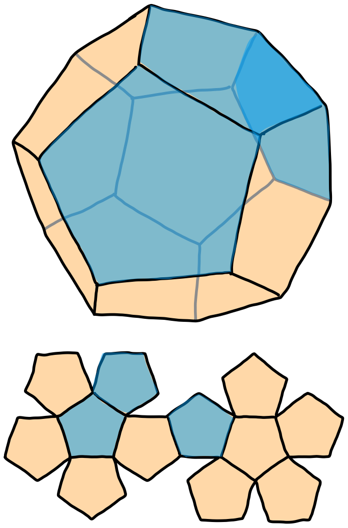
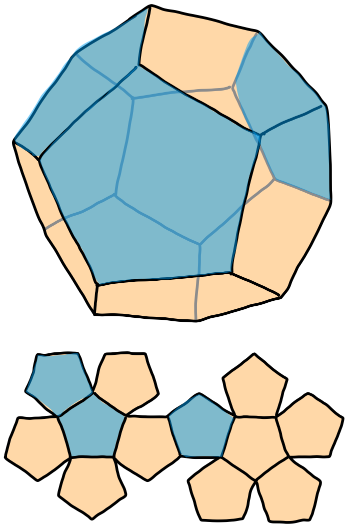
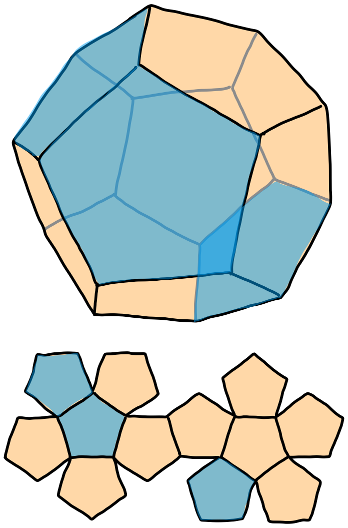
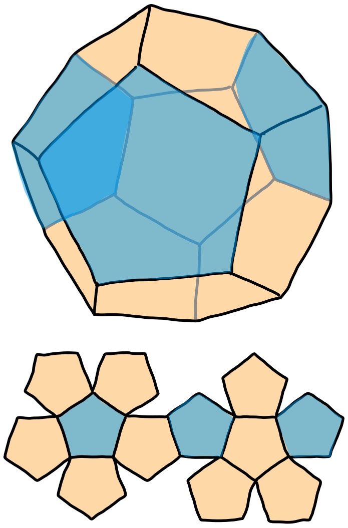
k=3 $\boxed{5}$



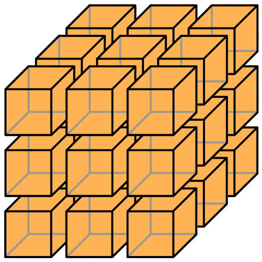


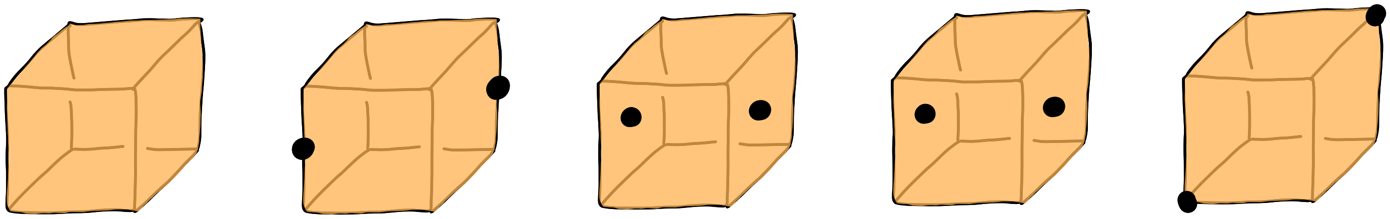


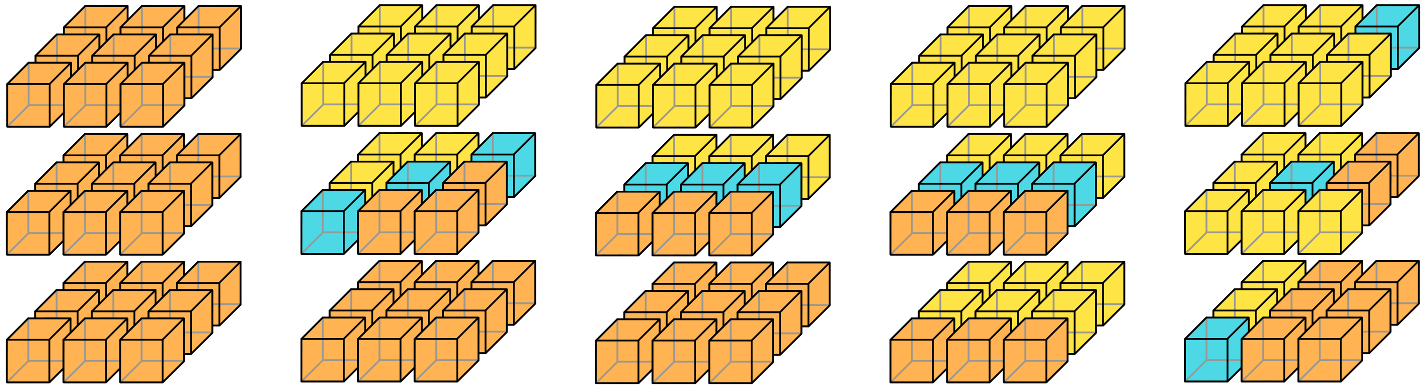




[5] How many ways can we choose two cubes from a $3 \times 3 \times 3$ array of 27 cubes, up to rotational symmetry? (This is not a *Rubik's Cube*. The symmetries are the 24 rotations we have studied of a solid cube.)



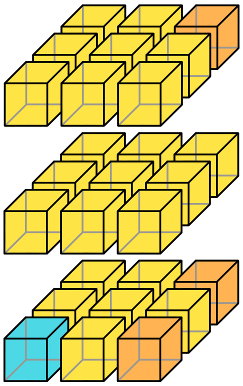




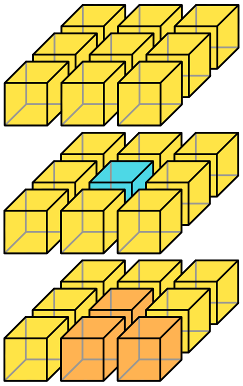
identity 1	½ turn	½ turn	¼ turn either way	⅓ turn either way
1	6	3	6	8
$\binom{27}{2} = \frac{27\cdot 26}{2\cdot 1}$ $= 351$	3 cubes on axis 12 pairs $\binom{3}{2}+12 = 15$	3 cubes on axis 12 pairs $\binom{3}{2}+12 = 15$	3 cubes on axis 6 quads $\binom{3}{2}=3$	3 cubes on axis 8 triplets $\binom{3}{2}=3$

$\frac{1}{24}(351+9\cdot15+14\cdot3) = 528/24 = \boxed{22}$ ways to pick two cubes

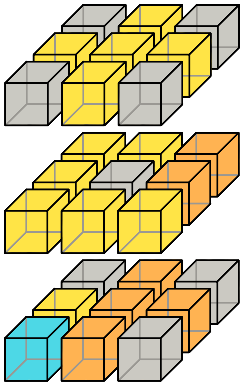
Check:



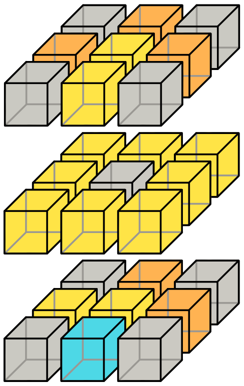
3 ways to
choose two
corners



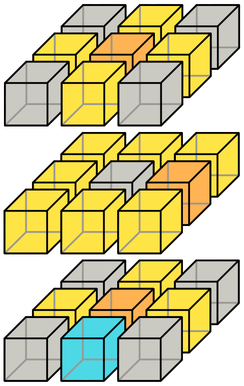
3 ways to
choose middle



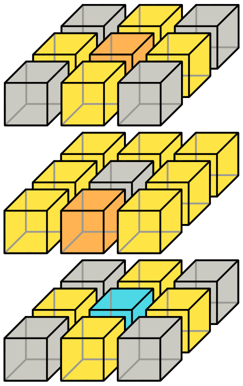
6 ways left to
choose one corner



5 ways to
choose two
edges
(as we saw before)



3 ways to
choose one edge,
one face



2 ways to
choose
two faces

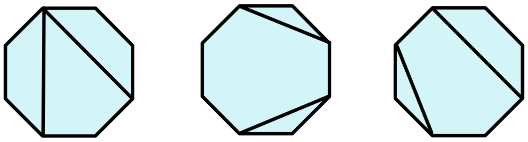
$3+7+2+5+3+2 = \boxed{22}$ ☑

Final Exam

Combinatorics, Dave Bayer, April 20-23, 2021

To receive full credit for correct answers, please show all work.

[1] How many ways can we dissect an octagon using 2 cuts? Provide a check of your answer.
(You may solve the problem two different ways, or classify the possibilities, or draw every possibility.)



[2] For each of the following Young tableaux, find the dissection of an n-gon given by Stanley's correspondence.

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4	5	6
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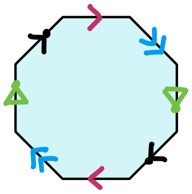
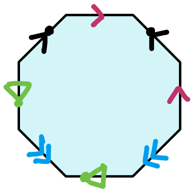
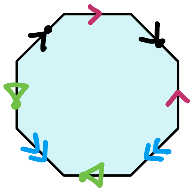
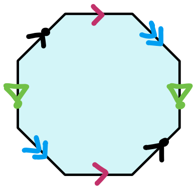
7		

[3] Identify each of the following surfaces from their gluing diagrams, computing their Euler characteristic and deciding whether or not they are orientable. Which two surfaces are homeomorphic (topologically equivalent)?

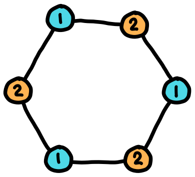


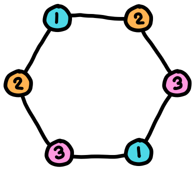


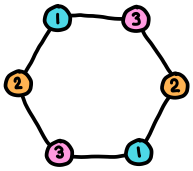




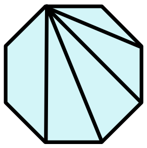
[4] How many ways can we properly color the vertices of a hexagon using n colors, up to rotational symmetry? Confirm your answer by drawing each of the possibilities for $n = 3$.
(For a proper coloring, adjacent vertices have distinct colors. You need not use every color.)

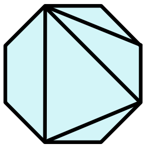


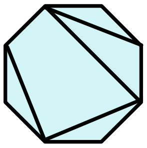




[5] How many ways can we dissect an octagon using 4 cuts, up to dihedral (rotations and flips) symmetry? Confirm your answer by drawing each of the possibilities. Which patterns are not chiral?







Final Exam

Combinatorics, Dave Bayer, April 20-23, 2021

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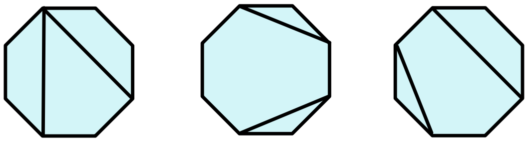

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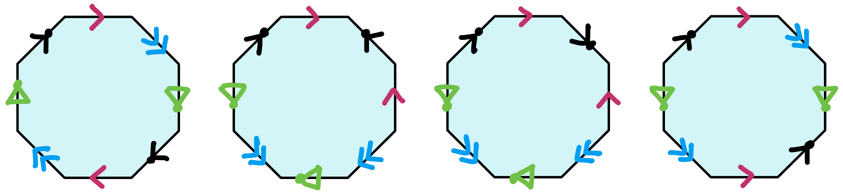

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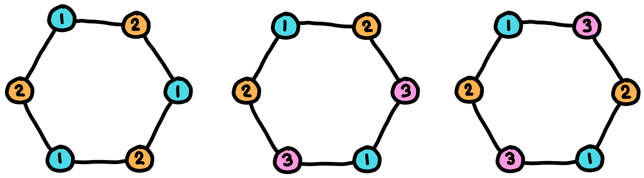

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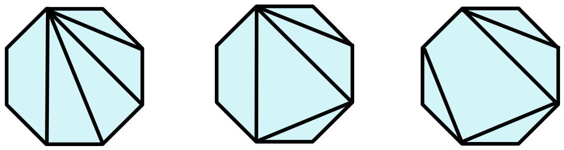

Final Exam

Combinatorics, Dave Bayer, April 20-23, 2021

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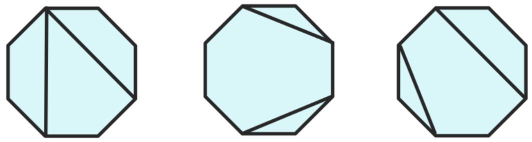


8 sides

$\binom{8}{2} = \frac{8\cdot7}{2\cdot1} = 28$ pairs of vertices

$\Rightarrow$ $28-8 = 20$ interior cuts

$\binom{20}{2} = \frac{20\cdot19}{2\cdot1} = 190$ pairs of cuts

$\binom{8}{4} = \frac{\overset{2}{\cancel{8}}\cdot7\cdot\cancel{6}\cdot5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1} = 70$ crossing pairs

$\Rightarrow$ $190-70 = \boxed{120}$ 2 cut dissections

Check: There should be $2\cdot120 = 240$ ordered dissections

n	4	5	6	7	8
$\binom{n}{2}$	6	10	15	21	28
interior cuts	2	5	9	14	20

8 rotations
14 second cuts

112

8 rotations
$2+9=11$ second cuts

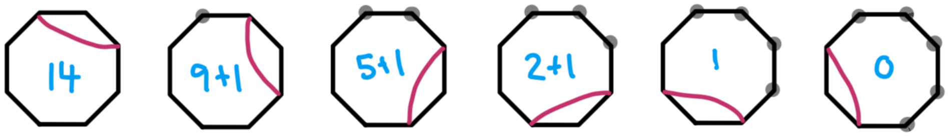
88

4 rotations
$5+5=10$ second cuts

40

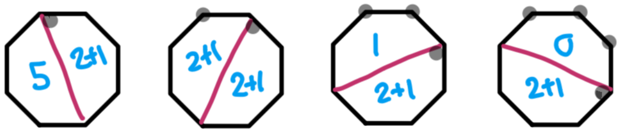
240 ✓

Check: Classify by position of first cut in sorting order:



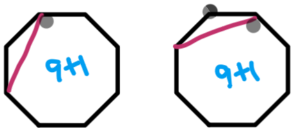
14+10+6+3+1+0 = 34

10+7+4+2+1 = 24



8+6+4+3 = 21

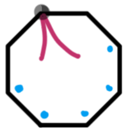
8+7+6 = 21



10+10 = 20

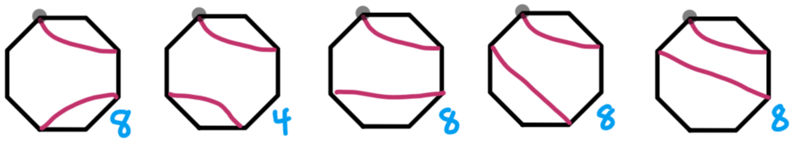
34+24+21+21+20 = 120 ☑

Check: Find up to rotational symmetry, and multiply by orbit sizes.

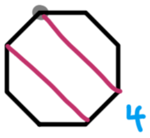


8 rotations
$\binom{5}{2}$ variants

80



36



80+36+4=120

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4	5	6
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8		

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7		

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4	5	6
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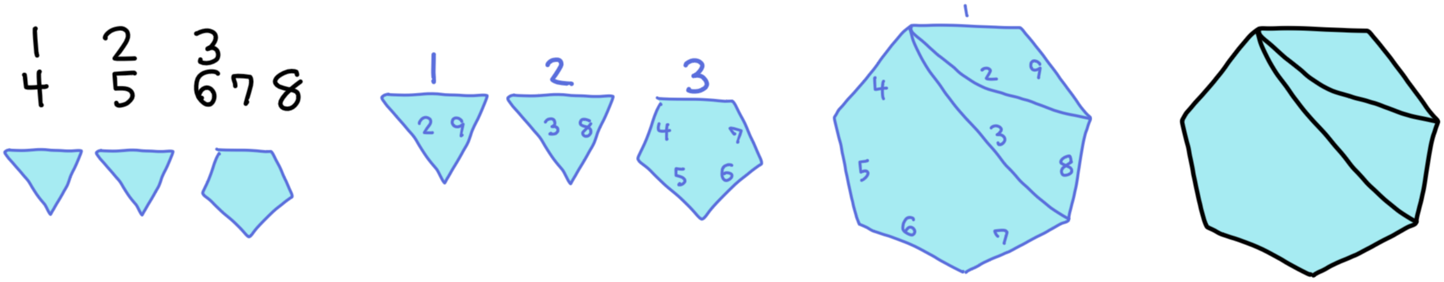
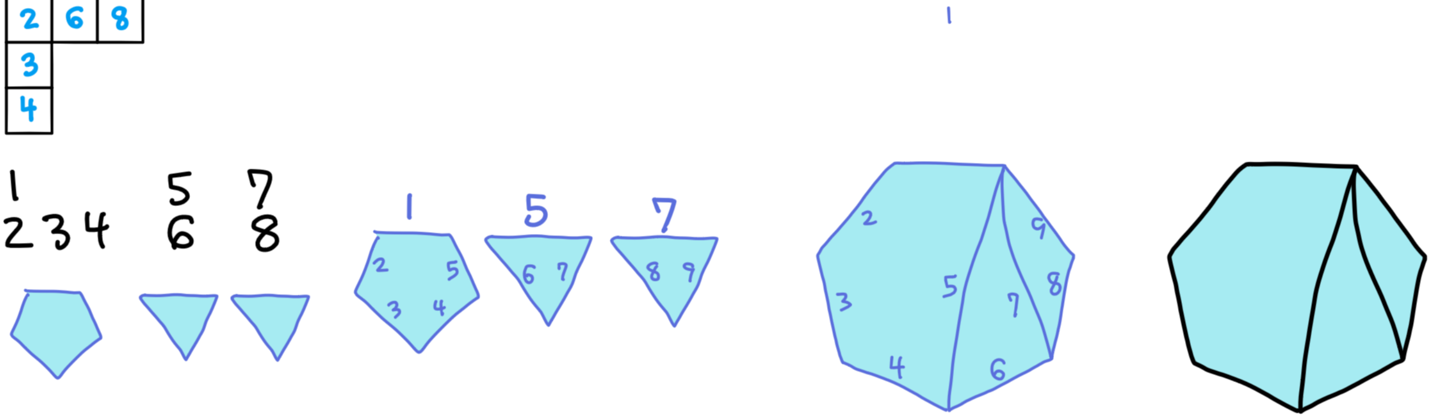
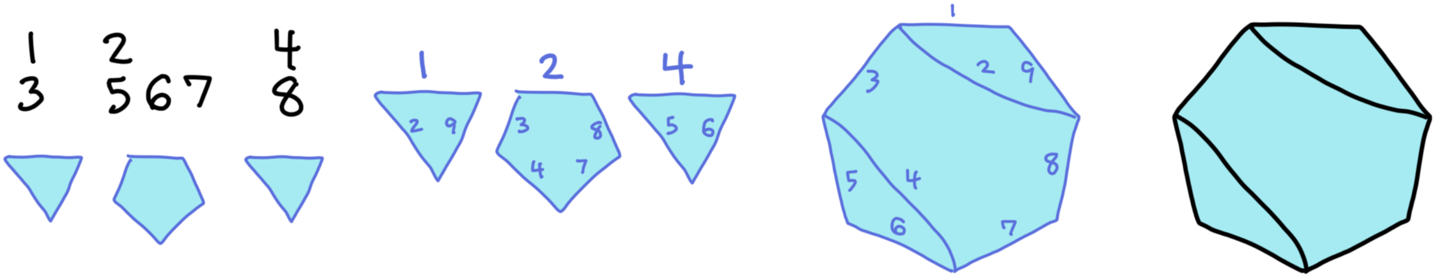
8		



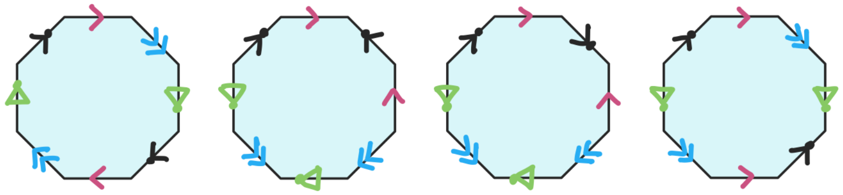
1	5	7
2	6	8
3		
4		

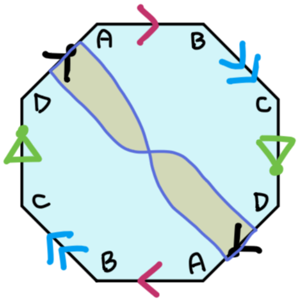


1	2	4
3	5	8
6		
7		



[3] Identify each of the following surfaces from their gluing diagrams, computing their Euler characteristic and deciding whether or not they are orientable. Which two surfaces are homeomorphic (topologically equivalent)?

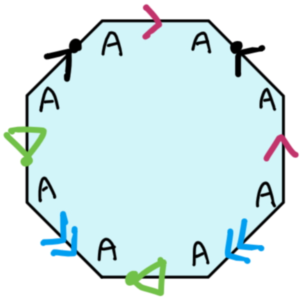




non-orientable

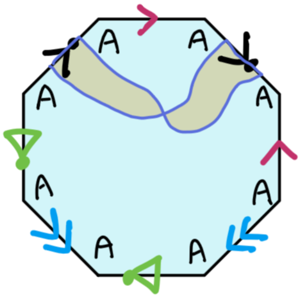
$\chi = 4 - 4 + 1 = 1$

⊛



orientable

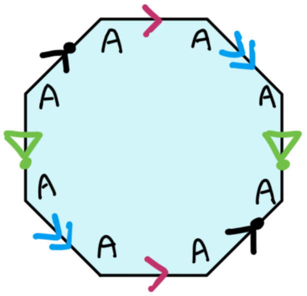
$\chi = 1 - 4 + 1 = -2$



non-orientable

$\chi = 1 - 4 + 1 = -2$

⊛

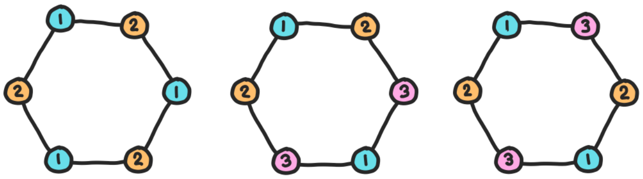


orientable

$\chi = 1 - 4 + 1 = -2$

⊛ Second and fourth surfaces are the same.

[4] How many ways can we properly color the vertices of a hexagon using n colors, up to rotational symmetry? Confirm your answer by drawing each of the possibilities for $n = 3$.
(For a proper coloring, adjacent vertices have distinct colors.)



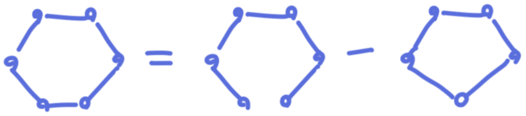
Let $f_k(n)$ be the chromatic polynomial of a k-cycle.

$$\boxed{f_k(n) = (n-1)^k + (-1)^k(n-1)}$$

Simplification of formula from class, we can prove by induction:

Basis: $k=2$  $f_2(n) = n(n-1) = (n-1)^2 + (n-1)$ ☑

Induction:



$$f_k(n) = n(n-1)^{k-1} - f_{k-1}(n)$$

$$(n-1)^k + (-1)^k(n-1) = n(n-1)^{k-1} - \left[(n-1)^{k-1} + (-1)^{k-1}(n-1)\right]$$ ☑

Apply Burnside's lemma $\frac{1}{|G|}$[#fixed patterns, each $g \in G$]

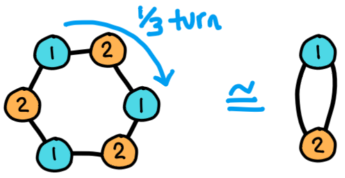
$G = \{0,1,2,3,4,5\}$ 6th turns $|G| = 6$

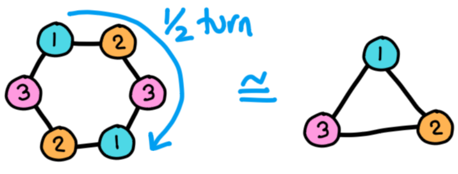
0: $f_6(n) = (n-1)^6 + (n-1)$

1,5: no proper colorings fixed under 1/6th turn

2,4: repeating pattern every 1/3 turn $\Leftrightarrow k=2$ $f_2(n) = (n-1)^2 + (n-1)$

3: repeating pattern every 1/2 turn $\Leftrightarrow k=3$ $f_3(n) = (n-1)^3 - (n-1)$

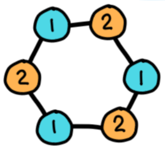




$$\boxed{f(n) = \frac{1}{6}\left[(n-1)^6 + (n-1)^3 + 2(n-1)^2 + 2(n-1)\right]}$$

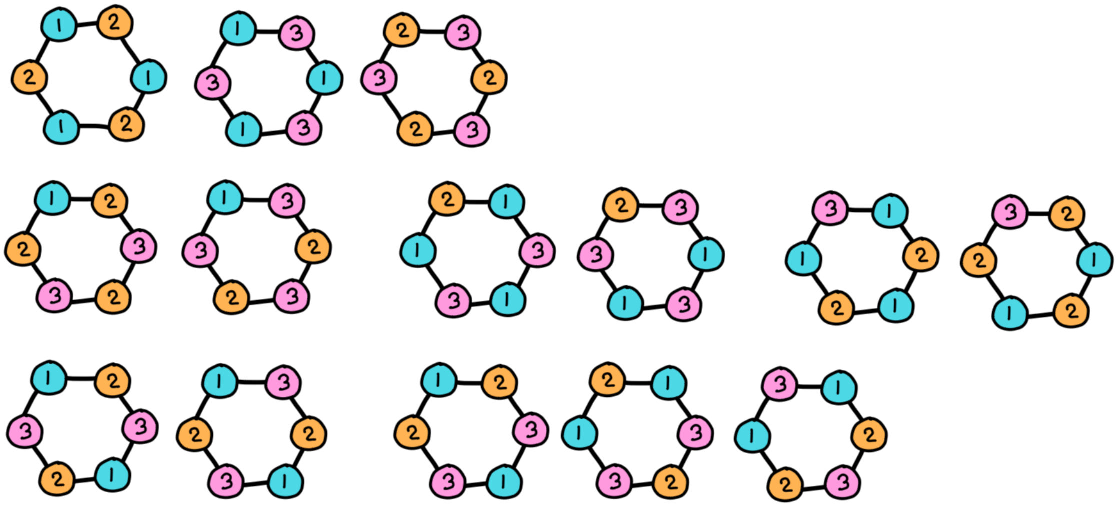
Check: $f(1) = 0$ ☑

$f(2) = 1$ ☑



n=2 only case,
up to symmetry

$$\boxed{f(3) = \frac{1}{6}\left[64 + 8 + 2\cdot 4 + 2\cdot 2\right] = 84/6 = 14}$$



What does f(n) look like expanded out?

$$f(n) = \frac{1}{6}\left[(n-1)^6 + (n-1)^3 + 2(n-1)^2 + 2(n-1)\right]$$

$$= \frac{1}{6}\left[n^6 - 6n^5 + 15n^4 - 19n^3 + 14n^2 - 5n\right]$$

	n^6	n^5	n^4	n^3	n^2	n	1
$(n-1)^6$	1	-6	15	-20	15	-6	1
$(n-1)^3$				1	-3	3	-1
$2(n-1)^2$					2	-4	2
$2(n-1)$						2	-2
	1	-6	15	-19	14	-5	0

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Check: f(1) = 0 ☑ 30 - 30

f(2) = 1 ☑

Check in binary

(For computer scientists, or anyone bored by conventional arithmetic)

64		1						
−6·32	-1	-1	0					
+15·16	1	1	1	1				
−19·8	-1	0	0	-1	-1			
+14·4			1	1	1	0		
−5·2					-1	0	-1	
	0^1	0^1	0	1	-1	0	-1	0
					1	0	-1	0
6						1	1	0 ☑

```
In[1]:= Try[f_, d_] := Module[{},
          Print[f /. m → n - 1 // Expand];
          Print[f /. n → m + 1 // Expand];
          Print[Table[f / d /. n → k /. m → k - 1, {k, 5}]];]
```

```
In[2]:= Try[m^6 + m^3 + 2 m^2 + 2 m, 6]
```

$-5\,n + 14\,n^2 - 19\,n^3 + 15\,n^4 - 6\,n^5 + n^6$

$2\,m + 2\,m^2 + m^3 + m^6$

```
{0, 1, 14, 130, 700}
```

```
In[3]:= Try[n m (m^4 - m^3 + m^2 + 2), 6]
```

$-5\,n + 14\,n^2 - 19\,n^3 + 15\,n^4 - 6\,n^5 + n^6$

$2\,m + 2\,m^2 + m^3 + m^6$

```
{0, 1, 14, 130, 700}
```

```
In[4]:= Try[n m^5 - n m (m^3 - m^2 + m - 1) + n m (n - 2) + 2 n m, 6]
```

$-5\,n + 14\,n^2 - 19\,n^3 + 15\,n^4 - 6\,n^5 + n^6$

$2\,m + 2\,m^2 + m^3 + m^6$

```
{0, 1, 14, 130, 700}
```

```
In[5]:= Try[m^6 + m + 2 n m  + n m (n - 2), 6]
```

$-5\,n + 14\,n^2 - 19\,n^3 + 15\,n^4 - 6\,n^5 + n^6$

$2\,m + 2\,m^2 + m^3 + m^6$

```
{0, 1, 14, 130, 700}
```

```
In[6]:= Try[(n^2 - n + 1) n m (n^2 - 4 n + 5), 6]
```

$-5\,n + 14\,n^2 - 19\,n^3 + 15\,n^4 - 6\,n^5 + n^6$

$2\,m + 2\,m^2 + m^3 + m^6$

```
{0, 1, 14, 130, 700}
```

```
In[7]:= Try[n (n - 1) ((n - 1)^4 - (n - 1)^3 + (n - 1)^2 - (n - 2)), 1]
```

$-5\,n + 15\,n^2 - 20\,n^3 + 15\,n^4 - 6\,n^5 + n^6$

$m + m^6$

```
{0, 2, 66, 732, 4100}
```

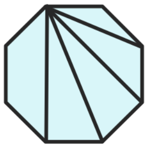
```
In[8]:= Try[2 m^3 - m, 1]
```

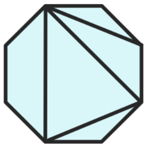
$-1 + 5\,n - 6\,n^2 + 2\,n^3$

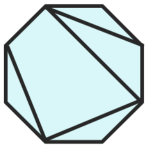
$-m + 2\,m^3$

```
{0, 1, 14, 51, 124}
```

[5] How many ways can we dissect an octagon using 4 cuts, up to dihedral (rotations and flips) symmetry? Confirm your answer by drawing each of the possibilities. Which patterns are not chiral?







Dihedral group G, $|G| = 16$ 8 rotations, 8 flips

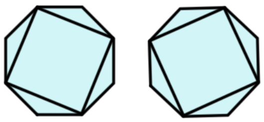
8^{th} turn rotations:

0: All raw dissections $\frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$

$n=8$, $k=4$ $\frac{1}{5}\binom{5}{4}\binom{11}{4} = \frac{11\cdot 10\cdot \overset{3}{\cancel{9}}\cdot\cancel{8}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = \boxed{330}$

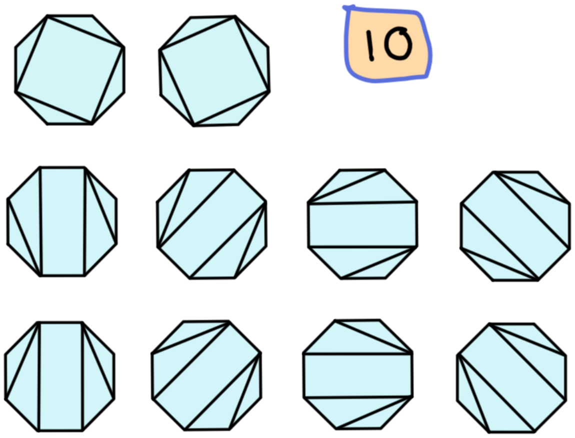
1, 3, 5, 7: No dissections fixed by eigth turn

2, 6:

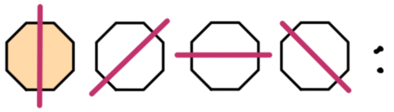


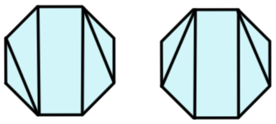
$\boxed{2}$

4:

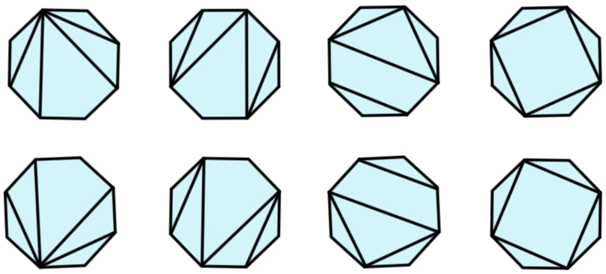


$\boxed{10}$





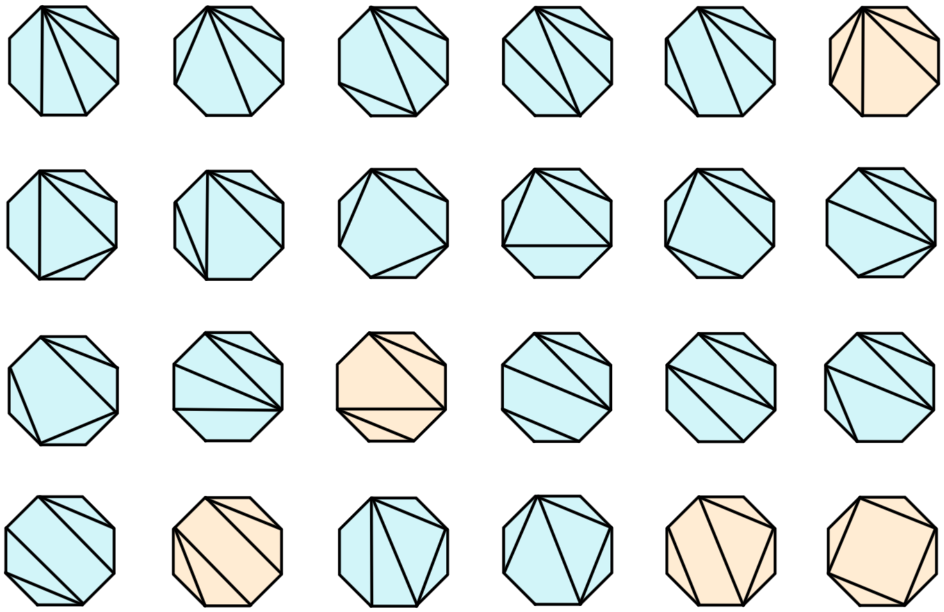
$\boxed{2}$



$\boxed{8}$

Burnside's formula:

$$\frac{1}{16}\left[330+\underbrace{2+2+10+2+2+2+2}_{22}+\underbrace{8+8+8+8}_{32}\right]=\frac{384}{16}=\boxed{24}$$



The following 5 dissections are not chiral: