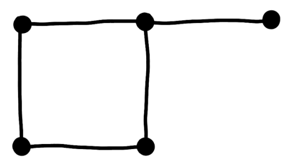
Exam 3

Combinatorics, Dave Bayer, April 26 - May 1, 2023

Please show all of your work. You will be graded for both your answers and your explanations.

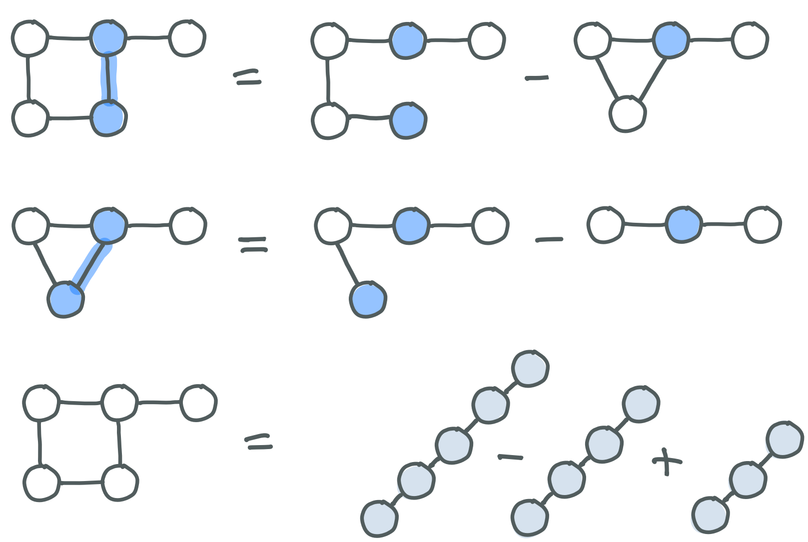
This test is open-book. You may use any resource such as my course materials, textbooks, or *The On-Line Encyclopedia of integer Sequences*. You may not receive help from another person.

[1] What is the chromatic polyomial of the following graph?



deletion-contraction:

(each graph stands for its polynomial)



So

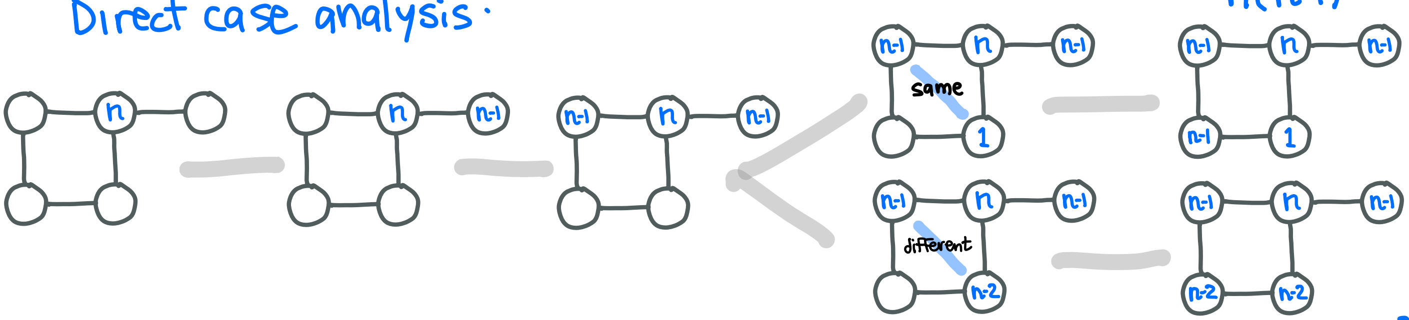
$x = n-1$

$$\boxed{\begin{gathered} n(n-1)^4 - n(n-1)^3 + n(n-1)^2 \\ (x+1)(x^4 - x^3 + x^2) \end{gathered}}$$

$n = x+1$

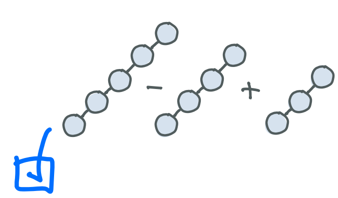
Direct case analysis:

$n(n-1)^3$

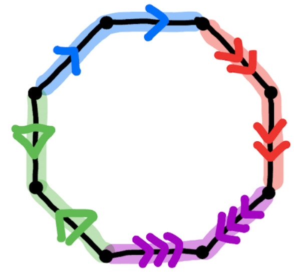


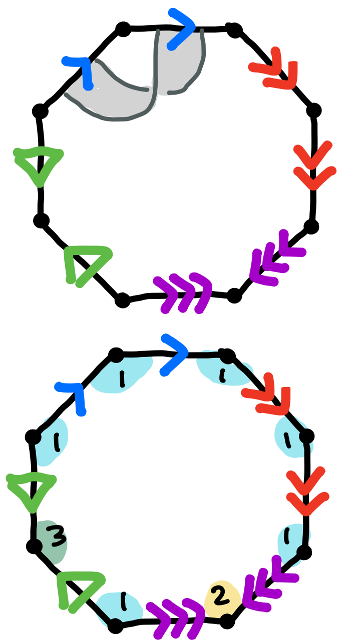
$n(n-1)^2(n-2)^2$

$$n\left[(n-1)^3 + (n-1)^2(n-2)^2\right] \xrightarrow{n = x+1} (x+1)\Big[\underbrace{x^3 + x^2(x-1)^2}_{x^4 - x^3 + x^2}\Big]$$



[2] Which topological surface is represented by the following gluing diagram?



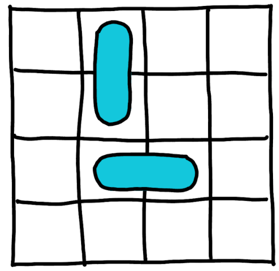


non-orientable

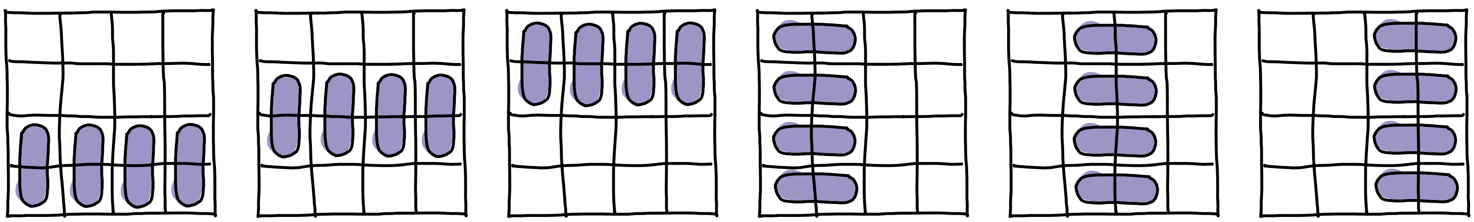
$\chi = v - e + f$
$= 3 - 4 + 1 = 0$

Klein bottle

[3] How many ways can we place a pair of 1×2 markers on a 4×4 grid?



There are 24 possible positions for a 1×2 marker on a 4×4 board:



We want to count unordered pairs of markers that do not overlap.

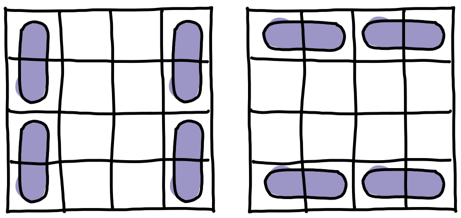
There are $24 \cdot 23 = 552$ ordered pairs of distinct markers, and

$$\binom{24}{2} = 276 \text{ unordered pairs.}$$

We want to subtract the pairs that overlap.

First approach Count overlaps, depending on position of first marker.

There are four cases: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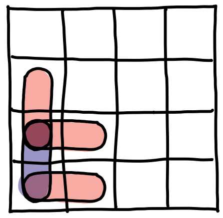
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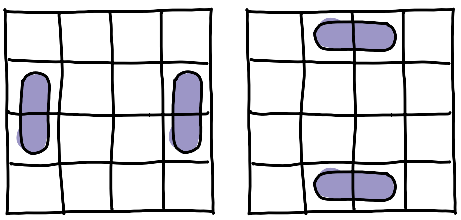
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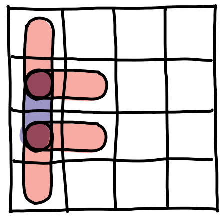
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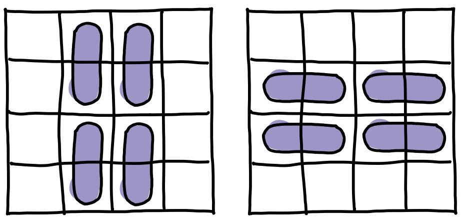
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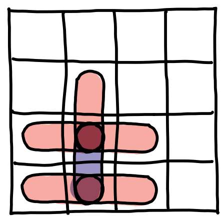
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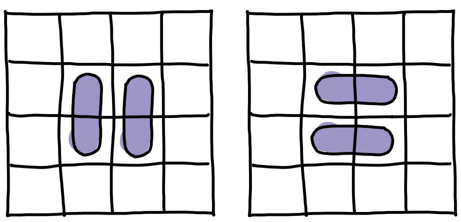
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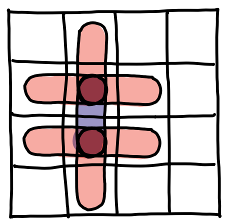
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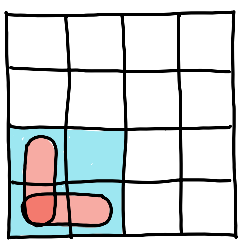
$$\underbrace{\begin{array}{c}8\cdot3+4\cdot4+8\cdot5+4\cdot6\\ 24+16+40+24\end{array}}_{104}$$

$$\underbrace{\begin{array}{c}(24\cdot23-104)/2\\ 552-104\end{array}}$$

$$448/2 = \boxed{224}$$

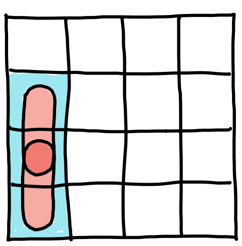
Second approach

Count possible configurations of unordered overlapping pairs



There are 9 positions for each of 4 rotations of an "elbow" overlap.

$$9\cdot4=36$$

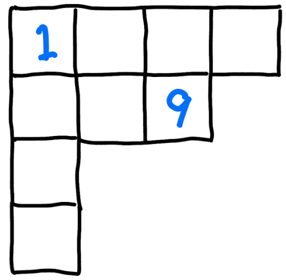


There are 8 positions for each of 2 rotations of a "straight" overlap.

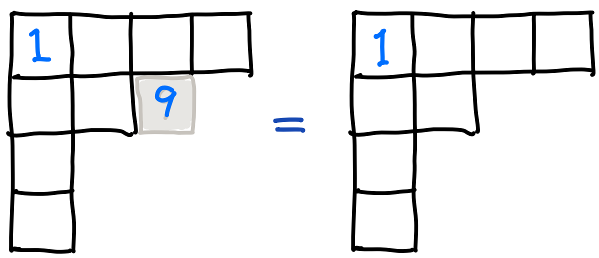
$$8\cdot2=16$$

$$\binom{24}{2}-36-16 = 276-36-16 = \boxed{224}$$

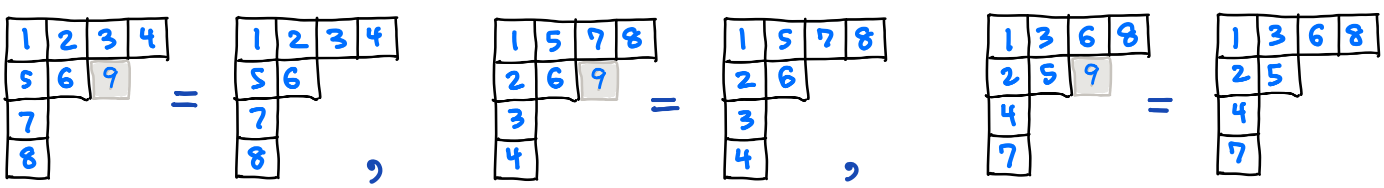
[4] How many ways can the following diagram be filled in to complete a Young tableau?



Every Young tableau starts with a 1 in the upper left corner. Each way of completing this diagram with a 9 as shown corresponds to a Young tableau on the diagram with that cell removed.

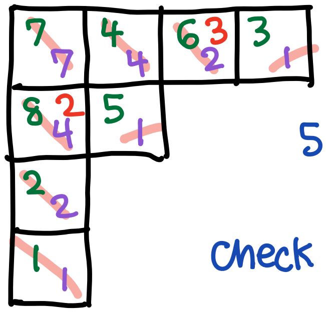


For example:



So the count is given by the hook length formula for the diagram with this cell removed.

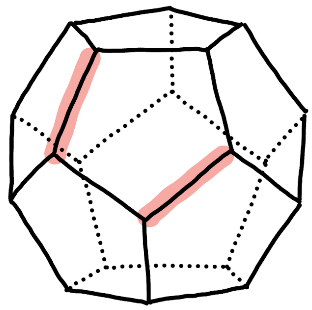
$$\frac{8!}{\prod \text{hook lengths}}$$

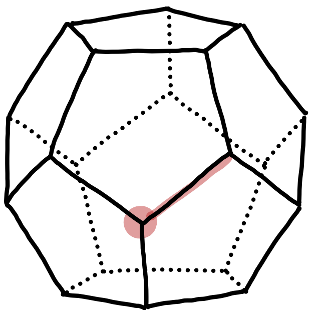


$5 \cdot 3 \cdot 3 \cdot 2 = \boxed{90}$

Check: $\frac{\overset{2}{8} \cdot 7 \cdot \overset{3}{6} \cdot \overset{5}{5} \cdot 4 \cdot \overset{3}{3} \cdot 2 \cdot 1}{7 \cdot 4 \cdot 2 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1}$

[5] How many ways can we choose a pair of edges of a dodecahedron, up to rotational symmetry?



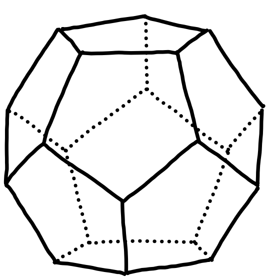


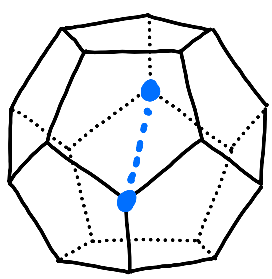
$\boxed{\text{Using Burnside's lemma}}$

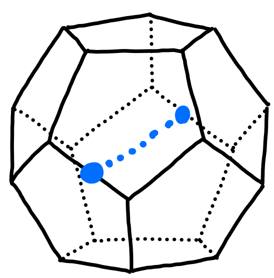
There are 20 vertices, 30 edges, 12 faces

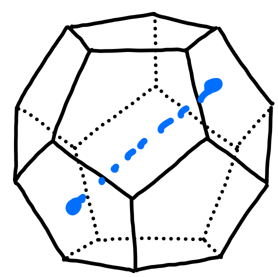
$\chi = v - e + f = 20 - 30 + 12 = 2$ ☑ sphere

$|G| = 20 \text{ vertices} \times 3 \text{ incident edges} = 60$







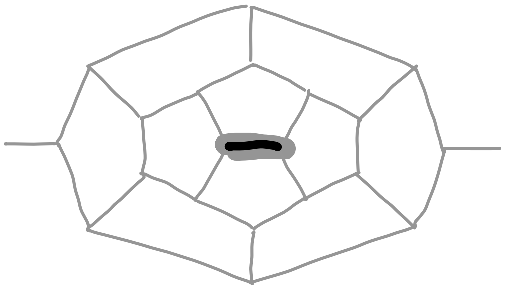


Identity	Vertex:Vertex 1/3 turn	Edge:Edge 1/2 turn	Face:Face 1,2,3,4 /5 turn
1	20	15	24
1 1 1 … 1	3 3 … 3	1 1 2 … 2	5 5 5 5 5 5
$\binom{30}{2}$	0	15	0

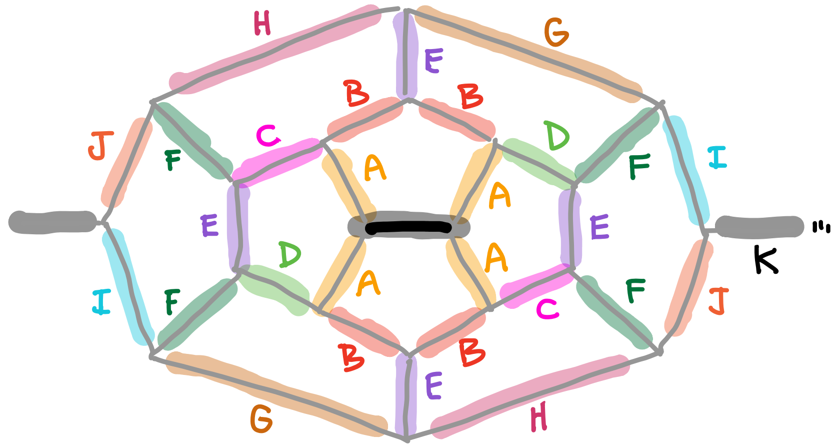
$$\left(\binom{30}{2} + 15 \cdot 15\right)/60 = 666/60 = \boxed{11}$$

$435 + 225$

Check by direct count:



How many different edges can be paired with this first edge?



Tricky to decide which pairs are the same.

Sometimes we see chirality.

$\boxed{11}$

A	B	C	D	E	F	G	H	I	J	K
4	4	2	2	4	4	2	2	2	2	1

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