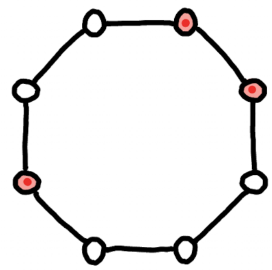
Exam 2

Combinatorics, Dave Bayer, April 6-10, 2023

Please show all of your work. You will be graded for both your answers and your explanations. You need not complete the entire exam; some questions are intended to be challenging.

This test is open-book. You may use any resource such as my course materials, textbooks, or *The On-Line Encyclopedia of integer Sequences.* You may not receive help from another person.

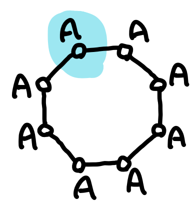
[1] How many ways can we choose three vertices of an octagon, up to rotation?



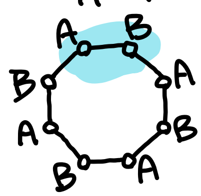
Turn /8	$\lvert X^g \rvert$
0	56
1,3,5,7	0
2,6	0
4	0

$$= \binom{8}{3} = \frac{8\cdot 7\cdot \cancel{6}}{\cancel{3}\cdot\cancel{2}\cdot 1}$$

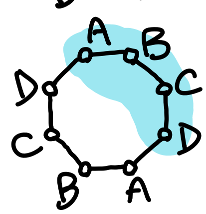
$$\frac{1}{|G|}\sum_{g\in G} |X^g| = 56/8 = \boxed{7}$$



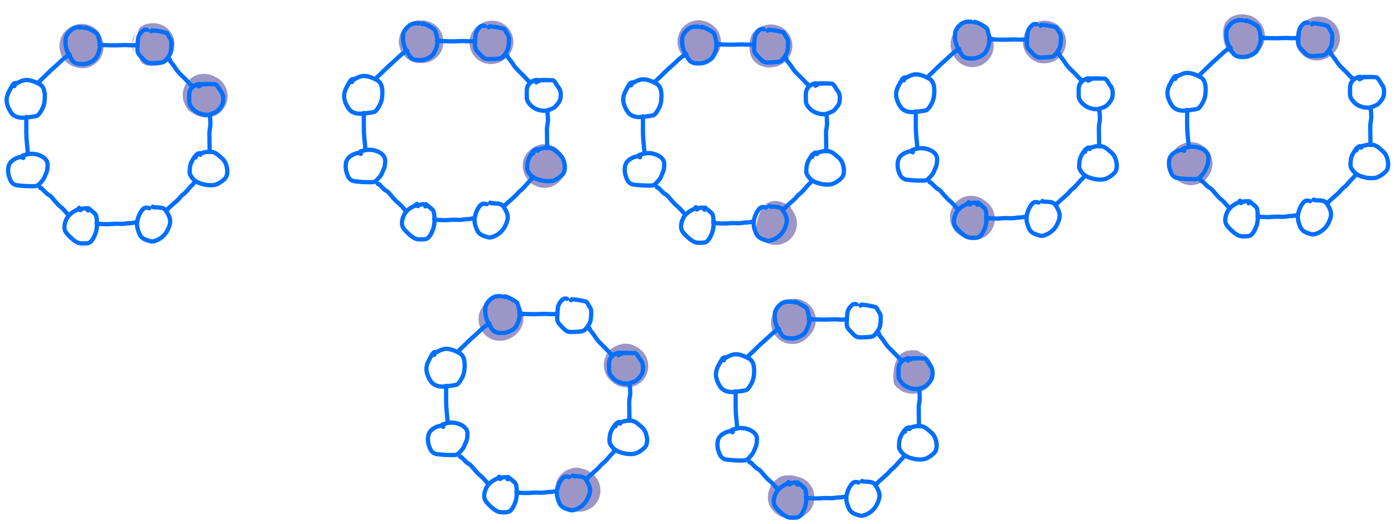
8



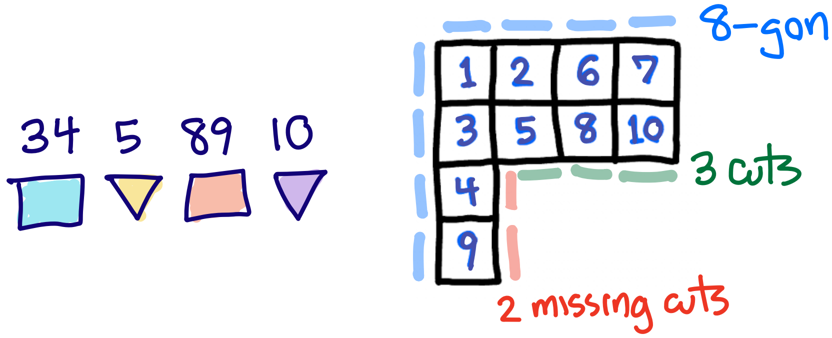
4 4

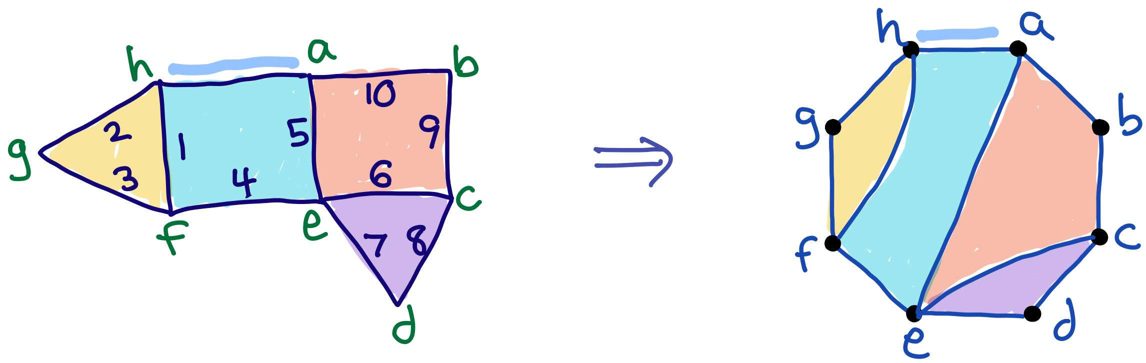


2 2 2 2

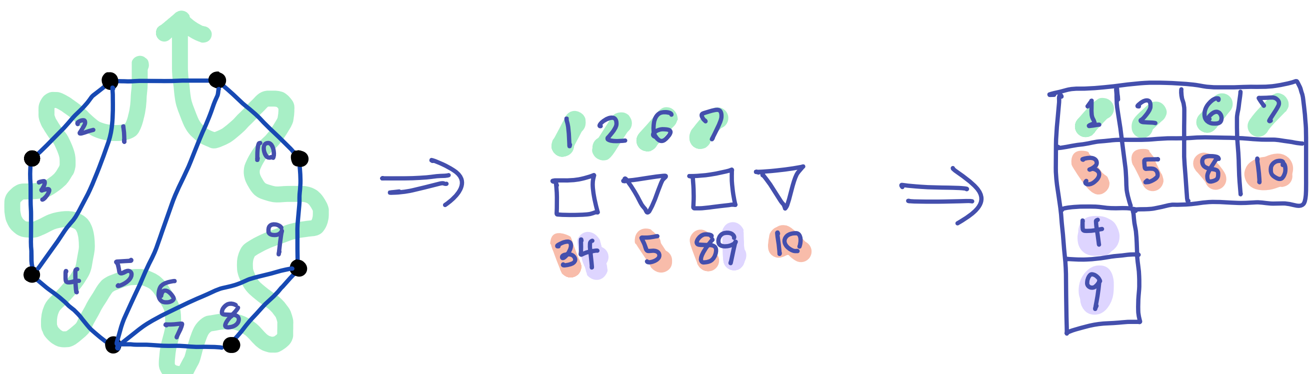


[2] Which dissection of a polygon corresponds to this Young tableau, under Stanley's correspondence?

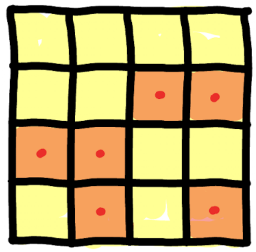


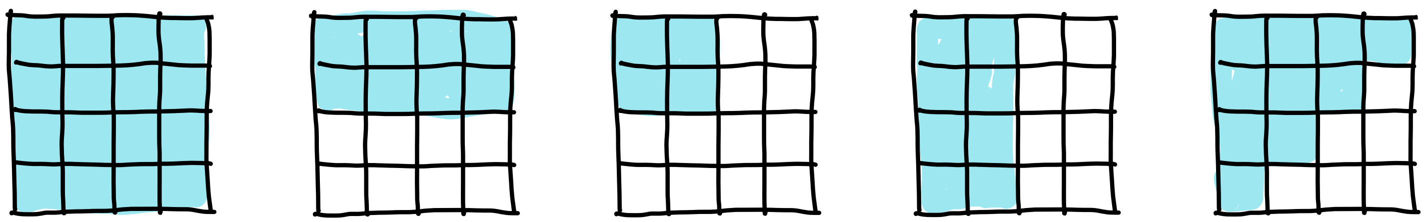


check:



[3] let $f(k)$ count the number of ways of coloring the squares of a 4×4 grid using at most k colors, up to the dihedral group D_4 of rotations and reflections of the square. What is $f(2)$? What can you say about $f(k)$?





1 — 16 choices

8 choices

4 choices

8 choices

10 choices

$$\boxed{f(k) = (k^{16} + 2k^{10} + 3k^8 + 2k^4)/8}$$

$$\boxed{f(2) = (2^{16} + 2 \cdot 2^{10} + 3 \cdot 2^8 + 2 \cdot 2^4)/8 = 8{,}548}$$

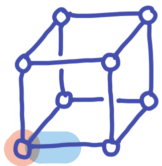
$$(65536 + 2 \cdot 1024 + 3 \cdot 256 + 2 \cdot 16)/8$$

		1	2	3
1	16	1	65,536	43,046,721
2	10	2	2,048	118,098
3	8	3	768	19,683
2	4	2	32	162
		1	**8,548**	**5,398,083**

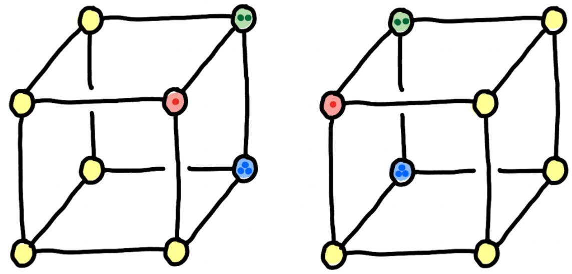
A217338 Number of inequivalent ways to color a 4 X 4 checkerboard using at most n colors allowing rotations and reflections.

0, 1, 8548, 5398083, 537157696, 19076074375, 352654485156, 4154189102413, 35184646816768,

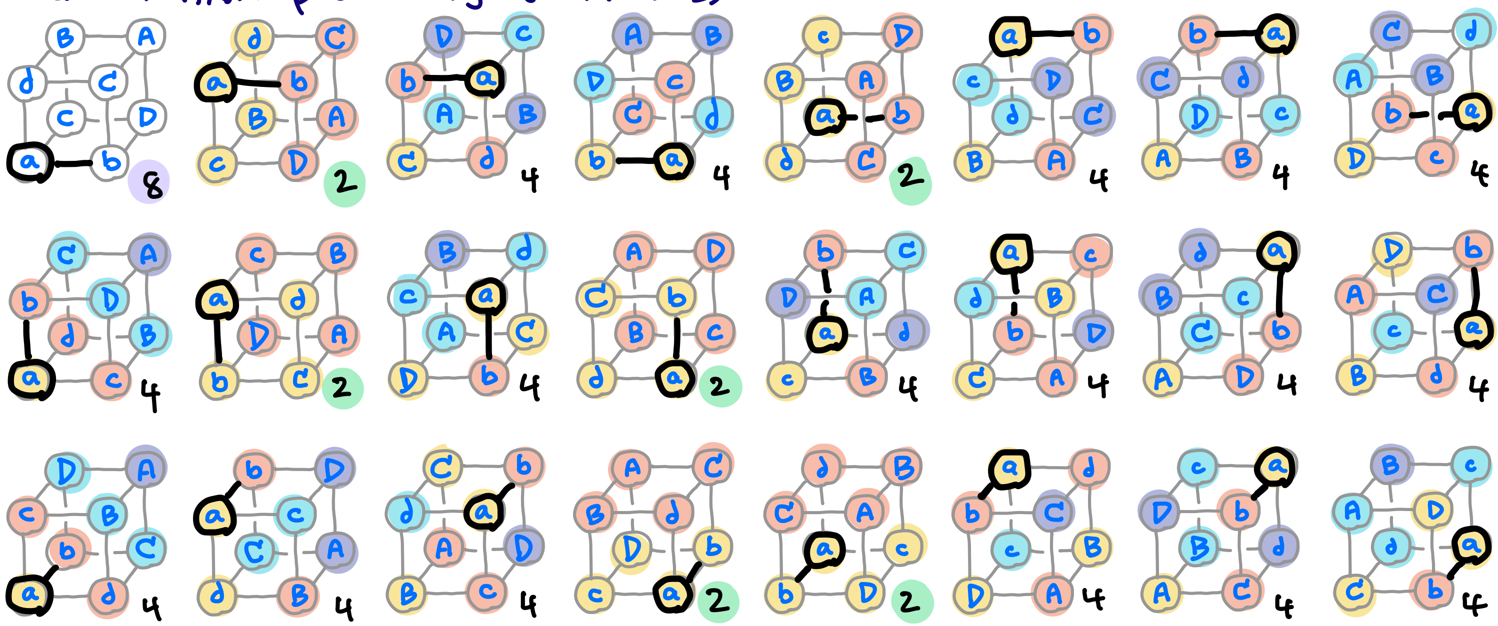
[4] Color the vertices of a cube using at most k colors, up to rotations of the cube. Let $f(k)$ count the number of *chiral pairs*: Mirror images that are not the same under rotation. What is $f(4)$? What can you say about $f(k)$?



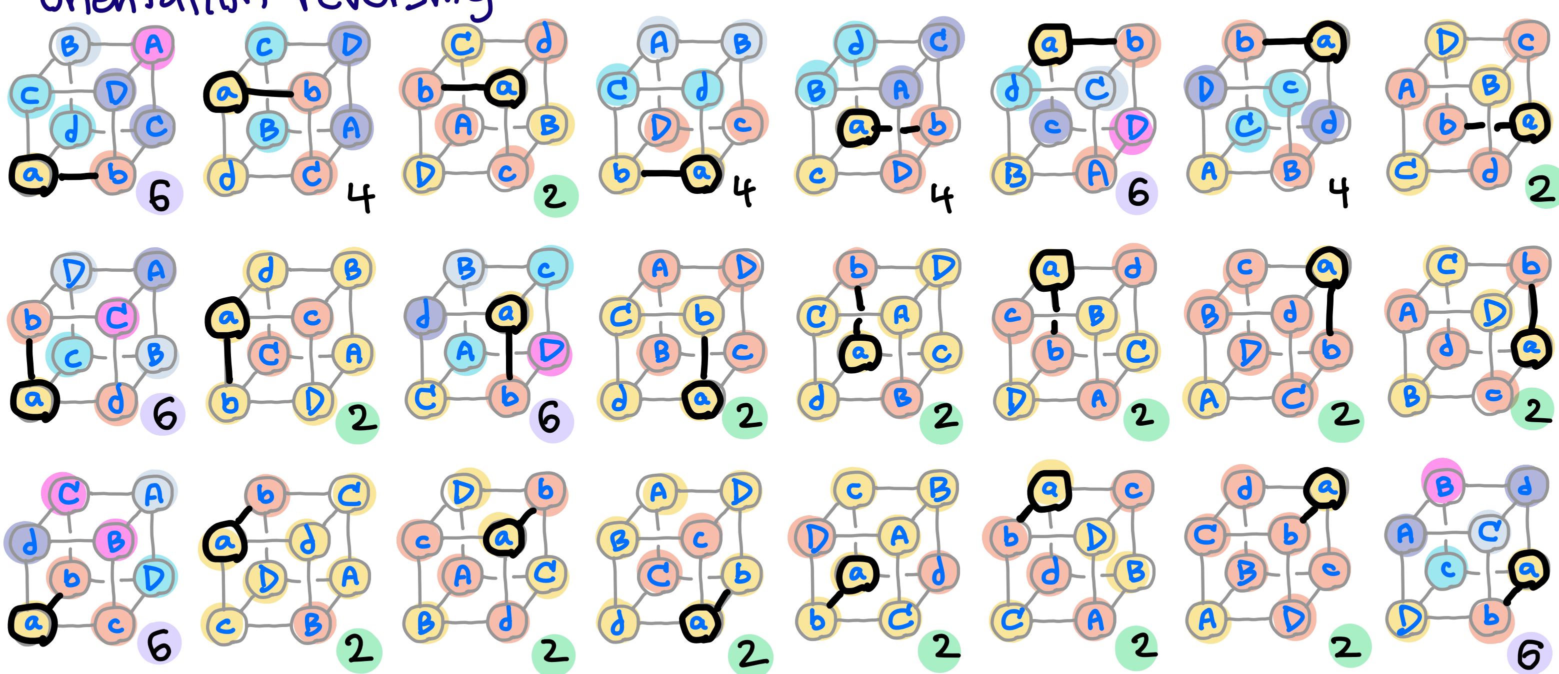
$8 \cdot 3 = 24$ rotations
24 reflections



orientation preserving (rotations)



orientation reversing



The above is a brute force check of the 48 possibilities, to get an answer without really understanding the group.

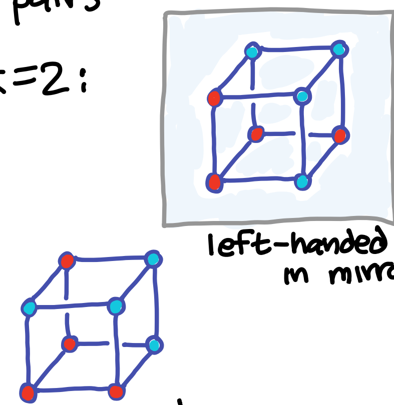
These counts agree with the following more careful analysis:

Let $g(k)$ = # colorings using $\le k$ colors, up to rotations

Let $h(k)$ = # colorings using $\le k$ colors, up to rotations and reflections

Then $f(k) = g(k) - h(k)$ = # chiral pairs

Example: There is one chiral pair for $k=2$:



left-handed in mirror

right-handed

So $f(2) = g(2) - h(2) = 1$

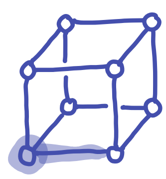
Rotations count these twice

Rotations and reflections count these once.

Let G = group of rotations of cube

Let H = group of rotations and reflections of cube

$|G| = 8 \cdot 3 = 24$



8 ways to mark a vertex

3 ways to mark an edge

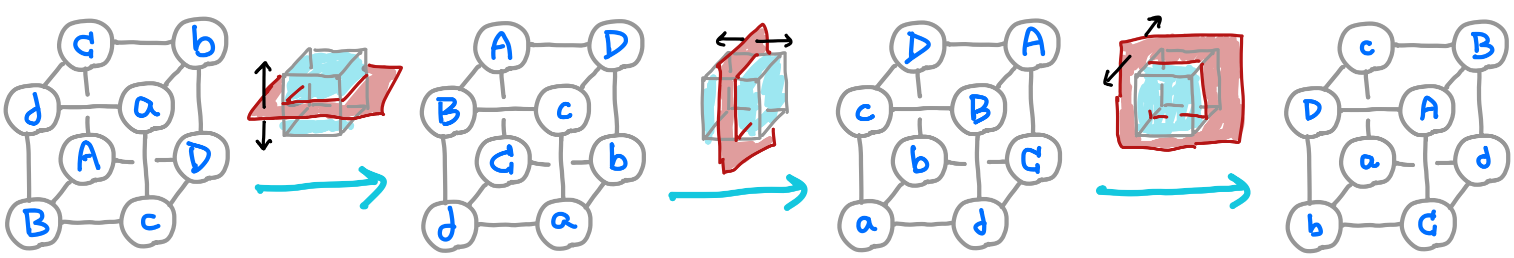
$|H| = 2|G| = 48$ $\quad G < H \quad H = G \cup Gh$

orientation preserving

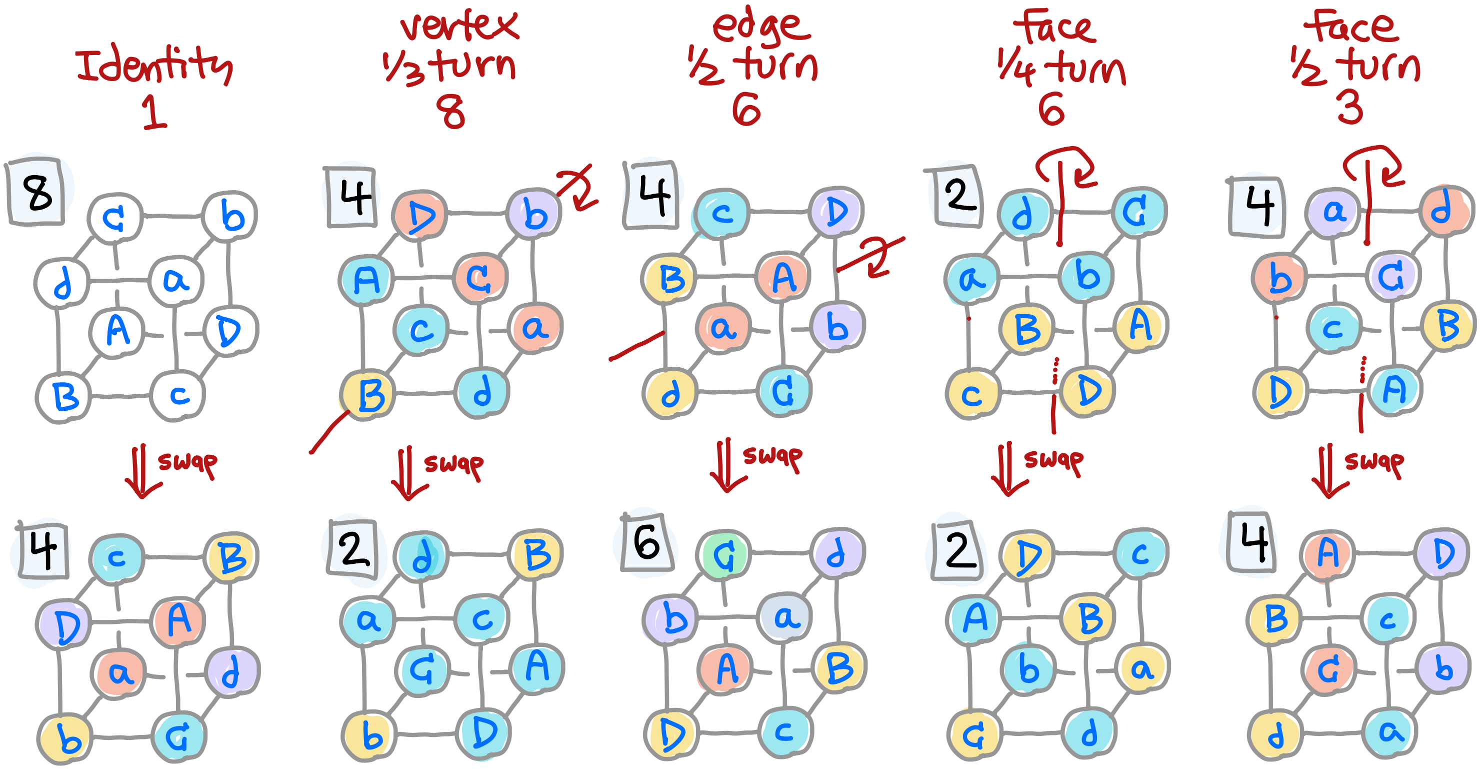
orientation reversing

The easiest way to understand H is to choose h that swaps all antipodal corners.

This can be understood as three reflections, through each coordinate plane.



We now study each kind of rotation, alone and followed by swapping antipodes



	8	6	4	2
G	1		17	6
Gh		6	4	14
H	1	6	21	20

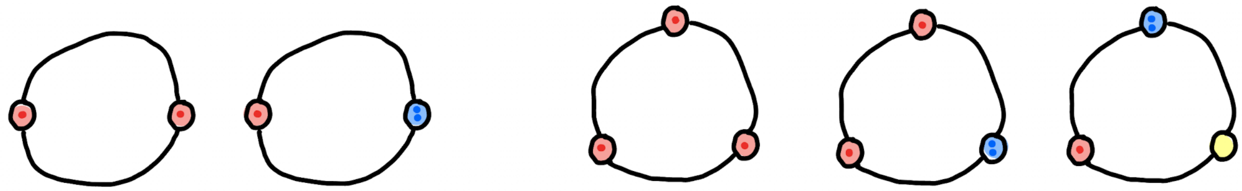
$$g(k) = (k^8 + 17k^4 + 6k^2)/24$$

$$h(k) = (k^8 + 6k^6 + 21k^4 + 20k^2)/48$$

$$\Rightarrow \boxed{\begin{array}{l} f(k) = (k^8 - 6k^6 + 13k^4 - 8k^2)/48 \\ f(4) = 920 \end{array}}$$

		1	2	3	4	5		
1	8	1	256	6561	65536	390625		
17	4	17	272	1377	4352	10625		
6	2	6	24	54	96	150		
		1	23	333	2916	16725	1,23,333,2916,16725	A000543
6	6	6	384	4374	24576	93750		
4	4	4	64	324	1024	2500		
14	2	14	56	126	224	350		
		1	22	267	1996	10375	1,22,267,1996,10375	A128766
		0	1	66	920	6350	0,1,66,920,6350	A337896

[5] Let $f(p)$ count the number of ways of coloring a p bead necklace using at most 3 interchangeable colors, up to rotation. In other words, we're partitioning the beads into up to 3 unnamed subsets, up to rotation. As shown, $f(2) = 2$ and $f(3) = 3$. What is $f(5)$? What can you say about $f(p)$, when p is prime?



The primes $p=2,3$ are special cases because 2 and 3 divide 6, the order of the permutation group S_3 interchanging the colors.
For prime $p \geq 5$, the group $C_p \times S_3$ has order $6p$, with fixed points

C_p \ S_3	( )	(123)	(132)	(12)	(13)	(23)
0	3^p	0	0	1	1	1
1	3	0	0	1	1	1
2	3	0	0	1	1	1
⋮						
p-1	3	0	0	1	1	1

so for prime $p \geq 5$,

$$\boxed{\begin{gathered} f(p) = \left[3^p + 3p + 3(p-1)\right]/6p \\ f(5) = [\underbrace{3^5}_{243} + \underbrace{3\cdot 5}_{15} + \underbrace{3\cdot 4}_{12}]/30 = 9 \end{gathered}}$$