Exam 1

Combinatorics, Dave Bayer, February 15-19, 2023

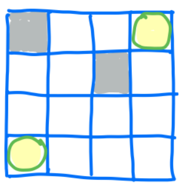
Please show all of your work. You will be graded for both your answers and your explanations. You need not complete the entire exam; some questions are intended to be challenging.

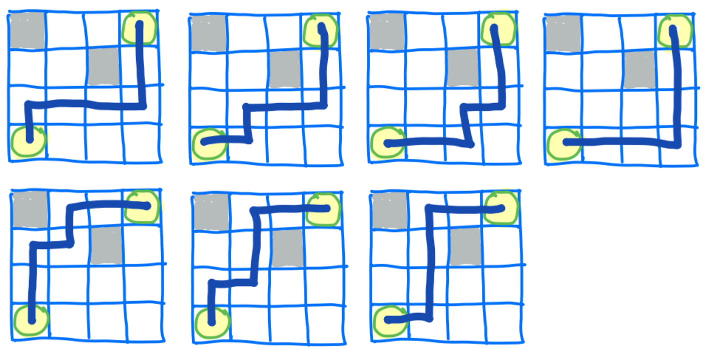
This test is open-book. You may use any resource such as my course materials, textbooks, or *The On-Line Encyclopedia of Integer Sequences*. You may not receive help from another person.

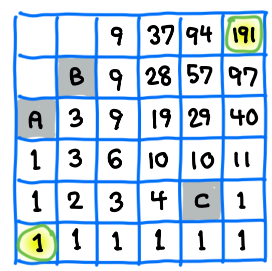
"What can you say about $f(n)$*?"* is up to you. There might be a formula. There might be a generating function. You might notice a pattern, or recognize the sequence.

Please match your understanding of my words with the examples, and contact me if you're concerned about any ambiguity.

[1] Moving up or to the right, for the smaller grid on the left there are seven paths between the marked corners that avoid the obstacles. For the larger grid on the right, how many paths avoid the obstacles?







$\emptyset \quad -A \quad -B \quad -C \quad +AB$

$$\binom{10}{5} - \binom{3}{0}\binom{7}{5} - \binom{5}{1}\binom{5}{4} - \binom{5}{4}\binom{5}{1} + \binom{3}{0}\binom{2}{1}\binom{5}{4}$$

$$252 - \underbrace{1\cdot 21 - 5\cdot 5 - 5\cdot 5}_{71} + \underbrace{1\cdot 2\cdot 5}_{10}$$

$$262 - 71 = \boxed{191} \checkmark$$

$\binom{2}{1} = 2$

$\binom{3}{0} = 1$

$\binom{5}{1} = 5$

$\binom{5}{4} = 5$

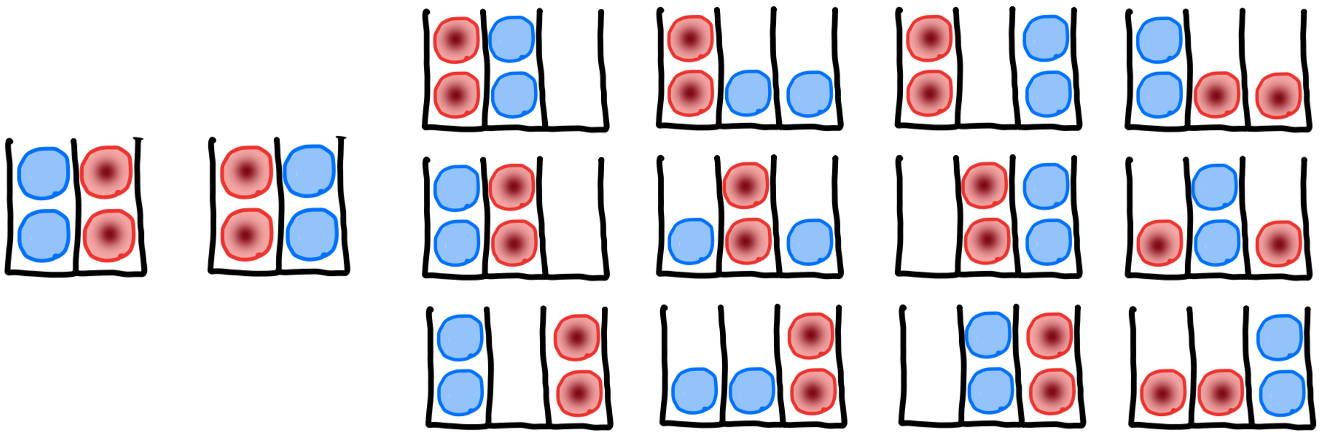
$\binom{7}{5} = \binom{7}{2} = \frac{7\cdot 6}{2\cdot 1} = 21$

$$\binom{10}{5} = \frac{\cancel{10}\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} \cdot 2\cdot 2$$

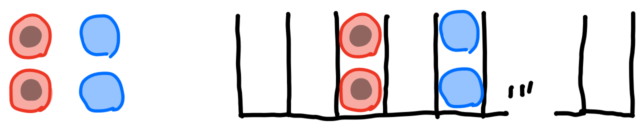
$= 9\cdot 7\cdot 2\cdot 2$

$= 63\cdot 4 = 252$

[2] Let $f(n)$ count the number of ways of placing two red balls and two blue balls in n bins, with the restriction that no bin can contain both a red and a blue ball. As shown, $f(2) = 2$ and $f(3) = 12$. What is $f(4)$? What can you say about $f(n)$?



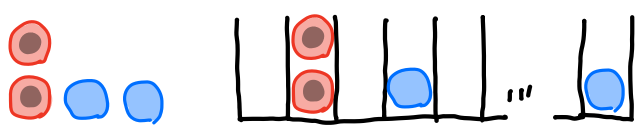
$n(n-1)$



n choices for

then $n-1$ choices for

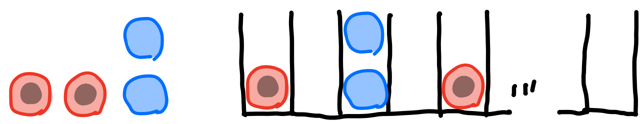
$\frac{n(n-1)(n-2)}{2}$



n choices for

then $\binom{n-1}{2}$ choices for

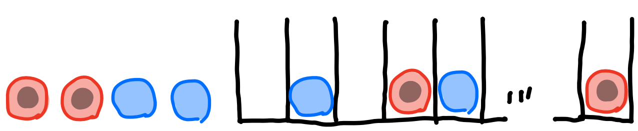
$\frac{n(n-1)(n-2)}{2}$



$\binom{n}{2}$ choices for

then $n-2$ choices for

$\frac{n(n-1)(n-2)(n-3)}{4}$



$\binom{n}{2}$ choices for

then $\binom{n-2}{2}$ choices for

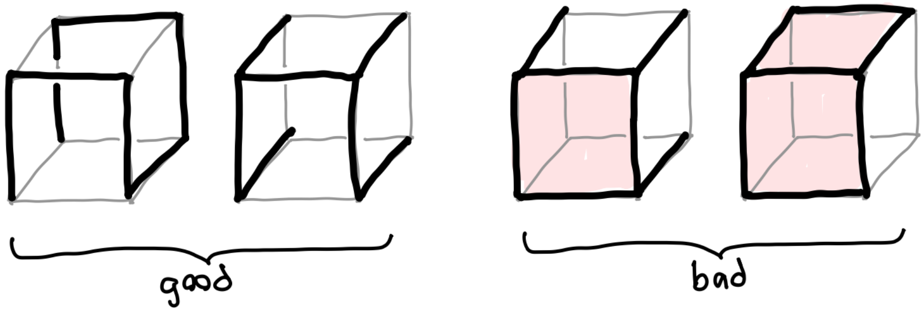
$$f(n) = n(n-1)\Big[\underbrace{1 + \frac{n-2}{2} + \frac{n-2}{2}}_{n-1} + \frac{(n-2)(n-3)}{4}\Big] = \boxed{n(n-1)\Big[(n-1) + \frac{(n-2)(n-3)}{4}\Big]}$$

$$f(2) = 2\cdot 1\Big[1 + \frac{0(-1)}{4}\Big] = 2 \ \checkmark$$

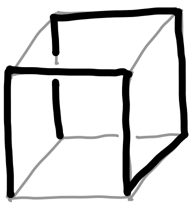
$$f(3) = 3\cdot 2\Big[2 + \frac{1\cdot 0}{4}\Big] = 12 \ \checkmark$$

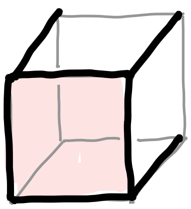
$$f(4) = 4\cdot 3\Big[3 + \frac{2\cdot 1}{4}\Big] = 36 + 6 = \boxed{42}$$

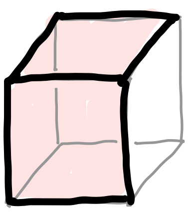
[3] How many ways can we mark seven edges of a cube, so that no square face has all four edges marked?



Inclusion-exclusion counting:

$\binom{12}{7}$  $\binom{12}{7} = \binom{12}{5} = \frac{\cancel{12}\cdot 11\cdot \cancel{10}\cdot 9\cdot 8}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 99\cdot 8 = 792$ markings in all

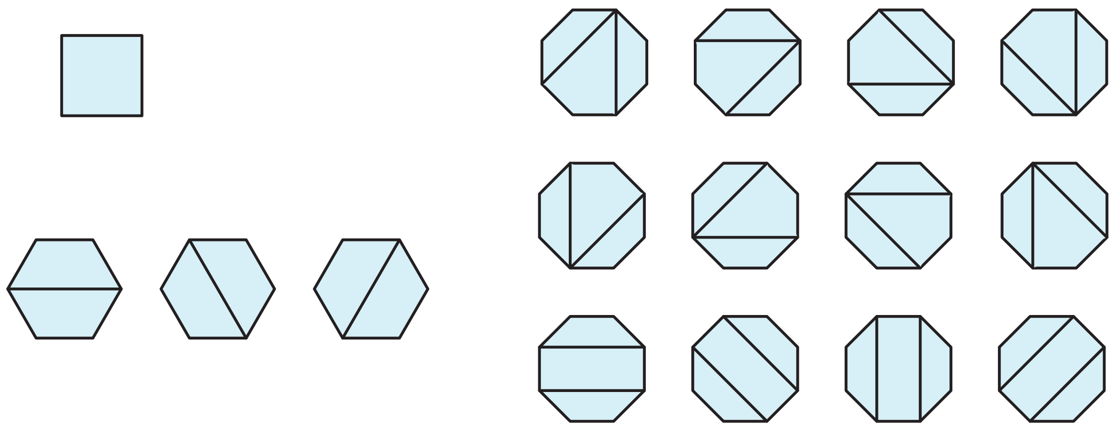
$-\,6\binom{8}{3}$  For each of 6 faces there are $\binom{8}{3} = \frac{8\cdot 7\cdot \cancel{6}}{\cancel{3}\cdot\cancel{2}\cdot 1} = 56$ markings including all edges of that face

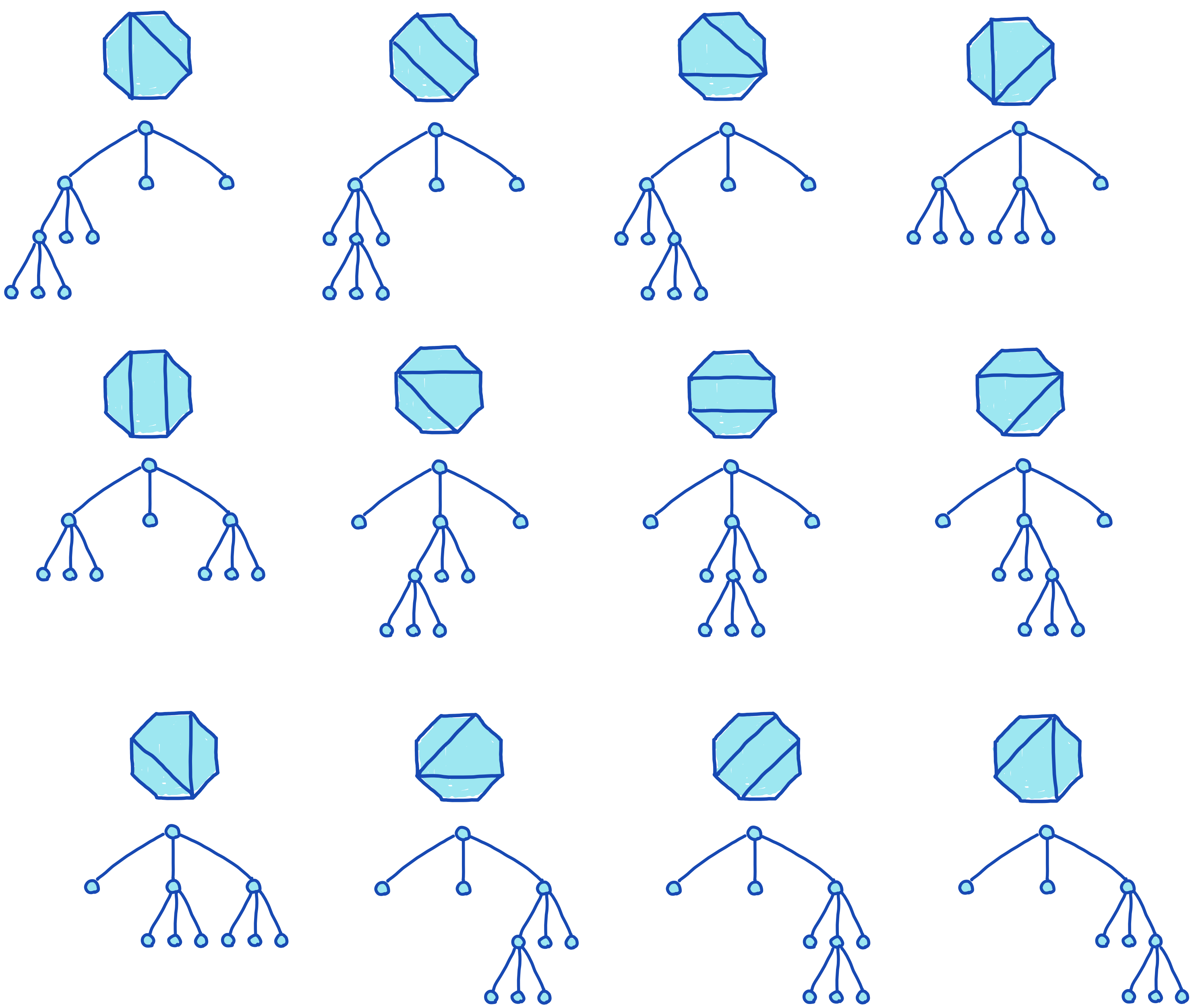
$+\,12$  For each of 12 edges there is one marking including all edges of both faces touching that edge

$$792 - 6\cdot 56 + 12 = \begin{array}{r} 792 \\ -336 \\ \hline 456 \\ +12 \\ \hline 468 \end{array}$$

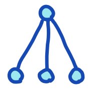
$\boxed{468 \text{ markings}}$

[4] Let $f(n)$ count the number of ways of dissecting a $2n$-gon into 4-sided regions. As shown, $f(2) = 1$, $f(3) = 3$, and $f(4) = 12$. What is $f(5)$? What can you say about $f(n)$?

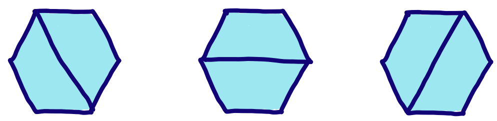




Like Catalan numbers, except we build size n objects from a triple of smaller objects of sizes adding up to $n-1$

Let m = # squares = # twigs 

n		2	3	4	5
m	0	1	2	3	4
#	1	1	3	12	55



$m=2$ (1,0,0) (0,1,0) (0,0,1)

$1\cdot1\cdot1 + 1\cdot1\cdot1 + 1\cdot1\cdot1 = 3$

$m=3$ (2,0,0) (1,1,0) (1,0,1) (0,2,0) (0,1,1) (0,0,2)

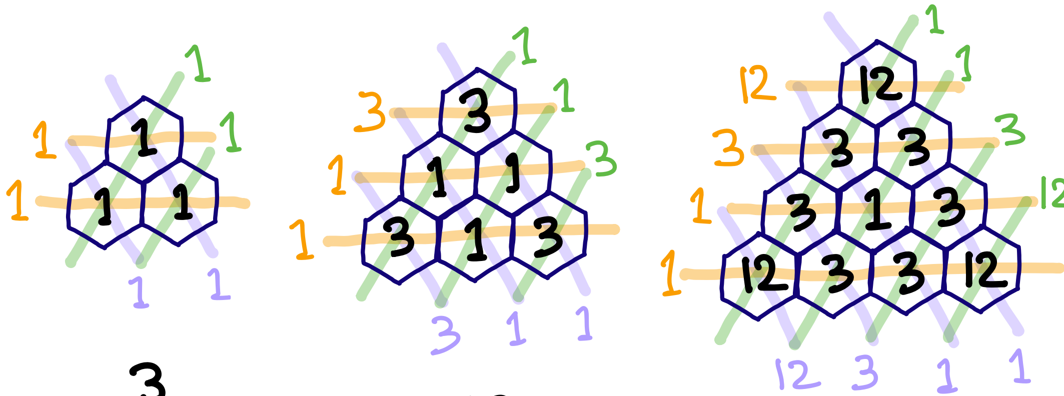
$3\cdot1\cdot1 + 1\cdot1\cdot1 + 1\cdot1\cdot1 + 1\cdot3\cdot1 + 1\cdot1\cdot1 + 1\cdot1\cdot3 = 12$

$m=4$ (3,0,0) (0,3,0) (0,0,3) (1,1,1) 55

$12\cdot1\cdot1$ $1\cdot12\cdot1$ $1\cdot1\cdot12$ $1\cdot1\cdot1$

(2,1,0) (2,0,1) (1,2,0) (0,2,1) (1,0,2) (0,1,2)

$3\cdot1\cdot1$ + $3\cdot1\cdot1$ $1\cdot3\cdot1$ $1\cdot3\cdot1$ $1\cdot1\cdot3$ $1\cdot1\cdot3$



3 12 55

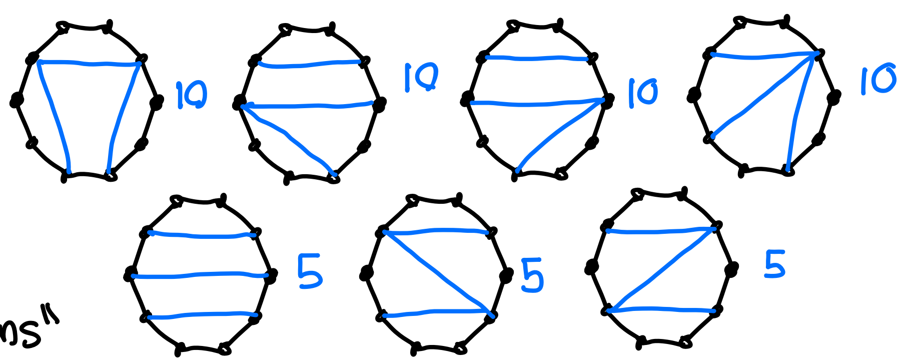
For Catalan numbers, the generating function $F(t)$ satisfies $F = 1 + tF^2$

Here, $\boxed{F = 1 + tF^3}$ by the same argument.

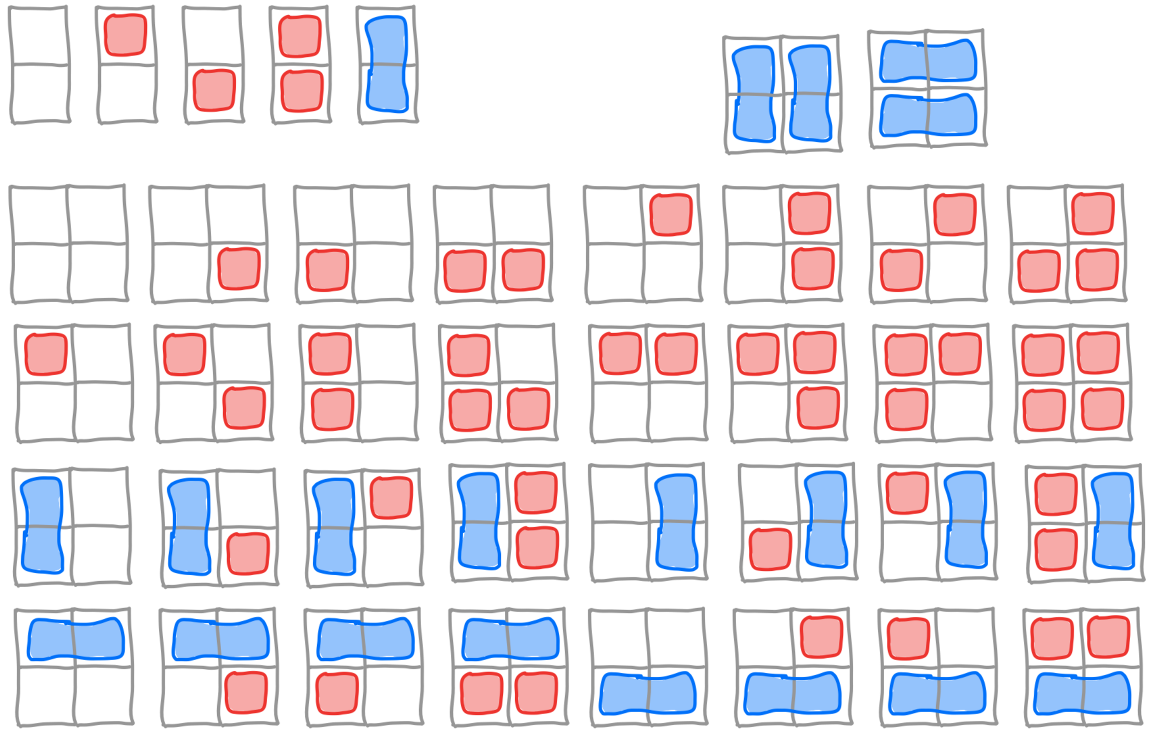
$\boxed{F(5) = 55}$

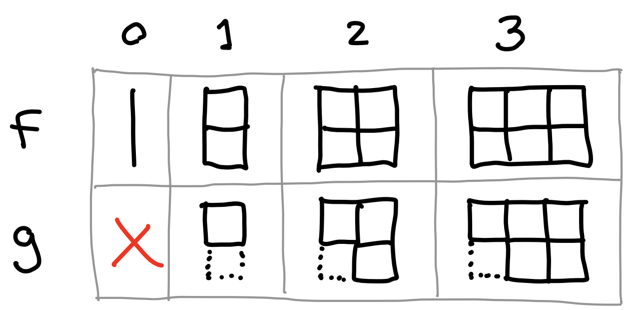
OEIS A001764

"complete quadrillages of 2n+2-gons"



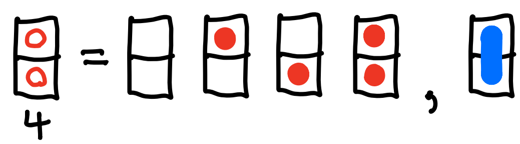
[5] Let $f(n)$ count the number of ways of tiling a $2 \times n$ grid using 1×1 and 1×2 tiles, where one is allowed to leave squares untiled. As shown, $f(1) = 5$, and $f(2) = 34$. What is $f(3)$? What can you say about $f(n)$?



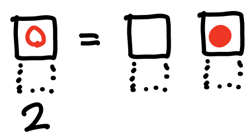


$f(0)=1$ $\quad$ $g(0)=0$

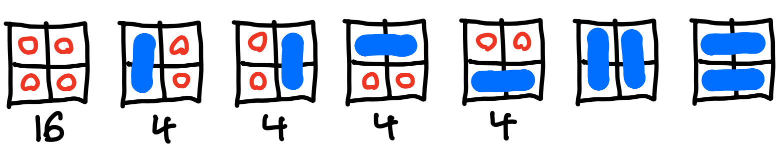
$f(1)=5$



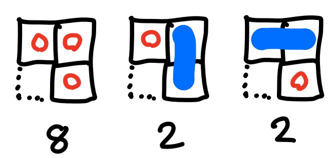
$g(1)=2$



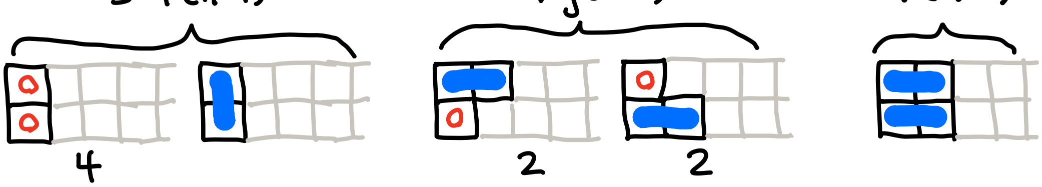
$f(2)=34$



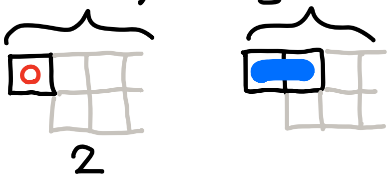
$g(2)=12$

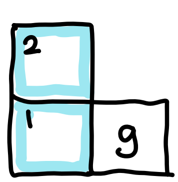


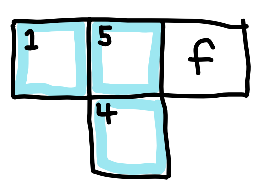
$f(n) = 5f(n-1) + 4g(n-1) + f(n-2)$

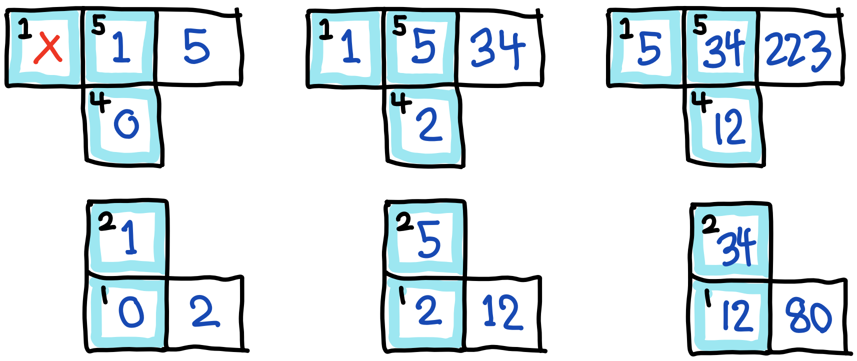


$g(n) = 2f(n-1) + g(n-1)$









	0	1	2	3
f	1	5	34	223
g	0	2	12	80

$$\boxed{f(3) = 223}$$

using generating functions:

$$F(t) = \sum_{n=0}^{\infty} f(n) t^n$$

$$G(t) = \sum_{n=0}^{\infty} g(n) t^n$$

$$\left\{ \begin{array}{l} F = 1 + 5tF + 4tG + t^2F \\ G = 0 + 2tF + tG \end{array} \right\}$$

$$\left\{ \begin{array}{rcl} (1-5t-t^2)F - 4tG & = & 1 \\ -2tF + (1-t)G & = & 0 \end{array} \right\}$$

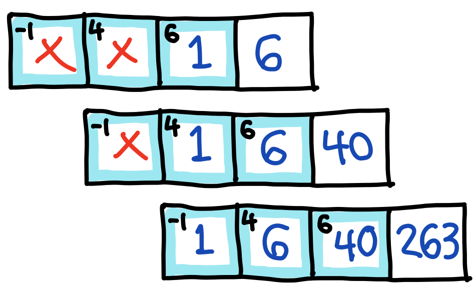
$$\begin{bmatrix} 1-5t-t^2 & -4t \\ -2t & 1-t \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Use Cramer's rule

$$F = \begin{vmatrix} 1 & -4t \\ 0 & 1-t \end{vmatrix} \Big/ \begin{vmatrix} 1-5t-t^2 & -4t \\ -2t & 1-t \end{vmatrix} = \boxed{\frac{1-t}{1-6t-4t^2+t^3}}$$

$$(1-5t-t^2)(1-t) - 8t^2 = 1-6t-4t^2+t^3$$

expand



	0	1	2	3
1	1	6	40	263
t		1	6	40
1-t	1	5	34	223

check using Mathematica

```
In[1]:= A = {{1 - 5 t - t^2, -4 t}, {-2 t, 1 - t}}; A // MatrixForm
```

Out[1]//MatrixForm=

$$\begin{pmatrix} 1 - 5t - t^2 & -4t \\ -2t & 1 - t \end{pmatrix}$$

```
In[2]:= F = LinearSolve[A, {1, 0}][[1]]
```

Out[2]= $\dfrac{1 - t}{1 - 6t - 4t^2 + t^3}$

```
In[3]:= Series[F, {t, 0, 6}]
```

Out[3]= $1 + 5t + 34t^2 + 223t^3 + 1469t^4 + 9672t^5 + 63685t^6 + O[t]^7$

$\boxed{\text{OEIS A102436}}$