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See Petersen – Eulerian Numbers, 5.8: pp 115–122

Or oeis.org A033282

$T(n,k)$ is the number of diagonal dissections of a convex n-gon into k+1 regions.

n	0	1	2	3	4	5	6
3	1						
4	1	2					
5	1	5	5				
6	1	9	21	14			
7	1	14	56	84	42		
8	1	20	120	300	330	132	
9	1	27	225	825	1485	1287	429

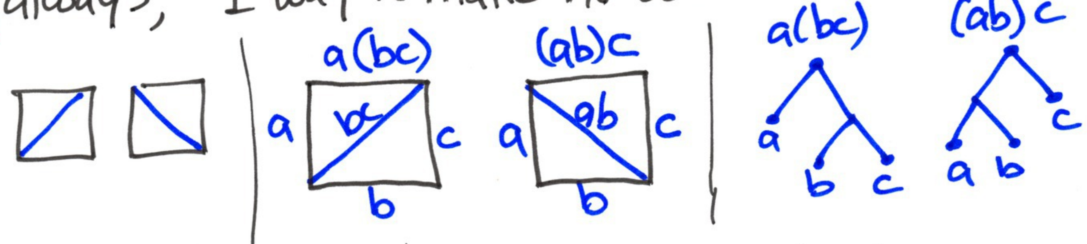
no cuts

catalan numbers

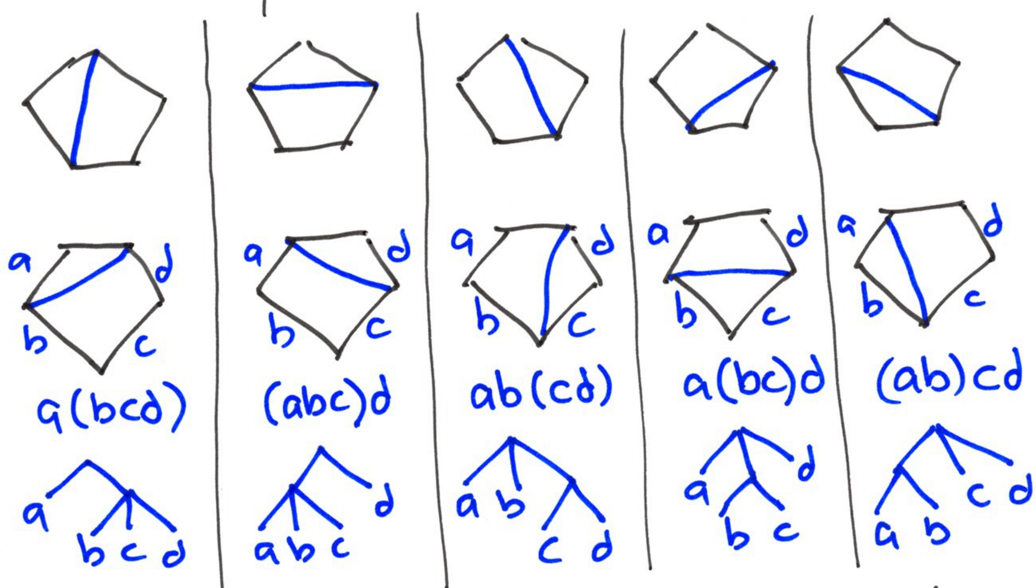
Or, rooted trees, or parenthesized (partially) expressions.

$\boxed{T(n,0)=1}$ always, 1 way to make no cuts.

$T(4,1)$:



$T(5,1)$:



k sets of parens on n−1 letters, or k interior nodes other than root, n−1 leaves.

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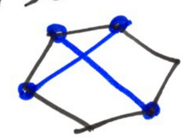
$$\boxed{T(n,1)} = \underbrace{\binom{n}{2}}_{\text{all edges}} - \underbrace{n}_{\text{border}}$$

n	3	4	5	6	...
$\binom{n}{2}$	3	6	10	15	
$\binom{n}{2}-n$	0	2	5	9	...

✓

$\boxed{T(n,2)}$ = all pairs of interior edges − pairs that cross.

$$= \underbrace{\binom{\binom{n}{2}-n}{2}}_{\text{choose pair of interior edges}} - \cancel{\binom{n}{2}-n}\ \underbrace{\binom{n}{4}}_{\text{one cross for every set of 4 vertices}}$$



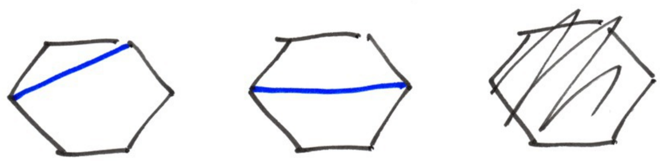
n	5	6	7	
$m = \binom{n}{2}-n$	5	9	14	...
$\binom{m}{2}$	10	36	91	
$\binom{n}{4}$	5	15	35	
$T(n,2)$	5	21	56	...

✓

$T(n,n-3)$ = Catalan numbers.

One instructive way to count (useful for other problems): Consider choices for first edge, how many ways to finish, divide by # choices for first edge (overcount).

$T(6,3)$:



$9 = 6 + 3$

9 choices for 1st cut.
6 divide into 3,5   3 divide into 4,4

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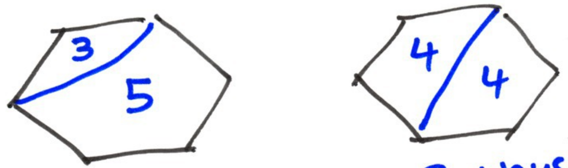
$n=6$

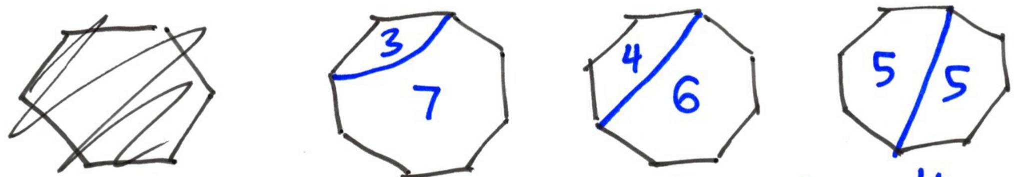


6 ways   3 ways

$$6\cdot 1\cdot 5 + 3\cdot 2\cdot 2 = 30+12 = 42$$

$$42 \div 3 = 14 \quad ✓ \quad T(6,3)$$

$n=8$



$8 + 8 + 4 = 20$

$$(8\cdot 1\cdot 42 + 8\cdot 2\cdot 14 + 4\cdot 5\cdot 5)/5$$

336   224   100

$$= 660/5 = 132 \quad ✓ \quad T(8,5)$$

$T(n,n-4)$ diagonal one in from Catalan numbers

One cut shy of full triangulation.

Count by starting with full triangulation, choose an edge to remove. Leaves a square, so two ways to reach each config. Overcounts, divide by 2.

$T(7,3) = 84.$

$$\begin{array}{rl} 42 & \text{full triangulations} \\ \times\ 4 & \text{choices of edge to remove} \\ \hline 168 & \div 2 = 84 \ ✓ \end{array}$$

$T(8,4) = 330$

$$\begin{array}{r} 132 \\ \times\ 5 \\ \hline 660 \div 2 = 330 \ ✓ \end{array}$$

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$T(n, n-5)$ two diagonals in from catalan numbers

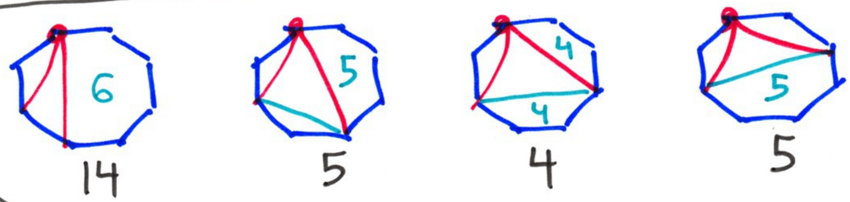
Remove <u>two</u> edges from full triangulation

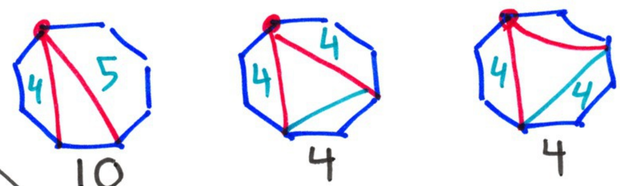
How much overcounting? <u>It depends</u>!

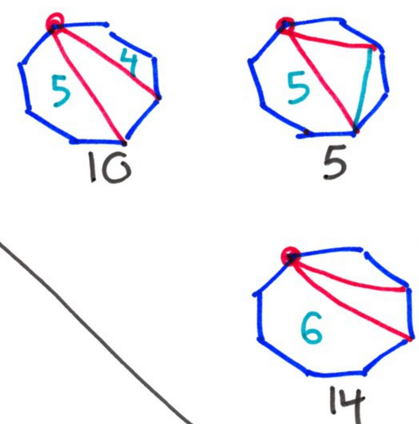
Are the edges adjacent, leaving a 5-gon? $\boxed{5x}$

Or separate, leaving two 4-gons? $\boxed{4x}$

So tricky subproblem: How many <u>adjacent</u> pairs of edges in full triangulations of an n-gon.







$75 \cdot 8 = 600$

full triangs × pairs

$= 132 \cdot 10 = 1320$

600 adjacent

$\Rightarrow$ 720 nonadjacent

$600 \div 5 = 120$

$720 \div 4 = 180$

$\boxed{300}$ ✓

all adjacent pairs meeting at one choice (out of 8) of vertex.

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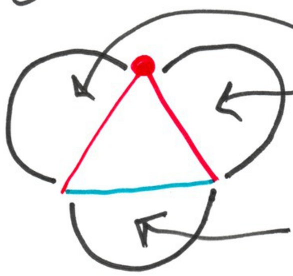
How can we make this computation easier?

Adjacent pair (rooted at a particular vertex, so we $\times n$ to use)

Has at least a triangle on either side, and nothing or at least a triangle attached to open end.



at least a triangle

nothing, or at least a triangle

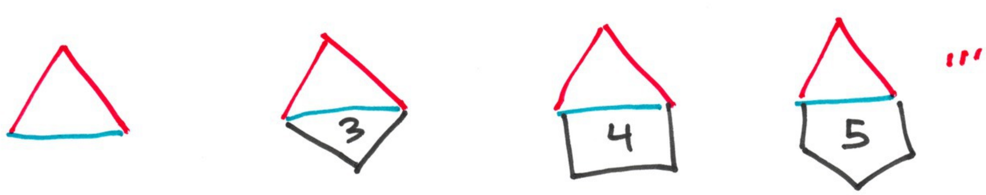
We need to keep track of how many interior edges we make, for each n has a total budget.

Better, count boundary edges, need to add up to n.

Left side: (or right side)

boundary edges K	2	3	4	5	6
# ways to fully triangulate (Catalan number)	1	2	5	14	42

Bottom:



K	1	2	3	4	...
# ways	1	1	2	5	

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How to efficently combine this data? "Multiplication" tables

		0	1	2	3	4	5	6		
		0	0	1	2	5	14	42		
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	
2	1	0	0	1	2	5	14	42		
3	2	0	0	2	4	10	28			
4	5	0	0	5	10	25				
5	14	0	0	14	28					
6	42	0	0	42						
		0	0							
		0								

live portion

(multiply not add!) working back to find mistake

combine left, right counts into left + right.

K	4	5	6	7	8	
#	1	4	14	48	~~150~~ 165	...

14 5
20 2
28 1
48 3
150 6 17
260 8 ✓
9
2340
2250+90 ✓

Now combine with bottom counts

		4	5	6	7	8
		1	4	14	48	~~150~~ 165
1	1	1	4	14	48	~~150~~ 165
2	1	1	4	14	48	150
3	2	2	8	28	96	300
4	5	5	25			
5	14	14				
6	42	42				
7	132					
8	429					

n	#	n·#
5	1	5
6	5	30
7	20	140
8	75	600
9	~~260~~ 275	~~2340~~ 2475

as before

(found mistake working back)

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Now we can compute $T(n,n-5)$ in bulk:

n	5	6	7	8	9
$T(n,n-5)$	1	9	56	300	1485
interior edges, full triangulation	2	3	4	5	6
pairs of "	1	3	6	10	15
full (catalan #)	5	14	42	132	429
product	5	42	252	1320	6435
A = adjacent pairs	5	30	140	600	2475
B = nonadjacent pairs	0	12	112	720	3960
$A \div 5$	1	6	28	120	495
$B \div 4$	0	3	28	180	990
sum	1	9	56	300	1485

Table of adjacent pairs best computed as "generating function"

$$\Big(\underbrace{t^2+2t^3+5t^4+14t^5+42t^6+\dots}_{\text{left, right counts}}\Big)^2\Big(\underbrace{t+t^2+2t^3+5t^4+14t^5+\dots}_{\text{bottom counts}}\Big)$$

$$= \left(t^5+5t^6+20t^7+75t^8+275t^9+\dots\right)$$

Power of t tracks number of boundary edges

Polynomial multiplication is exactly what we did before, just not in this language.

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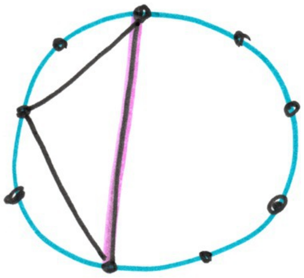
The one entry still out of reach:

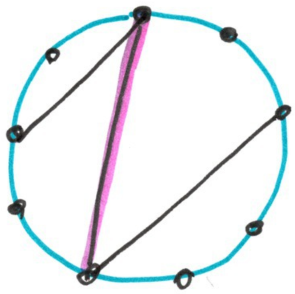
$$T(9,3) = 825$$

One approach: Triple count by picking one of three edges to place first, dividing problem into smaller problems.



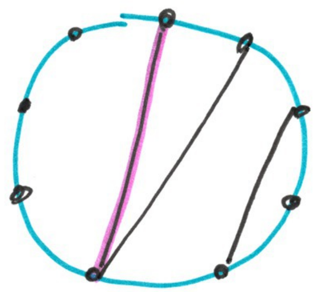
other 2 cuts on left:

$T(5,2) = 5$



one each:

$T(5,1)T(6,1)$
$= 5 \cdot 9 = 45$



both on right:

$T(6,2) = 21$

$$\underset{5 \cdot 1}{T(5,2)T(6,0)} + \underset{5 \cdot 9}{T(5,1)T(6,1)} + \underset{1 \cdot 21}{T(5,0)T(6,2)} = 71$$

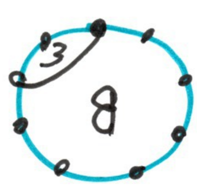
$T(5,-) \Rightarrow$

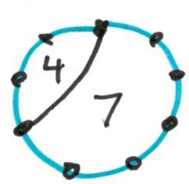
5	5	1
1	9	21

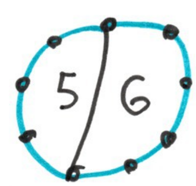
$\Leftarrow T(6,-)$

$5+45+21 = 71$

Now list all possibilities. 9 rotations of each.







9

0	0	1
1	20	120

120

+9

0	2	1
1	14	56

0 + 28 + 56 oops
~~85~~ 84

+9

5	5	1
1	9	21

5 + 45 + 21
71

$9(120 + \cancel{85}\ 84 + 71)/3 = \cancel{\ldots}\ 3 \cdot 275 = 825$ ✓

← undo the triple count

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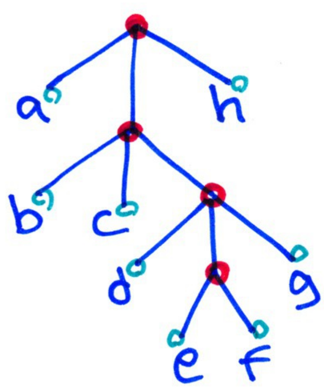
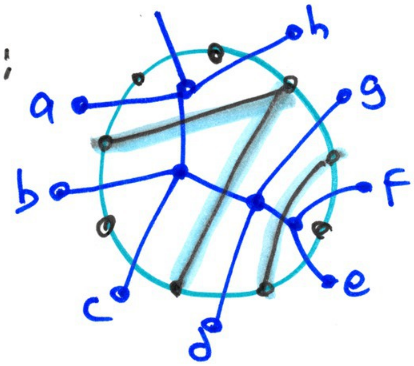
Another approach, trees.

$T(9,3)$ $\quad n-1 = 8$ leaves, $k+1$ inner nodes
$\quad k+1 = 4$ $\quad$ including root.

check by example:



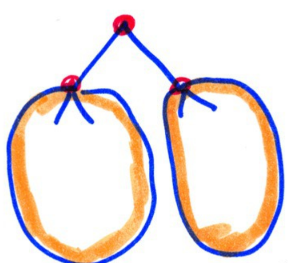


• 4 inner nodes ☑

∘ 8 leaves ☑

How can we break this apart?

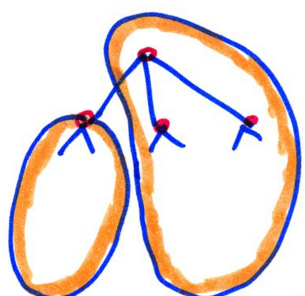
Two cases, root node is valence 2, or ≥ 3



$T(a,j)$ $\; T(b,\ell)$

$a+b = n+1$

$j+\ell = k-1$



$T(a,j)$ $\; T(b,\ell)$

$a+b = n+1$

$j+\ell = k$

$1 \leq a,b \leq n-1$

All these index shifts will drive us bonkers, so:

$\boxed{S(n,k) = \text{rooted trees, } n \text{ leaves, } k \text{ inner nodes}}$

$\quad \parallel$

$T(n+1, k-1)$

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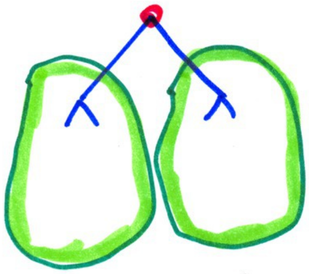
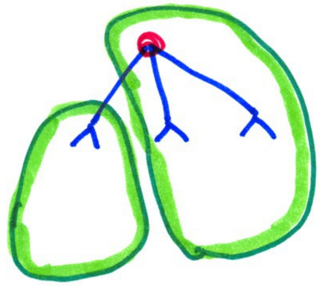



$S(n,k)$

n leaves

k inner nodes

$$S(a,i)\cdot S(b,j) + S(a,i)\cdot S(b,j)$$

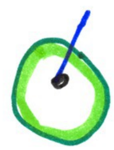
(sum over all ways)

$$\sum_{\substack{n=a+b\\k=i+j+1}} \qquad \sum_{\substack{n=a+b\\k=i+j}}$$

(all leaves found in one piece or other)

(did we add a new inner node?)

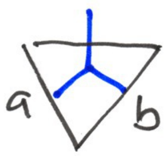
Extreme is adding single ~~leave~~ leaf.

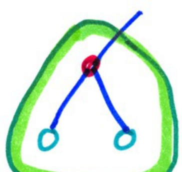


$S(1,0) = T(2,-1)$

wow

Next case is wishbone



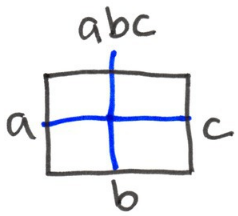


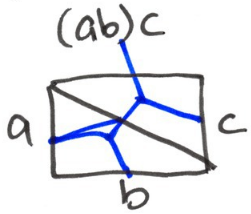
$S(2,1) = T(3,0)$

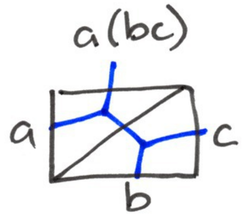
of course this is a Catalan number

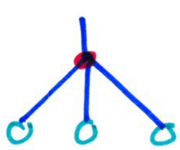
1, 1, 2, 5, 14, ..

Next cases are square

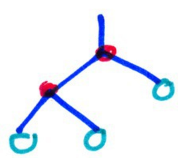
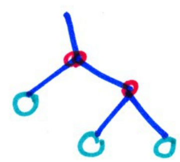








$S(3,1)=1$

$S(3,2) = 2$

Then

k	1	2	3	4
$S(4,k)$	1	5	5	
$S(5,k)$	1	9	21	14

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Let's practice, finding $S(5,3) = 21 = T(6,2)$

$b \backslash j$	0	1	2
1	1 $\frac{5}{5}$ 10	0 $\frac{1}{5}$	0 $\frac{0}{1}$
2	0 $\frac{2}{0}$	1 $\frac{1}{2}$ 3	0 $\frac{0}{1}$
3	0 $\frac{0}{0}$	1 $\frac{1}{0}$ 1	2 $\frac{0}{1}$ 2
4	0 $\frac{0}{0}$	1 $\frac{0}{0}$	5 $\frac{1}{0}$ 5

(If you program, this is much easier as code.)

$$S(b,j) \cdot \left(S(5-b, 2-j) + S(5-b, 3-j) \right) =$$

$10 + 3 + 1 + 2 + 5 = 21$ ✓

Now, redo $S(8,4) = 825 = T(9,3)$

$b \backslash j$	0	1	2	3
1	1 $\frac{120}{300}$ 420			
2		1 $\frac{14}{56}$ 70		
3		1 $\frac{9}{21}$ 30	2 $\frac{1}{9}$ 20	
4		1 $\frac{5}{5}$ 10	5 $\frac{1}{5}$ 30	5 1 5
5		1 2 2	9 $\frac{1}{2}$ 27	21 1 21
6		1	14 1 14	56 1 56
7		1	20	120 1 120

$$S(b,j)\left(S(8-b, 3-j) + S(8-b, 4-j) \right) =$$

$420 + 112 + 91 + 202 = \boxed{825}$ ✓

I learn from students. How would you do this!?