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How many ways can we draw arcs connecting vertices on perimeter of a circle, that don't cross?
(ok to use boundary arcs in this problem)

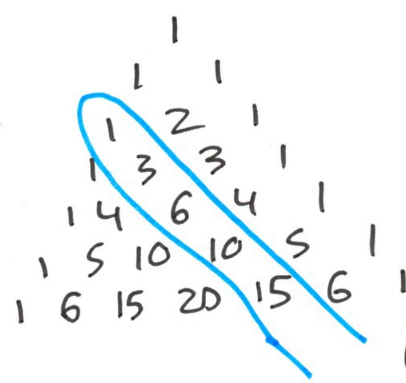
$n = 2$ vertices
$k = 1$ arc

$n=3$
$k=1$

$k=1$

In general for $k=1$, there are $\binom{n}{2}$ available arcs, we choose one.

n	2	3	4	5	6	...
$k=1$	1	3	6	10	15	



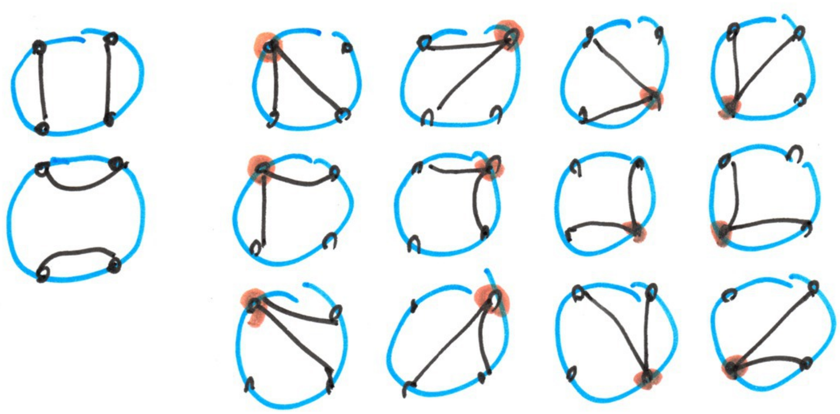
$\binom{n}{2}$ diagonal of pascal's triangle.

$k=2$

(note arcs can share a vertex in this version)

$n=3$

$n=4$



Two cases: use 3, or 4 vertices

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V=4 vertices: choose $\binom{n}{4}$ set of vertices,
2 out of 3 pairings don't cross

so $2\binom{n}{4}$

V=3 vertices.
2 approaches. [1] pick vertex where arcs meet, then subset for other ends

$$\underbrace{n\binom{n-1}{2}}_{\text{direct count}} = \underbrace{3\binom{n}{3}}_{\text{simplification. Huh?}} = \cancel{3}\,\frac{n(n-1)(n-2)}{\cancel{3}\cdot 2\cdot 1}$$

or [2] choose subset of three vertices, then which vertex has both arcs.

---

so $f(n) = 2\binom{n}{4} + 3\binom{n}{3}$

$f(n) = 2\binom{n}{4} + 3\binom{n}{3}$	n
$3 = 2\cdot 0 + 3\cdot 1$	3
$14 = 2\cdot 1 + 3\cdot 4$	4
$40 = 2\cdot 5 + 3\cdot 10$	5

K=2 alternate approach: choose any pair of arcs, then subtract those that cross

when we choose $\binom{n}{4}$ a set of 4 vertices, one of three pairings <u>do</u> cross.

$\binom{n}{2}$ ~~pairs of~~ arcs

$\binom{\binom{n}{2}}{2}$ pairs of arcs

$$\boxed{\binom{\binom{n}{2}}{2} - \binom{n}{4}}$$??

check this.

$n=5$: $\binom{\binom{n}{2}}{2} - \binom{n}{4} = \binom{10}{2} - \binom{5}{4} = 45 - 5 = 40$ ✓

In general,

$$\binom{\binom{n}{2}}{2} - \binom{n}{4} \stackrel{?}{=} 2\binom{n}{4} + 3\binom{n}{3}$$

$$\frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}-1\right)}{2} - \frac{n(n-1)(n-2)(n-3)}{24} \stackrel{?}{=} ////$$

This is a mess to reduce, but clearly both sides are degree 4 polynomials in n. If they agree on 5 values, they're identical.

n	0	1	2	3	4
$\binom{n}{2}$	0	0	1	3	6
$\binom{n}{3}$	0	0	0	1	4
$\binom{n}{4}$	0	0	0	0	1
$\binom{\binom{n}{2}}{2}$	0	0	0	3	15
$\binom{\binom{n}{2}}{2} - \binom{n}{4}$	0	0	0	3	14
$2\binom{n}{4}+3\binom{n}{3}$	0	0	0	3	14

} ✓

This is far easier for confirming formulas agree.

Note that polynomials in n can always be written as combinations of

$$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \binom{n}{3}\ \binom{n}{4} \ldots$$

so $\boxed{2\binom{n}{4}+3\binom{n}{3}}$ is a very nice way to write this

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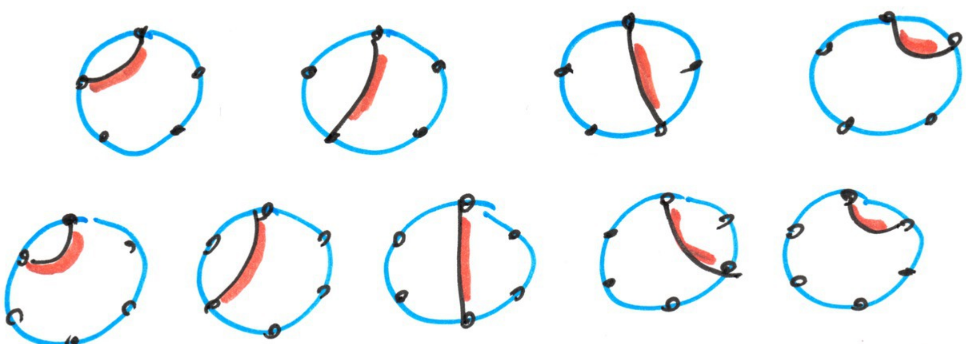
we want still more ways to do this, to have choices as the problem gets harder e.g. $k \geq 3$.

Over count: $k=2$ with a chosen edge counts each configuration twice.

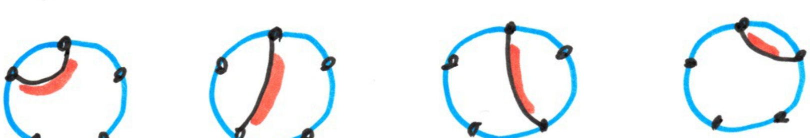
Chosen edge can be viewed as a cut into smaller problems, with smaller k.



Even cleaner (to avoid awkward middle case when n even) 4x overcount, pick first <u>arc</u> and make it sided.
(which way does fuzz face?)

Now we add a second arc on one side or other, taking care to not reuse same arc. m vertices on a half

m	2	3	4	5	6
$\binom{m}{2}-1$	0	2	5	9	14



\# 5 + 5 + 5 + 5 = $2\binom{5}{2}$ ✓

0+9 + 2+5 + 5+2 + 9+0 #

45 35 35 45 = $4 * 40$ ✓

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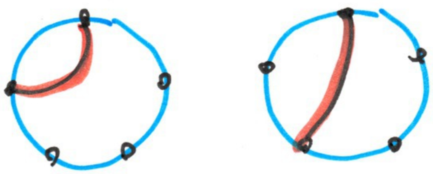
of course, middle arc still needs special care for n even, fewer of them.

So double count,



$\#$ 5 + 5 = $\binom{5}{2}$ choices for first cut

$\times$ 0+9   2+5   choices for second cut

45 + 35 = 80 ÷2 = **40** ✓

we are double counting

Try this for n= 6

$\#$ 6 + 6 + 3 = $\binom{6}{2} = 15$ choices for first cut

0+14   2+9   5+5

84   66   30 = 180 ÷2 = 90

double counting

check: $2\binom{6}{4} + 3\binom{6}{3} =$

$2\cdot 15 + 3\cdot 20 = 30 + 60 = 90$ ✓

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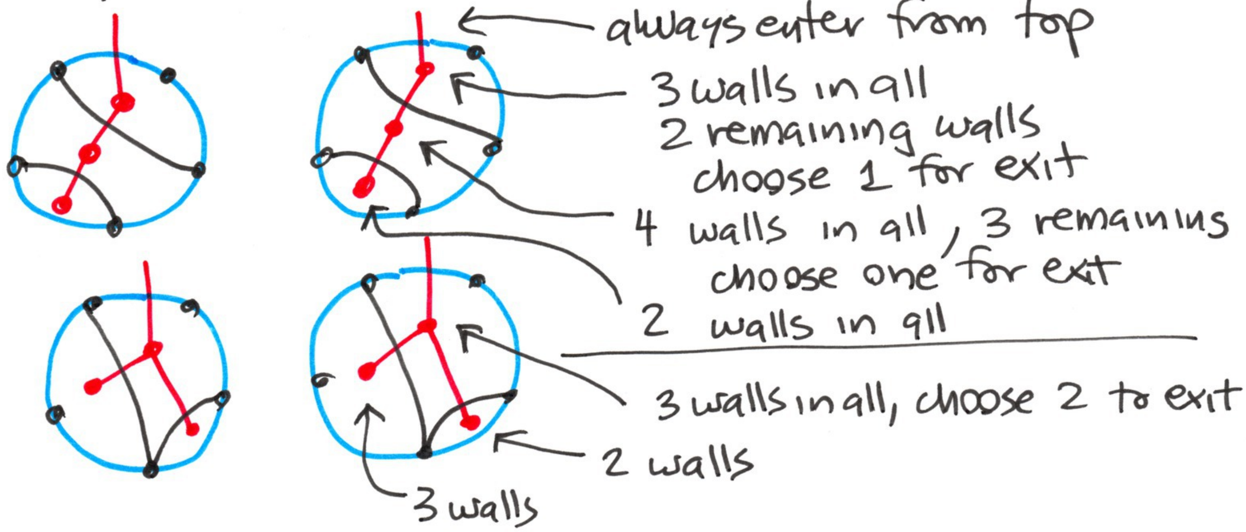
Yet another approach:

Draw a ~~tree~~ tree through diagram.

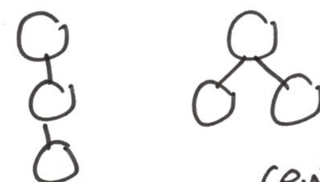
Classify then count possible trees.



$n = 5$ vertices, $k = 2$ cuts (interior rooms ≥ 3 walls)

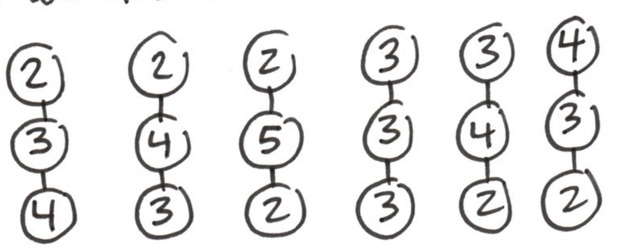
$5 + 4 = 9$ walls, in 3 rooms, each room ≥ 2 walls

Two basic types of trees:



(center at top)

If we add room sizes



1·2	1·3	1·4	2·2	2·3	3·2
2	3	4	4	6	6

25

( on next page)

Trick is now to count how many ways each diagram arises.

Exercise, I'm not fully explaining.

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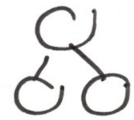
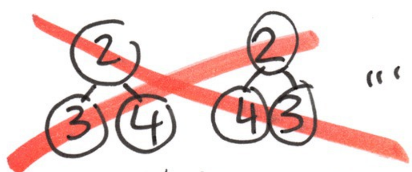
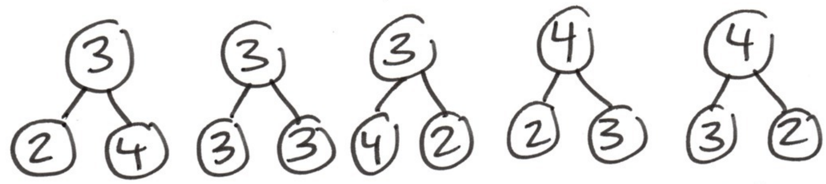
Now do other case:





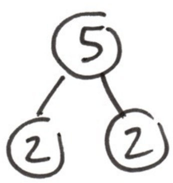
...

can't happen
we came in one wall,
need two more walls
to leave through



$\binom{2}{2}$ $\binom{2}{2}$ $\binom{2}{2}$ $\binom{3}{2}$ $\binom{3}{2}$

1 1 1 3 3



$\binom{4}{2}$

6

} new case we almost forgot

$1+1+1+3+3+6 = 15 \ldots +25 = \boxed{40}$ ✓

Exercise: Do $k=3$ cuts, by any method you can get to work.

Exercise: A version of this problem cuts n-gons, where we aren't allowed to use a side of the n-gon itself.

Exercise: A version of this problem has n even, but no two arcs can use the same vertex.

Web exercise: 3, 14, 40, 90
Look up our data in online encyclopedia of integer sequences (oeis.org)