Prize Exam 2002
Solutions

1. Let f be a smooth function defined on $(-\infty, \infty)$ satisfying $f''(x) = xf(x)$, $f(0) = 0$ and $f'(0) = 1$. Show that f is positive on $(0, \infty)$. What is the limit of f as x approaches ∞?

f increases rapidly to infinity.

Let Z be $\{x \geq 0 | f(x) = 0\}$. By the continuity of f, Z is closed. Since $f'(0) > 0$, 0 is an isolated point of Z, so $Y = Z \setminus \{0\}$ is also closed. We first want to show that Y is empty.

Assume not. Then there is a least element of Y, its infimum, $y_0 > 0$. By Rolle's Theorem, for some c, $0 < c < y_0$ and $f'(c) = 0$. However, since $f \geq 0$ on $(0, c)$, $(f')' = f'' = xf > 0$ on $(0, c)$, it must be that $-1 = f'(c) - f'(0) \geq 0$. This contradicts our assumption that Y is nonempty, so $f(x) > 0$ for all $x > 0$.

f increases to infinity since $f'' > 0$ for all $x > 0$, hence $f'(x) > f'(0) = 1$ for all $x > 0$.

2. Show that the maximum and minimum of $ax^2 + 2bxy + cy^2$ on the unit circle $x^2 + y^2 = 1$ are the eigenvalues of the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

The method of Lagrange multipliers shows that the critical points of f on a level set of g occur when the gradients of f and g are parallel, e.g., for some λ, $\nabla f = \lambda \nabla g$. In this case, we can take $g(x, y) = x^2 + y^2$ and $f(x, y) = ax^2 + 2bxy + cy^2$.

$\nabla f(x, y) = (2ax + 2by, 2bx + 2cy)$

$\nabla g(x, y) = (2x, 2y)$

The equation $(2ax + 2by, 2bx + 2cy) = \lambda(2x, 2y)$ may be rewritten

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

or

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

So the possible values of λ are the eigenvalues of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, and the corresponding values of (x, y) are the eigenvectors of length 1. Finally, we must verify that for a given solution λ corresponding to the extremum (x, y), $\lambda = ax^2 + 2bxy + cy^2$.

$ax^2 + 2bxy + cy^2 = x(ax + by) + y(bx + cy) = x(\lambda x) + y(\lambda y) = \lambda(x^2 + y^2) = \lambda$.

3. Consider a point P inside a regular n-gon. Let $d_1, \ldots d_n$ be the distances from P to the lines which define the sides of the n-gon. Show that $d_1 + d_2 + \ldots + d_n$ is independent of the choice of P.

In fact, this is true for all convex equilateral n-gons, not just those that are regular. Let s be the length of each side. Given P, triangulate the n-gon by adding line segments connecting P to the vertices. The area of the ith triangle is $\frac{1}{2}(base)(height) = \frac{1}{2} s d_i$. So the total area is $\frac{1}{2} s(d_1 + d_2 + \ldots + d_n)$. Since the area of the n-gon does not depend on P, neither does $d_1 + d_2 + \ldots + d_n$.

4. A triangle in the plane has side lengths a, b, c. Its vertices all lie on a circle of diameter d. Show that the area of the triangle is $\frac{abc}{2d}$.

The area of the triangle is $ab \sin(\theta)/2$, where θ is the angle between the sides of lengths a and b. If θ is inscribed in a circle, the measure of the opposing arc is 2θ radians. The length of the secant chord is both c and $2\sin(\theta)$ radians, so $c = 2\sin(\theta) r = \sin(\theta) d$. Putting these together produces the formula.

5. What is the convex hull of the graph of $y = x^3$? (The convex hull of a set is the smallest convex set containing the given set.)

The convex hull is the entire plane. To check this, it suffices to show that every point in the plane is contained in a line segment whose endpoints are on the graph. For any (a, b), choose m so that $b+m > (a+1)^3$ and $b - m < (a-1)^3$. Then the line ℓ through (a, b) of slope m is above the graph at $x = a+1$, and below the graph at $x = a - 1$. As x increases to infinity, x^3 increases more rapidly than any linear function, so there must be an intersection of ℓ with the graph for some point with $x > a+1$. Similarly, there must be an intersection of ℓ with the graph at some point with $x < a-1$. (a, b) is between these points, hence is in the convex hull of the graph of $y = x^3$.

6. Consider the 3×3 matrices with entries in $Z_{/2}$. How many have multiplicative inverses?

$168 = 7 * 6 * 4$. In fact, these matrices form the second smallest nonabelian simple group. The matrices with multiplicative inverses are precisely those whose rows are linearly independent. There are 7 choices for the first row r_1, every nonzero vector. For any choice of r_1, for r_2 to be linearly independent with r_1, r_2 can be anything except for r_1 and $(0,0,0)$, so there are 6 choices for r_2. For any choice of r_1 and r_2, there are 4 choices for r_3, as the forbidden vectors are those in the subspace spanned by r_1 and r_2, $\{(0,0,0), r_1, r_2, r_1 + r_2\}$.

7. Which rectangles can be tiled by 5×7 and 7×5 rectangles?

The possible dimensions are of the forms $5a \times 7b$, $7a \times 5b$, $35a \times (5m + 7n)$, and $(5m + 7n) \times 35a$, for a, b, m, n nonnegative integers. These are precisely the rectangles whose areas are divisible by 35 such that each side is expressible as a nonnegative linear combination of 5 and 7. The first condition is obviously necessary. The second condition follows from the fact that a tiling of a rectangle produces a tiling of each edge by intervals of length 5 and 7.

To show sufficiency in the first two cases is easy. To show that one can tile $35a \times (5m+7n)$, concatenate tilings of $35a \times 5m$ and $35a \times 7n$. Similarly tile the last case. So these conditions are necessary and sufficient.

8. What is the greatest common factor of all values of

$$(x-2)(x-1)^2x^3(x+1)^2(x+2), \quad x \text{ an integer?}$$

The greatest common factor is the value at $x = 3$, $2^6 3^3 5 = 8640$. Certainly any common factor must divide this, so it suffices to prove that every value of the polynomial at an integer x must be divisible by $2^6 3^3 5$. This can be checked prime by prime: 5 is a factor of $(x-2)(x-1)(x)(x+1)(x+2)$, 3 is a factor of $(x-2)(x-1)(x)$, $(x-1)(x)(x+1)$, and $(x)(x+1)(x+2)$, and 8 is a factor of $(x-2)(x-1)(x)(x+1)$ (as it is 24 times $x+1$ choose 4) and $(x-1)(x)(x+1)(x+2)$. So $2^6 3^3 5$ is the greatest common factor of $(x-2)(x-1)^2(x)^3(x+1)^2(x+2)$.

9. Show that if a and b are between -10^{10} and 10^{10} and not both 0, then $|a + b\sqrt[3]{2}| > 10^{-100}$. You may assume that $\sqrt[3]{2}$ is irrational.

Suppose $a + b\sqrt[3]{2} = \epsilon$. We want to show that $|\epsilon| > 10^{-100}$.

Method 1:

$b\sqrt[3]{2} = -a + \epsilon$

$2b^3 = -a^3 + 3a^2\epsilon - 3a\epsilon^2 + \epsilon^3$

$2b^3 + a^3 = \epsilon(3a^2 - 3a\epsilon + \epsilon^2)$

The left hand side is an integer, hence has magnitude at least 1 since it is not 0. Since

$|3a^2 - 3a\epsilon + \epsilon^2| < 10^{21}$ if ϵ is small,

$|\epsilon| > 10^{-21} > 10^{-100}$.

Method 2:

Let ω be a complex cube root of 1. Consider

$(a + b\sqrt[3]{2})(a + \omega b\sqrt[3]{2})(a + \omega^2 b\sqrt[3]{2}) = a^3 + 2b^3$.

The right hand side is a nonzero integer, hence has magnitude at least 1. The magnitudes of $(a + \omega b\sqrt[3]{2})$ and $(a + \omega^2 b\sqrt[3]{2})$ are each at most 10^{11}, so $|\epsilon| > 10^{-22} > 10^{-100}$.