

Columbia-Barnard
MATHEMATICS PRIZE EXAM

March 26, 2002

Please print your name:

Indicate school:

☐ First-year ☐ Sophomore ☐ Junior ☐ Senior

Expected date of graduation:

This is a three-hour exam. Please print your name on each booklet that you hand in. Submit your paper even if you have done no more than one or two problems. However, work as many problems as you can because partial credit will be given for significant progress made on a problem.

Mathematics Prize Exam
Tuesday, March 26, 2002

1. Let f be a smooth function defined on $(-\infty, \infty)$ satisfying $f''(x) = xf(x)$, $f(0) = 0$ and $f'(0) = 1$. Show that f is positive on $(0, \infty)$. What is the limit of f as x approaches ∞ ?
2. Show that the maximum and minimum of $ax^2 + 2bxy + cy^2$ on the unit circle $x^2 + y^2 = 1$ are the eigenvalues of the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.
3. Consider a point P inside a regular n -gon. Let d_1, \dots, d_n be the distances from P to the lines which define the sides of the n -gon. Show that $d_1 + d_2 + \dots + d_n$ is independent of the choice of P .
4. A triangle in the plane has side lengths a, b, c . Its vertices all lie on a circle of diameter d . Show that the area of the triangle is $\frac{abc}{2d}$.
5. What is the convex hull of the graph of $y = x^3$? (The convex hull of a set is the smallest convex set containing the given set.)
6. Consider the 3×3 matrices with entries in $\mathbb{Z}/2$. How many have multiplicative inverses?
7. Which rectangles can be tiled by 5×7 and 7×5 rectangles?
8. What is the greatest common factor of all values of

$$(x-2)(x-1)^2x^3(x+1)^2(x+2), \quad x \text{ an integer?}$$

9. Show that if a and b are between -10^{10} and 10^{10} and not both 0, then $|a + b\sqrt[3]{2}| > 10^{-100}$. You may assume that $\sqrt[3]{2}$ is irrational.

End of Exam