$\boxed{\text{27 Sep 2016}}$ Using change of coords to find a matrix

Find A so $A\begin{bmatrix}1\\-1\end{bmatrix} = 2\begin{bmatrix}1\\-1\end{bmatrix}$ $A\begin{bmatrix}1\\2\end{bmatrix} = -1\begin{bmatrix}1\\2\end{bmatrix}$

⊛ "Reversing the check"

check: $\underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{A}\begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}2 & -1\\-2 & -2\end{bmatrix}$

$$A = \begin{bmatrix}2 & -1\\-2 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix}^{-1}$$

$\begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}/3 = \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}/3$

$$A = \begin{bmatrix}2 & -1\\-2 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}/3 = \begin{bmatrix}3 & -3\\-6 & 0\end{bmatrix}/3 = \begin{bmatrix}1 & -1\\-2 & 0\end{bmatrix}$$

$\begin{bmatrix}a & c\\b & d\end{bmatrix}^{-1} = \begin{bmatrix}d & -c\\-b & a\end{bmatrix}/(ad-bc)$

⊛ $S = \{e_1, e_2\}$ standard coords

$V = \{v_1, v_2\}$ new coords

$\begin{bmatrix}a\\b\end{bmatrix}_S = ae_1 + be_2 = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$ standard coords implied

$\begin{bmatrix}a\\b\end{bmatrix}_V = av_1 + bv_2 = a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}1\\2\end{bmatrix}$

we now mark which coords we're using, so we can switch

$$V = \begin{cases} v_1 = \begin{bmatrix}1\\-1\end{bmatrix}_S = \begin{bmatrix}1\\0\end{bmatrix}_V \\ v_2 = \begin{bmatrix}1\\2\end{bmatrix}_S = \begin{bmatrix}0\\1\end{bmatrix}_V \end{cases}$$

L is our map

$L : \mathbb{R}^2 \to \mathbb{R}^2$

$Av_1 = 2v_1 \quad Av_2 = -v_2$

$$\underset{S\leftarrow S}{\overset{L}{\begin{bmatrix}1 & -1\\-2 & 0\end{bmatrix}}} = \underset{S\leftarrow V}{\overset{Id}{\begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix}}}\underset{V\leftarrow V}{\overset{L}{\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}}}\underset{V\leftarrow S}{\overset{Id}{\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}/3}}$$

abstract maps

which coords? out ← in

$$A = \underset{S\leftarrow V}{\overset{L}{\begin{bmatrix} 2 & -1 \\ -2 & -2 \end{bmatrix}}} \underset{V\leftarrow S}{\overset{Id}{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}/3}}$$

We should use L here
A is L in S-coords
D is L in V-coords

$$L\begin{bmatrix} 1 \\ -1 \end{bmatrix}_S = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}_S \qquad L\begin{bmatrix} 1 \\ 0 \end{bmatrix}_V = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}_V$$

$$L\begin{bmatrix} 1 \\ 2 \end{bmatrix}_S = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix}_S \qquad L\begin{bmatrix} 0 \\ 1 \end{bmatrix}_V = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix}_V$$

oops

not S!

$$\begin{bmatrix} r \\ s \end{bmatrix}_V = rv_1 + sv_2 \qquad L\begin{bmatrix} r \\ s \end{bmatrix} = rL\begin{bmatrix} 1 \\ 0 \end{bmatrix}_V + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}_V$$

$$= r\begin{bmatrix} 2 \\ 0 \end{bmatrix}_V + s\begin{bmatrix} 0 \\ -1 \end{bmatrix}_V$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 2r \\ -s \end{bmatrix}$$

This showed us the change of coord was same as check.

effect on	A	e^{At}	I	$=J$	$+K$	$2J-K$ $=A$
$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$	2	e^{2t}	1	1	0	2
$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$	-1	e^{-t}	1	0	1	-1
in V coords	$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ (2,-1)		$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (1,1)	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (1,0)	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (0,1)	$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ (2,-1)

$$J\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad J\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$J\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}/3 = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}/3$$

$$J : \mathbb{R}^2 \to \mathbb{R}^2$$

rank 1 matrix

*	2	-1
1	2	-1
-1	-2	1

$$\underbrace{\underset{2\times 1}{\begin{bmatrix}1\\-1\end{bmatrix}}\underset{1\times 2}{\begin{bmatrix}2&-1\end{bmatrix}}}_{2\times 2}$$

Image of J is all multiples of $\begin{bmatrix}1\\-1\end{bmatrix}$

$$J\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}2&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$J\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}2&-1\end{bmatrix}/3\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

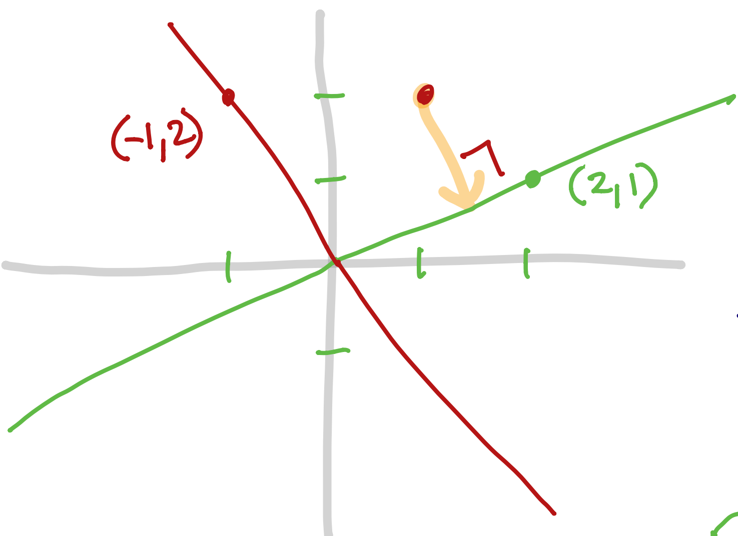
$$J\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}2&-1\end{bmatrix}/3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$K = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix}/3 = \begin{bmatrix}1&1\\2&2\end{bmatrix}/3$$

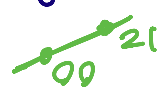
$$\underset{J}{\begin{bmatrix}2&-1\\-2&1\end{bmatrix}/3} + \underset{K}{\begin{bmatrix}1&1\\2&2\end{bmatrix}/3} = \begin{bmatrix}3&0\\0&3\end{bmatrix}/3 = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I \quad ✓$$

$$\underset{A}{\begin{bmatrix}1&-1\\-2&0\end{bmatrix}} = 2\,\underset{J}{\begin{bmatrix}2&-1\\-2&1\end{bmatrix}/3} - 1\,\underset{K}{\begin{bmatrix}1&1\\2&2\end{bmatrix}/3} = \begin{bmatrix}3&-3\\-6&0\end{bmatrix}/3 \quad ✓$$

$$e^{At} = e^{2t}\,J + e^{-t}\,K$$
$$A = 2\,J + (-1)\,K$$



find B so B projects $\perp$ onto this line



$$B\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \qquad Bv_1 = v_1$$

$$B\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad Bv_2 = 0v_2$$

$$B = \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2&1\end{bmatrix}/5 = \begin{bmatrix}4&2\\2&1\end{bmatrix}/5$$

$$\begin{bmatrix}4&2\\2&1\end{bmatrix}/5\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix}/5 = \begin{bmatrix}2\\1\end{bmatrix} \; ✓$$

$$\begin{bmatrix}4&2\\2&1\end{bmatrix}/5\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}/5 = \begin{bmatrix}0\\0\end{bmatrix} \; ✓$$

$$\boxed{A = rJ + sK}$$

What it means to be a basis.

W vector space — can add two vectors
multiply by scalars
rules work as expected

vectors in W:
$V = \{v_1, v_2, \dots v_n\}$ is a basis for W

$\iff$ every $w \in W$ can be written

$w = r_1 v_1 + r_2 v_2 + \dots + r_n v_n$ in exactly one way

$\iff$ can write any $w = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}_V$

$\iff$ $\mathbb{R}^n \longrightarrow W$ is a bijection 1:1 onto

$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \longmapsto r_1 v_1 + r_2 v_2 + \dots + r_n v_n$

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example: all real polynomials in x of degree ≤ 2

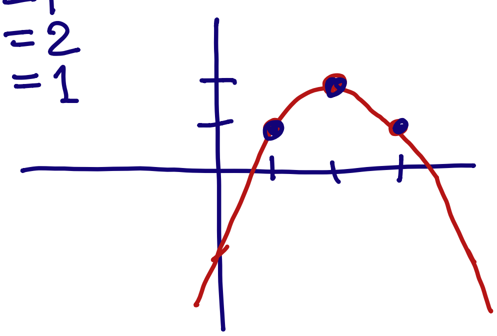
$\mathbb{R}^3 \cong W = \{ a + bx + cx^2 \}$ for all $a, b, c \in \mathbb{R}$

$V = \begin{cases} v_1 = 1 \\ v_2 = x \\ v_3 = x^2 \end{cases}$ $\qquad \mathbb{R}^3 \longrightarrow W$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto a + bx + cx^2$

Find $f(x)$ of degree ≤ 2 so

$f(1) = 1$
$f(2) = 2$
$f(3) = 1$



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

we can solve for a, b, c

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 9 & 1 \end{array}\right] \xrightarrow[\textcircled{3} \leftarrow \textcircled{3} - \textcircled{1}]{\textcircled{2} \leftarrow \textcircled{2} - \textcircled{1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 0 \end{array}\right] \xrightarrow{\textcircled{3} \leftarrow \textcircled{3} - 2\textcircled{2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{\textcircled{3} \leftarrow \frac{1}{2}\textcircled{3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[\textcircled{2} \leftarrow \textcircled{2} - 3\textcircled{3}]{\textcircled{1} \leftarrow \textcircled{1} - \textcircled{3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{\textcircled{1} \leftarrow \textcircled{1} - \textcircled{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \begin{array}{l} a = -2 \\ b = 4 \\ c = -1 \end{array}$$

$f(x) = -2 + 4x - x^2$

$f(1) = -2 + 4 - 1 = 1$ ✓

$f(2) = -2 + 8 - 4 = 2$ ✓

$f(3) = -2 + 12 - 9 = 1$ ✓

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basis:
- ☐ every w can be written at least one way
  $\iff$ basis spans W
- ☐ every w can be written at most one way
  $\iff$ basis is linearly independent
- ☐ every w can be written exactly one way

spans — we can write any $w \in W$
as $w = r_1 v_1 + \dots + r_n v_n$ somehow

$$w = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}_V \qquad \text{at least one way}$$

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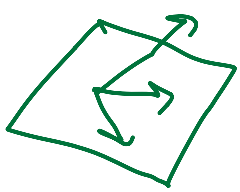
example in $\mathbb{R}^3$ $\quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ spans $\mathbb{R}^3$



third vector sticks out of plane

eg

$$r\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - r\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ -r \end{bmatrix}$$

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second example in $\mathbb{R}^3$ — don't span $\mathbb{R}^3$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

third vector in plane of first two

$\boxed{\text{linearly dependent}}$

$$r_1v_1 + r_2v_2 + \dots + r_nv_n = 0 \quad \text{not all } r_i = 0$$

$$0v_1 + 0v_2 + \dots + 0v_n = 0$$

two ways to write zero

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_V = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}_V$$

two ways to write zero
$\Rightarrow$ we're got a problem

$$\underset{v_1, v_2, \dots v_n}{\mathbb{R}^n} \longrightarrow W$$

if collapsing, seen at origin

$$\begin{aligned} + \ & (w = r_1v_1 + r_2v_2 + \dots + r_nv_n) \\ - \ & (w = s_1v_1 + s_2v_2 + \dots + s_nv_n) \end{aligned} \quad \text{and } r_1 \neq s_1$$

subtract $\quad 0 = \underbrace{(r_1 - s_1)}_{\text{not zero}}v_1 + (r_2 - s_2)v_2 + \dots + (r_n - s_n)v_n$