Sep 22, 2016

Reversing the problem of solving a system of equations

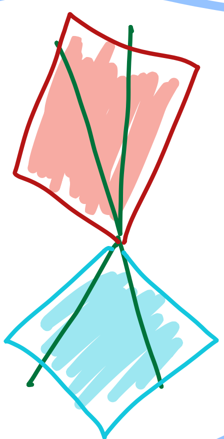
Intersect two affine subspaces

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \Bigg(\overbrace{\underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}}_{\text{particular}} + \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 4 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}_{\text{homogeneous}}}^{(w,x,y,z)} \Bigg) = \begin{bmatrix} \quad \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & -4 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 4 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

transpose

$$\begin{bmatrix} 2 & 1 & 4 & 0 \\ 3 & 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -4 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



$\mathbb{R}^4$

$$② \leftarrow ② - ①$$

$$\begin{bmatrix} 2 & 1 & 4 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$1 \quad -2 \quad -1 \quad = 0$$

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Franklin Barbeau

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & -4 & 1 & -1 \end{bmatrix} \Bigg(\overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 4 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}^{(w,x,y,z)} \Bigg) = \begin{bmatrix} -6 \\ -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$$

problem was to solve

[1]

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & -4 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$$

Practice Problems
Feb 12, 2011
[1], [5]

[5]

codim 2

V | W

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 0 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

oops

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

parametrizations

equs

we want $V \cap W$ intersection

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} i \\ j \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} i \\ j \\ 4 \\ 0 \end{bmatrix}$$

stack both sets of equations, and solve

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 0 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

set both parametrizations equal to each other

$$\leadsto \begin{bmatrix} -2 & -1 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

solutions tell us how to move (s,t) and (p,q) together to trace out same points in intersection

plug in parametrization for V into equations for W

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 0 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

plug in

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 0 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix}4\\0\end{bmatrix} + \begin{bmatrix}0&0\\-4&4\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix} \qquad \begin{bmatrix}s\\t\end{bmatrix} = \underbrace{\begin{bmatrix}0\\0\end{bmatrix}}_{\text{particular}} + \underbrace{\begin{bmatrix}1\\1\end{bmatrix}r}_{\text{homogeneous}}$$

$$\begin{bmatrix}0&0\\-4&4\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

disregard

plug in to our parametrization

$$\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\0\\0\\2\end{bmatrix} + \begin{bmatrix}-2&-1\\3&0\\0&3\\-1&-2\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix} \qquad \begin{bmatrix}s\\t\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}r$$

plug in

$$\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\0\\0\\2\end{bmatrix} + \begin{bmatrix}-2&-1\\3&0\\0&3\\-1&-2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}r$$

$$\boxed{\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\0\\0\\2\end{bmatrix} + \begin{bmatrix}-3\\3\\3\\-3\end{bmatrix}r}$$

} common line in V, W

we see in W by $\begin{bmatrix}s\\t\end{bmatrix} = \begin{bmatrix}r\\r\end{bmatrix}$ ✓

Is our line also in V? check by plugging in

$$\begin{bmatrix}1&1&1&1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix} \qquad \begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\0\\0\\2\end{bmatrix} + \begin{bmatrix}-3\\3\\3\\-3\end{bmatrix}r$$

plug in

$$\begin{bmatrix}1&1&1&1\\1&-1&1&-1\end{bmatrix}\left(\begin{bmatrix}2\\0\\0\\2\end{bmatrix} + \begin{bmatrix}-3\\3\\3\\-3\end{bmatrix}r\right) \stackrel{?}{=} \begin{bmatrix}4\\0\end{bmatrix}$$

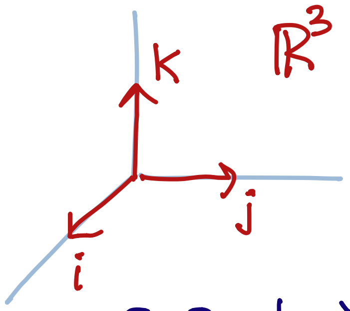
$$\begin{bmatrix}4\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix}r = \begin{bmatrix}4\\0\end{bmatrix} \quad ✓$$

44:00

linear independence

basis for a vector space

we want the freedom to use any coordinates we want

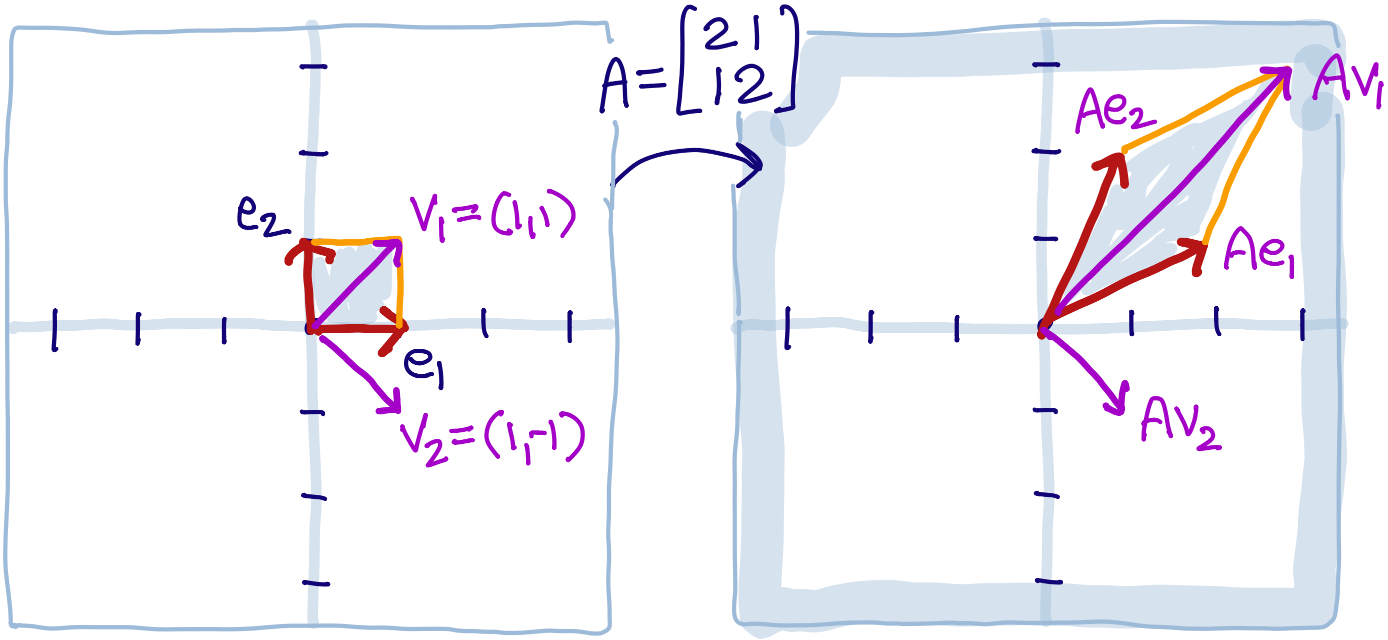


$\mathbb{R}^3$

$$(2,3,4) = (2,3,4)_S = \underbrace{2i+3j+4k}_{\text{what our notation means}}$$

so we remember we're in standard coords

S for standard coords



$L:\mathbb{R}^2\to\mathbb{R}^2$ is abstract map. A is matrix in S-coordinates

D is matrix in V-coordinates

$$Le_1 = 2e_1+e_2$$
$$Le_2 + e_1+2e_2$$

L what map

$$A=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

$S \leftarrow S$

output language | input language

$$Lv_1 = 3v_1$$
$$Lv_2 = 1v_2$$

L what map

$$D=\begin{bmatrix}3&0\\0&1\end{bmatrix}$$

$V \leftarrow V$

output language | input language

write using $\{e_1, e_2\} = S$

notation $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_S = (2,1)_S = 2e_1 + e_2$

---

write using $\{v_1, v_2\} = V$

notation $\begin{bmatrix} -1 \\ 3 \end{bmatrix}_V = (-1,3)_V = -v_1 + 3v_2$

---

linear map determined by what it does to a basis.

(once we know effect on two vectors, we know all linear combinations.)

Again we use L for abstract map
A in S-coords
D in V-coords

$$Lv_1 = 3v_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}_V$$

$$Lv_2 = v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_V$$

$$L(rv_1 + sv_2) = rLv_1 + sLv_2 = r\begin{bmatrix} 3 \\ 0 \end{bmatrix}_V + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}_V$$

$$L\begin{bmatrix} r \\ s \end{bmatrix}_V = \underbrace{r\begin{bmatrix} 3 \\ 0 \end{bmatrix}_V + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}_V}_{\text{exploded view of matrix multiply}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

L what map

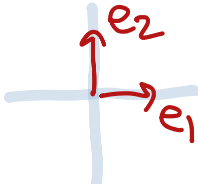
$$D = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{V \leftarrow V}\underbrace{\begin{bmatrix} r \\ s \end{bmatrix}}_{V}$$

output language

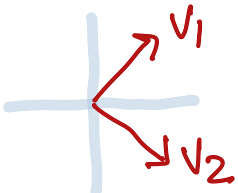
input language

let's call this matrix representation for the linear map L in V-coordinates "D"

$S \begin{cases} e_1 \\ e_2 \end{cases}$



$V \begin{cases} v_1 \\ v_2 \end{cases}$



How to convert from V to S?

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix}_V = \begin{bmatrix}1\\1\end{bmatrix}_S$$

$$v_2 = \begin{bmatrix}0\\1\end{bmatrix}_V = \begin{bmatrix}1\\-1\end{bmatrix}_S$$

call this "Id" for identity

translate V to S

$$\underset{V}{\begin{bmatrix}a\\b\end{bmatrix}} = \underset{V}{\begin{bmatrix}1\\0\end{bmatrix}}a + \underset{V}{\begin{bmatrix}0\\1\end{bmatrix}}b = \underset{S}{\begin{bmatrix}1\\1\end{bmatrix}}a + \underset{S}{\begin{bmatrix}1\\-1\end{bmatrix}}b = \underset{S\leftarrow V}{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}\underset{V}{\begin{bmatrix}a\\b\end{bmatrix}}$$

translate V to S: $\underset{S\leftarrow V}{\overset{Id}{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}} = C$

do L using S coords: $\underset{S\leftarrow S}{\overset{L}{\begin{bmatrix}2&1\\1&2\end{bmatrix}}} = A$

do L using V coords: $\underset{V\leftarrow V}{\overset{L}{\begin{bmatrix}3&0\\0&1\end{bmatrix}}} = D$

$$\underset{\substack{S\leftarrow S\\ A}}{\overset{L}{\begin{bmatrix}2&1\\1&2\end{bmatrix}}} = \underset{\substack{S\leftarrow V\\ C}}{\overset{Id}{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}}\underset{\substack{V\leftarrow V\\ D}}{\overset{L}{\begin{bmatrix}3&0\\0&1\end{bmatrix}}}\underset{\substack{V\leftarrow S\\ C^{-1}}}{\overset{Id}{\begin{bmatrix}+1&+1\\+1&-1\end{bmatrix}}}/(\div 2)$$

check:

$$\underset{\substack{S\leftarrow V\\ C}}{\overset{Id}{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}}\underset{\substack{V\leftarrow S\\ C^{-1}}}{\overset{Id}{\begin{bmatrix}+1&+1\\+1&-1\end{bmatrix}}}/(\div 2) = \begin{bmatrix}2&0\\0&2\end{bmatrix}/2 = \begin{bmatrix}1&0\\0&1\end{bmatrix} \checkmark$$

check:

$$\underset{\substack{S\leftarrow S\\ A}}{\overset{L}{\begin{bmatrix}2&1\\1&2\end{bmatrix}}} = \underset{\substack{S\leftarrow V\\ C}}{\overset{Id}{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}} \circ \underbrace{\underset{\substack{V\leftarrow V\\ D}}{\overset{L}{\begin{bmatrix}3&0\\0&1\end{bmatrix}}} \circ \underset{\substack{V\leftarrow S\\ C^{-1}}}{\overset{Id}{\begin{bmatrix}+1&+1\\+1&-1\end{bmatrix}}}/(\div 2)}_{\underset{V\leftarrow S}{\overset{L}{\begin{bmatrix}3&3\\1&-1\end{bmatrix}}}/2}$$

$$✓\quad \underset{S\leftarrow S}{\overset{L}{\begin{bmatrix}4&2\\2&4\end{bmatrix}}}/2$$

e^{At} for a matrix : goal at end of course

$$e^x = 1 + x + x^2/2 + x^3/6 + \dots$$

need formula for A^n

$$A = CDC^{-1}$$
$$A^2 = CD\boxed{\cancel{C^{-1}C}}_{I}DC^{-1}$$
$$= CD^2C^{-1}$$

$$A^n = \underbrace{CDC^{-1}}_{1}\underbrace{CDC^{-1}}_{2}\underbrace{CDC^{-1}}_{3}\dots\underbrace{CDC^{-1}}_{n} = CD^nC^{-1}$$

$$e^{At} = I + At + A^2t^2/2 + A^3t^3/6 + \dots$$

use our formula for A^n to compute e^{At}