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Cross product

$$(a,b,c)\times(d,e,f) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{bmatrix} = \begin{matrix} +\mathbf{i}\begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\mathbf{j}\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ +\mathbf{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{matrix} = \left(\begin{vmatrix} b & c \\ e & f \end{vmatrix}, -\begin{vmatrix} a & c \\ d & f \end{vmatrix}, \begin{vmatrix} a & b \\ d & e \end{vmatrix}\right)$$

$$r\mathbf{i} + s\mathbf{j} + t\mathbf{k} = (r,s,t)$$

Dot product $(x,y,z)\cdot(r,s,t) = rx + sy + tz$

$$(x,y,z)\cdot(r\mathbf{i} + s\mathbf{j} + t\mathbf{k})$$

$$= (x,y,z)\cdot(r\overset{x}{\mathbf{i}} + s\overset{y}{\mathbf{j}} + t\overset{z}{\mathbf{k}}) = rx + sy + tz$$

Triple product $(x,y,z)\cdot[(a,b,c)\times(d,e,f)]$

$$(x,y,z)\cdot\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{bmatrix} = (x,y,z)\cdot\det\begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{bmatrix} \end{array} = \det\begin{bmatrix} x & y & z \\ a & b & c \\ d & e & f \end{bmatrix}$$

Discussion of determinant as signed volume.
Cross product as vector to measure height,
scaled to multiply by base area.

Cramer's rule:

Solve for y

$$\underbrace{\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & r & g \\ b & s & h \\ c & t & i \end{vmatrix}}{\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}} = (r,s,t) \cdot \frac{\begin{vmatrix} a & \mathbf{i} & g \\ b & \mathbf{j} & h \\ c & \mathbf{k} & i \end{vmatrix}}{\det(A)}$$

$$= (r,s,t) \cdot \left(-\begin{vmatrix} b & h \\ c & i \end{vmatrix}, +\begin{vmatrix} a & g \\ c & i \end{vmatrix}, -\begin{vmatrix} a & g \\ b & h \end{vmatrix} \right) \Big/ \det(A)$$

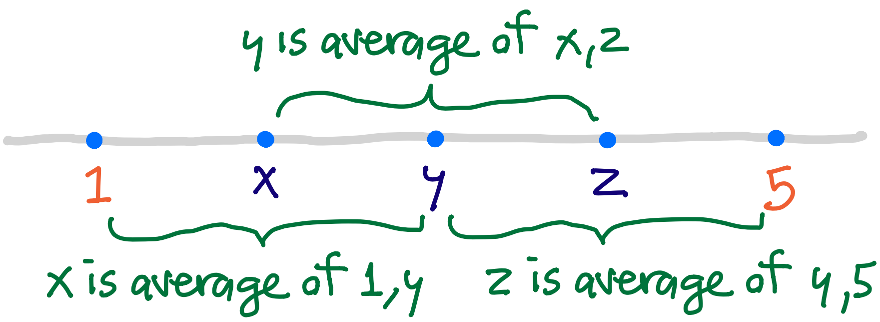
Or (another way to use same minors):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}^{-1} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} e & h \\ f & i \end{vmatrix} & -\begin{vmatrix} d & g \\ f & i \end{vmatrix} & +\begin{vmatrix} d & g \\ e & h \end{vmatrix} \\ -\begin{vmatrix} b & h \\ c & i \end{vmatrix} & +\begin{vmatrix} a & g \\ c & i \end{vmatrix} & -\begin{vmatrix} a & g \\ b & h \end{vmatrix} \\ +\begin{vmatrix} b & e \\ c & f \end{vmatrix} & -\begin{vmatrix} a & d \\ c & f \end{vmatrix} & +\begin{vmatrix} a & d \\ b & e \end{vmatrix} \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} \Big/ \det(A)$$

$$= (r,s,t) \cdot \left(-\begin{vmatrix} b & h \\ c & i \end{vmatrix}, +\begin{vmatrix} a & g \\ c & i \end{vmatrix}, -\begin{vmatrix} a & g \\ b & h \end{vmatrix} \right) \Big/ \det(A)$$

Example of Cramer's rule

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \qquad \left\{\begin{aligned} 2x - y &= 1 \\ -x + 2y - z &= 0 \\ -y + 2z &= 5 \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= (1+y)/2 \\ y &= (x+z)/2 \\ z &= (y+5)/2 \end{aligned}\right\}$$



$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\frac{y}{z} = \frac{\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 5 & 2 \end{bmatrix} \big/ |A|}{\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 5 \end{bmatrix} \big/ |A|} = \frac{2\overbrace{\begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix}}^{5} + 1\overbrace{\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix}}^{2}}{2\underbrace{\begin{vmatrix} 2 & 0 \\ -1 & 5 \end{vmatrix}}_{20} + 1\underbrace{\begin{vmatrix} -1 & 1 \\ -1 & 5 \end{vmatrix}}_{-4}} = \frac{12}{16} = \frac{3}{4} \ \boxed{\checkmark}$$

expand down first columns

Any linear map $\mathbb{R}^2 \to \mathbb{R}^n$ is a matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix} \qquad L\begin{bmatrix} r \\ s \end{bmatrix} = y$$

$$L\begin{bmatrix} r \\ s \end{bmatrix} = \frac{\begin{vmatrix} a & r \\ b & s \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} = \frac{-rb + sa}{ad - bc} = \frac{\begin{bmatrix} -b & a \end{bmatrix}}{ad - bc}\begin{bmatrix} r \\ s \end{bmatrix}$$

$$L\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \Big/ ad-bc \begin{bmatrix} r \\ s \end{bmatrix} = \frac{\begin{bmatrix} -b & a \end{bmatrix}}{ad - bc}\begin{bmatrix} r \\ s \end{bmatrix}$$

Rules for determinant	Rules for row reduction
• $\det(I) = 1$	• end with I
• swap = negate	• swap two rows
• linear in each row (col)	• multiply row by r add multiple of one row to another

$$\begin{vmatrix} a & b & c \\ d & e & f \\ rg & rh & ri \end{vmatrix} = r \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

determinant is linear in any row or column

$$\underbrace{\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}}_{16-4=12} = 2 \underbrace{\begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix}}_{2(8-2)=12} = 2 \cdot 2 \underbrace{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}_{2 \cdot 2 \cdot 3 = 12}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 + r1 \end{array} \begin{vmatrix} a & b & c \\ d & e & f \\ \underbrace{g+ra \quad h+rb \quad i+rc}_{[g\ h\ i] + r[a\ b\ c]} \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + r \cancel{\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix}}$$

same

So this step doesn't change the determinant

$$\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \xrightarrow{2 \leftarrow 2 - 1} \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} \xrightarrow{1 \leftrightarrow 2} -\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} \xrightarrow{2 \leftarrow 2 - 2 \cdot 1} -\begin{vmatrix} 1 & 4 \\ 0 & -7 \end{vmatrix} \xrightarrow{2 \leftarrow -\frac{1}{7} 2} 7\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} \xrightarrow{1 \leftarrow 1 - 4 \cdot 2} 7\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

7	7	7	7	7	7

$\boxed{4\times4}$

$$\begin{vmatrix}3&3&3&0\\3&3&0&3\\3&0&3&3\\0&3&3&3\end{vmatrix} = 3^4\begin{vmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{vmatrix} = 3^4\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix}$$

$$= 3^4\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\3&3&3&3\end{vmatrix} = 3^5\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&1\end{vmatrix} = 3^5\begin{vmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\1&1&1&1\end{vmatrix}$$

$$= -3^5\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&1&1&1\end{vmatrix} = -3^5\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix} = -3^5$$

$\boxed{n\times n}$

$$\begin{vmatrix}3&3&\cdots&3&0\\3&3&\cdots&0&3\\\vdots&\vdots&&\vdots&\vdots\\3&0&\cdots&3&3\\0&3&\cdots&3&3\end{vmatrix} = \boxed{3^n}\begin{vmatrix}1&1&\cdots&1&0\\1&1&\cdots&0&1\\\vdots&\vdots&&\vdots&\vdots\\1&0&\cdots&1&1\\0&1&\cdots&1&1\end{vmatrix} = \boxed{(-1)^{\lfloor\frac{n}{2}\rfloor}}\,\boxed{3^n}\begin{vmatrix}0&1&\cdots&1&1\\1&0&\cdots&1&1\\\vdots&\vdots&&\vdots&\vdots\\1&1&\cdots&0&1\\1&1&\cdots&1&0\end{vmatrix}$$

$$= \boxed{(-1)^{\lfloor\frac{n}{2}\rfloor}}\,\boxed{3^n}\begin{vmatrix}0&1&\cdots&1&1\\1&0&\cdots&1&1\\\vdots&\vdots&&\vdots&\vdots\\1&1&\cdots&0&1\\n-1&n-1&\cdots&n-1&n-1\end{vmatrix} = \boxed{(n-1)}\,\boxed{(-1)^{\lfloor\frac{n}{2}\rfloor}}\,\boxed{3^n}\begin{vmatrix}0&1&\cdots&1&1\\1&0&\cdots&1&1\\\vdots&\vdots&&\vdots&\vdots\\1&1&\cdots&0&1\\1&1&\cdots&1&1\end{vmatrix} = \boxed{(n-1)}\,\boxed{(-1)^{\lfloor\frac{n}{2}\rfloor}}\,\boxed{3^n}\begin{vmatrix}-1&0&\cdots&0&0\\0&-1&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&-1&0\\1&1&\cdots&1&1\end{vmatrix}$$

$$= \boxed{(-1)^{(n-1)}}\,\boxed{(n-1)}\,\boxed{(-1)^{\lfloor\frac{n}{2}\rfloor}}\,\boxed{3^n}\begin{vmatrix}1&0&\cdots&0&0\\0&1&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&1&0\\1&1&\cdots&1&1\end{vmatrix} = \boxed{(-1)^{(n-1)}}\,\boxed{(n-1)}\,\boxed{(-1)^{\lfloor\frac{n}{2}\rfloor}}\,\boxed{3^n}\begin{vmatrix}1&0&\cdots&0&0\\0&1&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&1&0\\0&0&\cdots&0&1\end{vmatrix} = \boxed{(-1)^{(n-1)}}\,\boxed{(n-1)}\,\boxed{(-1)^{\lfloor\frac{n}{2}\rfloor}}\,\boxed{3^n}$$

Introduce eigenvalues and eigenvectors

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ in standard coords, $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ in eigencoords

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_1 = 3$	$Av_1 = \lambda_1 v_1$
$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda_2 = 1$	$Av_2 = \lambda_2 v_2$

$\left.\begin{array}{ll} v & \text{eigenvector} \\ \lambda & \text{eigenvalue} \end{array}\right\}$ corresponding

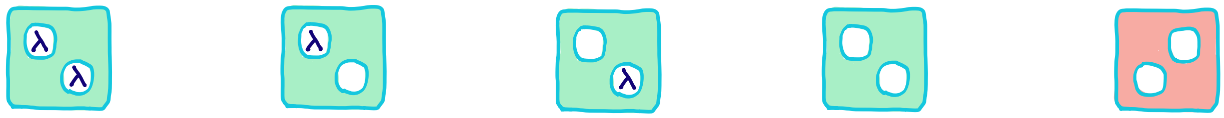
$\Leftrightarrow \boxed{Av = \lambda v} \Leftrightarrow Av = (\lambda I)v \Leftrightarrow \boxed{(A-\lambda I)v = 0}$

A	v	λ	$(A-\lambda I)v=0$
$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	3	$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	1	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
won't work:		2	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cancel{\begin{bmatrix} \ \\ \ \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Want $\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$

$$|A-\lambda I| = (2-\lambda)(2-\lambda) - 1\cdot 1$$
$$= \lambda^2 - 4\lambda + 3$$
$$= (\lambda-3)(\lambda-1) = 0$$
$$\lambda = 3, 1$$

$$\begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix} = 0 =$$



$$\underbrace{\begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix}}_{\lambda^2} \quad \underbrace{\begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix} \quad \begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix}}_{-(a+d)\lambda} \quad \underbrace{\begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix} \quad \begin{vmatrix} a-\lambda & c \\ b & d-\lambda \end{vmatrix}}_{+(ad-bc)}$$

$$\lambda^2 - \underbrace{(a+d)}_{\text{trace}(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)}$$

sum of diagonal entries

Suppose roots are r, s (eigenvalues)

$$\begin{aligned}
&(\lambda - r)(\lambda - s) \\
&= \lambda^2 - \boxed{(r+s)}\lambda + \boxed{rs} \\
&= \lambda^2 - \boxed{(a+d)}\lambda + \boxed{(ad-bc)} \\
&= \lambda^2 - \boxed{\text{trace}(A)}\lambda + \boxed{\det(A)}
\end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \text{trace}(A) &= \text{sum of eigenvalues} \\ \det(A) &= \text{product of eigenvalues} \end{aligned}}$$

We can skip the polynomial, and work from this.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{aligned} \text{trace}(A) &= \boxed{4 = r+s} \\ \det(A) &= \boxed{3 = rs} \end{aligned} \quad \Rightarrow r, s = 1, 3$$