Oct 20, 2016

Fast way to compute 3x3 inverses

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}^{-1} = \begin{bmatrix} ei-fh & -di+fg & dh-eg \\ -bi+ch & ai-cg & -ah+bg \\ bf-ce & -af+cd & ae-bd \end{bmatrix} / \det(A)$$

recall 3x3 determinant:

$$\underbrace{\begin{matrix} a & d & g \\ b & e & h \\ c & f & i \end{matrix}}_{A} \quad \underbrace{\begin{matrix} a & d \\ b & e \\ c & f \end{matrix}}_{A}$$

Do the same thing in both directions, after transposing:

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ a & b & c & a & b \\ d & e & f & d & e \end{matrix}$$

$$\begin{array}{c|cccc|} & a & b & c & a & b \\ \hline d & e & f & d & e \\ g & h & i & g & h \\ a & b & c & a & b \\ d & e & f & d & e \\ \hline \end{array}$$

$$\begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei-fh \qquad \begin{vmatrix} f & d \\ i & g \end{vmatrix} = -di+fg \qquad \begin{vmatrix} d & e \\ g & h \end{vmatrix} = dh-eg$$

$$\begin{vmatrix} h & i \\ b & c \end{vmatrix} = -bi+ch \qquad \begin{vmatrix} i & g \\ c & a \end{vmatrix} = ai-cg \qquad \begin{vmatrix} g & h \\ a & b \end{vmatrix} = -ah+bg$$

$$\begin{vmatrix} b & c \\ e & f \end{vmatrix} = bf-ce \qquad \begin{vmatrix} c & a \\ f & d \end{vmatrix} = -af+cd \qquad \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae-bd$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 1 & 2 & 5 \end{bmatrix}}_{A^{-1}} \Big/ 2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccccc} 2 & -1 & 0 & 2 & -1 \\ 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 2 & -1 \\ 1 & 2 & -1 & 1 & 2 \end{array}$$

$$\begin{array}{cccc} e & f & d & e \\ h & i & g & h \\ b & c & a & b \\ e & f & d & e \end{array} \qquad \begin{vmatrix} h & i \\ b & c \end{vmatrix} = \begin{array}{ccc} a & d & g \\ b & e & h \\ c & f & i \end{array} \qquad \begin{array}{ccc} + & \boxed{-} & + \\ - & + & - \\ + & - & + \end{array}$$

"torus" view

order swapped but sign is $\boxed{-}$

Orthogonal projection onto plane $x+y+z=0$ in $\mathbb{R}^3$

$\vec{n} = (1,1,1) \mapsto 0$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\ \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} Av_1 &= v_1 \\ Av_2 &= v_2 \\ Av_3 &= 0 \end{aligned}}$$

$S = \{e_1, e_2, e_3\} \qquad V = \{v_1, v_2, v_3\}$

$$\underset{\substack{S \leftarrow S \\ A}}{\overset{L}{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 3}} = \underset{\substack{S \leftarrow V \\ C}}{\overset{Id}{\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}}} \underset{\substack{V \leftarrow V \\ D}}{\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}} \underset{\substack{V \leftarrow S \\ C^{-1}}}{\overset{Id}{\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \Big/ 3}}$$

$$\begin{array}{ccccc} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \end{array}$$

$$= 1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \end{bmatrix} \Big/ 3 + 1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} \Big/ 3$$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}/3 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ -1 & -1 & 2 \end{bmatrix}/3 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}/3$$

Check:

$$A = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}/3}_{I} - \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}/3}_{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}/3} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}/3$$

orthogonal projection onto (1,1,1)

Check:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}/3 \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{v_1 \; v_2 \; v_3} = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -3 & 0 \end{bmatrix}/3 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{v_1 \; v_2 \; 0}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}/3 \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}/3 = 1 \qquad \begin{bmatrix} 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}/3 = 0 \qquad \begin{bmatrix} 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}/3 = 0$$

projection $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{bmatrix} 2 & -1 & -1 \end{bmatrix}/3$ preserves $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, zeros $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Determinant from a recurrence relation

$n=0$	1	2	3	4
$[\,]$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1&1\\-1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\-1&1&1\\0&-1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\-1&1&1&0\\0&-1&1&1\\0&0&-1&1\end{bmatrix}$
det $=1$	1	2	3	5

Expand by minors 1^{st} col

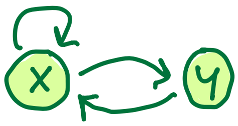
$$\begin{bmatrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{bmatrix}$$

$$\underbrace{\begin{vmatrix}1&1&0&0\\-1&1&1&0\\0&-1&1&1\\0&0&-1&1\end{vmatrix}}_{f(4)} = 1\underbrace{\begin{vmatrix}1&1&0\\-1&1&1\\0&-1&1\end{vmatrix}}_{f(3)} - (-1)\underbrace{\begin{vmatrix}1&0&0\\-1&1&1\\0&-1&1\end{vmatrix}}_{+f(2)\ \left\{\begin{vmatrix}1&1\\-1&1\end{vmatrix}\right.}$$

$$f(4) = f(3) + f(2)$$

$$\boxed{f(0)=1 \quad f(1)=1 \quad f(n) = f(n-1) + f(n-2)}$$

n	$f(n)$
0	1
1	1
2	2
3	3
4	5
5	8



$f(5)$ = Paths of length 5, x to x

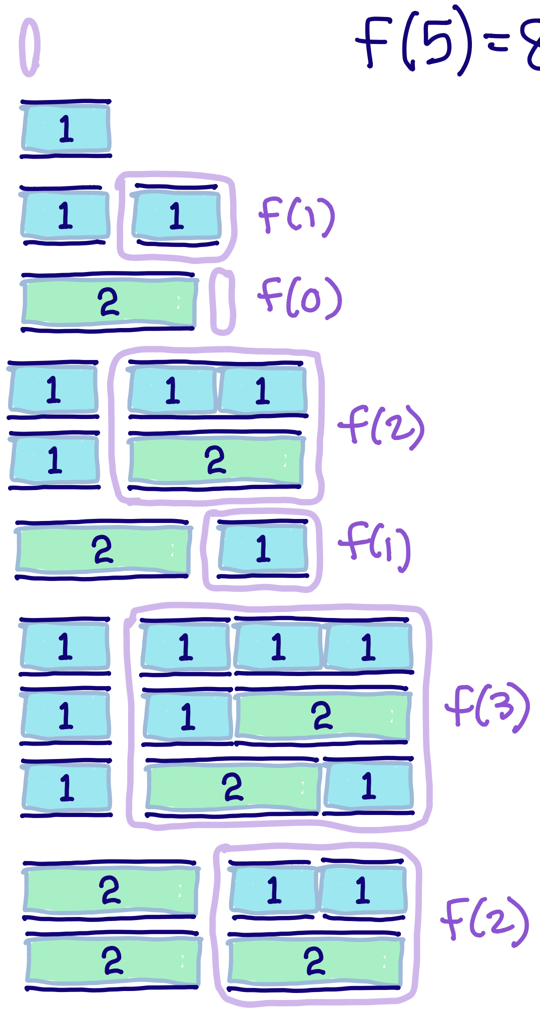
$f(0)=1$

$f(1)=1$

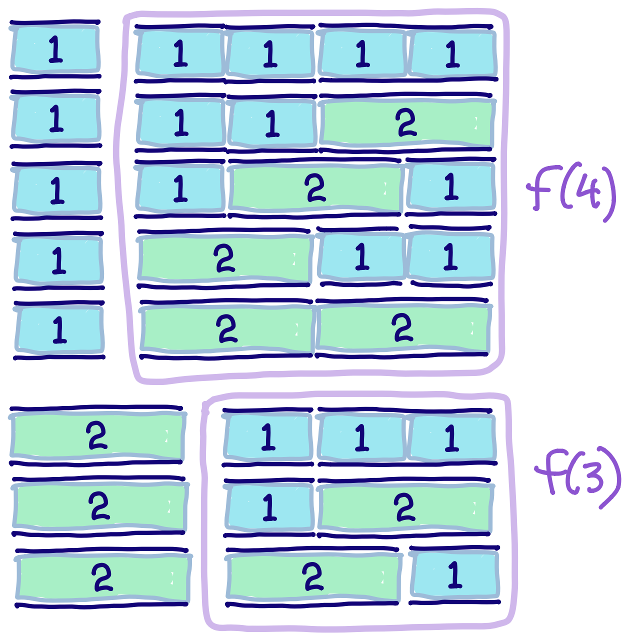
$f(2)=2$

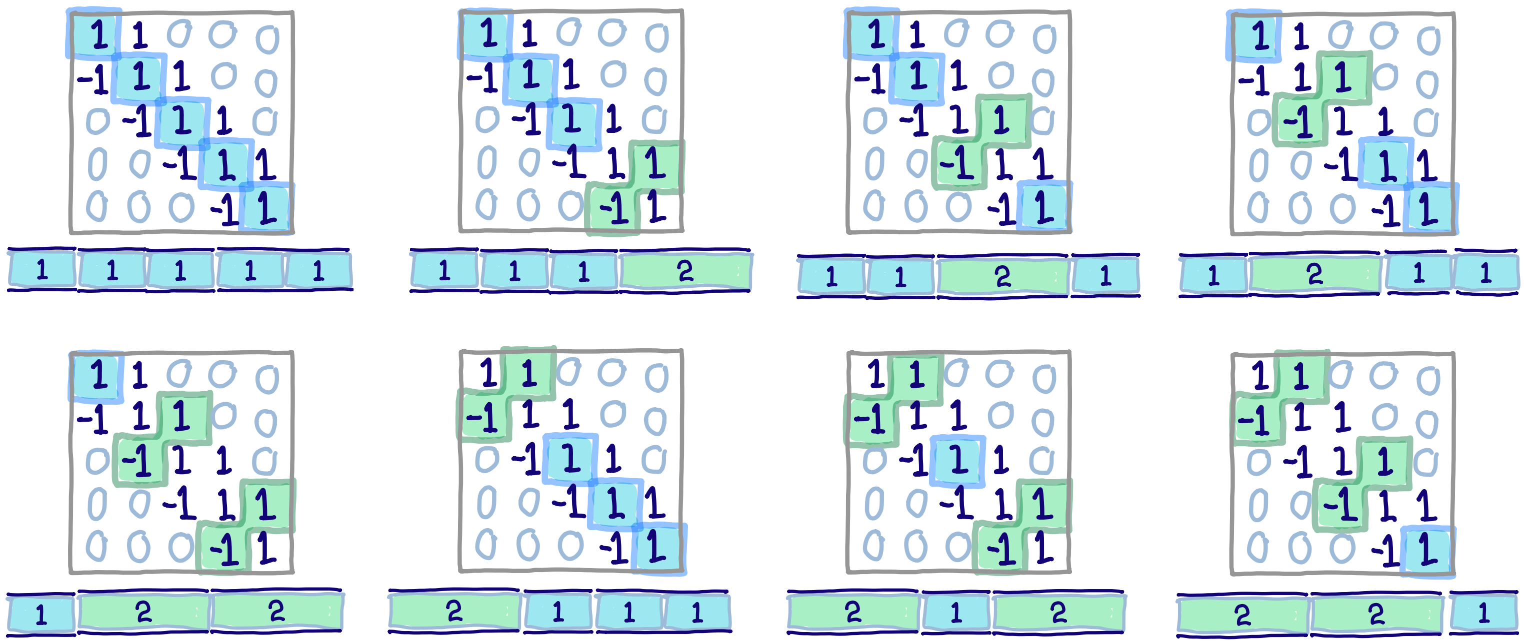
$f(3)=3$

$f(4)=5$



$f(5)=8$





8 of the 120 terms of this determinant are 1. The rest are zero.

$$f(n) = a\,f(n-1) + b\,f(n-2)$$

Guess that the recurrence is of this form. Find a, b.

n	$f(n) = a\,f(n-1) + b\,f(n-2)$
2	$2 = 1a + 1b$
3	$3 = 2a + 1b$
4	$5 = 3a + 2b$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \qquad a = b = 1$$

$$\begin{bmatrix} f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$\begin{bmatrix} f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f(1) \\ f(2) \end{bmatrix}$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

Formula for n^{th} power of matrix gives formula for $f(n)$.

Eigenvalues will give us this formula.

Cramer's rule

$$\begin{bmatrix} a & d & g\\ b & e & h\\ c & f & i\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}r\\s\\t\end{bmatrix}$$

$$x=\frac{\begin{vmatrix} r & d & g\\ s & e & h\\ t & f & i\end{vmatrix}}{\begin{vmatrix} a & d & g\\ b & e & h\\ c & f & i\end{vmatrix}} \qquad y=\frac{\begin{vmatrix} a & r & g\\ b & s & h\\ c & t & i\end{vmatrix}}{\begin{vmatrix} a & d & g\\ b & e & h\\ c & f & i\end{vmatrix}} \qquad z=\frac{\begin{vmatrix} a & d & r\\ b & e & s\\ c & f & t\end{vmatrix}}{\begin{vmatrix} a & d & g\\ b & e & h\\ c & f & i\end{vmatrix}}$$

why?

$$\begin{bmatrix}r\\s\\t\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}x+\begin{bmatrix}d\\e\\f\end{bmatrix}y+\begin{bmatrix}g\\h\\i\end{bmatrix}z$$

Use linearity

$$\begin{vmatrix} r & d & g\\ s & e & h\\ t & f & i\end{vmatrix}=\begin{vmatrix} \boxed{\begin{bmatrix}a\\b\\c\end{bmatrix}x+\begin{bmatrix}d\\e\\f\end{bmatrix}y+\begin{bmatrix}g\\h\\i\end{bmatrix}z} & \begin{matrix}d\\e\\f\end{matrix} & \begin{matrix}g\\h\\i\end{matrix}\end{vmatrix}$$

$$=x\begin{vmatrix} a & d & g\\ b & e & h\\ c & f & i\end{vmatrix}+y\underbrace{\cancel{\begin{vmatrix} d & d & g\\ e & e & h\\ f & f & i\end{vmatrix}}}_{\substack{\text{same}\\ 0}}+z\underbrace{\cancel{\begin{vmatrix} g & d & g\\ h & e & h\\ i & f & i\end{vmatrix}}}_{\substack{\text{same}\\ 0}}$$