$\boxed{\text{Oct 6, 2016}}$ (Discussion of scanned exam grading)
Do each problem on its own sheet

Gram-Schmidt process

$w+2x+y+3z=0$ defines hyperplane $\underset{\text{3-dim}}{H} \subset \underset{4}{\mathbb{R}^4}$

Find orthogonal basis for H, then extend to $\perp$ basis for all of $\mathbb{R}^4$

$(1,2,1,3)\cdot(w,x,y,z)=0$ $\quad$ $v_4=(1,2,1,3)$ will be our fourth vector

$v_1=(2,-1,0,0)\in H$ ✓
$v_2=(0,0,3,-1)\in H$ ✓
$v_3=(0,1,-2,0)\in H$

$\}$ $(a,b)\cdot(b,-a)=0$ pattern
$(1,2)\cdot(2,-1)=0$
$(1,3)\cdot(3,-1)=0$

is not yet $\perp$ to v_1,v_2, tedious to fix.

instead reverse the check:

$$\underbrace{\begin{bmatrix} v_4 \\ v_1 \\ v_2 \end{bmatrix}\begin{bmatrix} v_3 \end{bmatrix}}_{\text{checks that } v_3\perp v_4,v_1,v_2}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} v_3 \end{bmatrix}}_{\text{so solve for } v_3}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ 2a \\ \\ \\ \end{bmatrix}=\begin{bmatrix} \\ 0 \\ \\ \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} \\ \\ b \\ 3b \end{bmatrix}=\begin{bmatrix} \\ \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ 2a \\ b \\ 3b \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ -1 \\ -3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5a+10b=0$
$(5,10)\cdot(a,b)=0$
$(5,10)\cdot(10,-5)=0$
$(5,10)\cdot(2,-1)=0$
$a=2,\ b=-1$

$$\boxed{\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 2 & 4 & -1 & -3 \\ \hline 1 & 2 & 1 & 3 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l}\\ \\ \\ \end{array}\right\} \perp \text{ basis for } H \\ \left.\right\} \text{extend to } \perp \text{ basis for } \mathbb{R}^4 \end{array}}$$

Inner products

dot product $v \cdot w = 0 \iff v \perp w$ in ordinary sense

conjugant gradient uses generalization

(see http://numerical.recipes/)

inner product

$$\langle (a,b,c),(d,e,f)\rangle = (a,b,c)\cdot(d,e,f) = [a\ b\ c]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}d\\e\\f\end{bmatrix}$$

or

$$\boxed{\langle (a,b,c),(d,e,f)\rangle = [a\ b\ c]\begin{bmatrix}2&1&0\\1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}d\\e\\f\end{bmatrix}}\ \circledast$$

Problem: Extend $(1,1,1)$ to an orthogonal basis for $\mathbb{R}^3$ using the second inner product above

$$\langle v,w\rangle = [\quad v \quad]\begin{bmatrix}2&1&0\\1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}w\end{bmatrix}$$

$$w_2 = v_2 - \left(\frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle}\right) w_1$$

(what Gram-Schmidt looks like for a general inner product)

$$\boxed{\begin{aligned} v_1 &= (1,1,1)\\ v_2 &= (1,-1,0)\\ v_3 &= (3,-4,3)\end{aligned}}$$

$$\underbrace{[1\ 1\ 1]}_{v_1}\begin{bmatrix}2&1&0\\1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}v_2\end{bmatrix} = [0]$$

$$\Rightarrow [3\ 3\ 1]\begin{bmatrix}v_2\end{bmatrix} = [0] \Rightarrow [3\ 3\ 1]\begin{bmatrix}1\\-1\\0\end{bmatrix} = [0]$$

$$\begin{bmatrix}1&1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}2&1&0\\1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}v_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3&3&1\\1&0&-1\end{bmatrix}\begin{bmatrix}v_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}3&3&1\\1&0&-1\end{bmatrix}\begin{bmatrix}1\\ \cancel{\ }\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3&3&1\\1&0&-1\end{bmatrix}\begin{bmatrix}3\\ \cancel{\ }\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}3&3&1\\1&0&-1\end{bmatrix}\begin{bmatrix}3\\-4\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\circledast$ Issue: Our matrix isn't "positive definite" so it doesn't give a nice inner product.

$V = \{ax^2+bx+c\} \cong \mathbb{R}^3$

polynomials of deg ≤ 2 $\quad (a,b,c)$

$W \subset V$ subspace (plane)

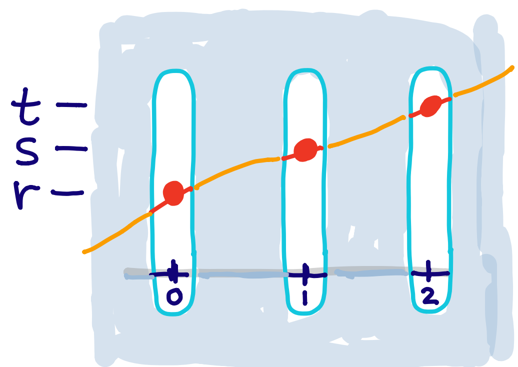
If $f(x) = ax^2+bx+c$, $\quad f(1)=0$ is one condition

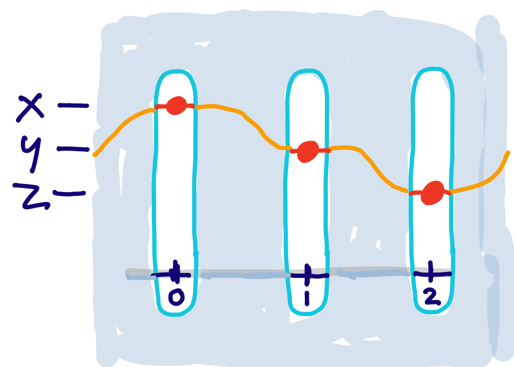
$$f(x) = a(1)^2 + b(1) + c = a+b+c = 0$$

Use an integral as a more natural inner product:

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)\,g(x)\,dx$$

If $\{0,1,2\}$ were only 3 points in existence (rest of $\mathbb{R}$ disappears)



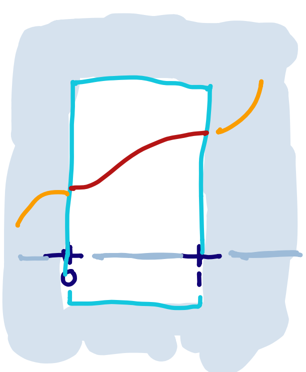


$f(0) = r$
$f(1) = s$ $\quad$ "$f = (r,s,t)$"
$f(2) = t$

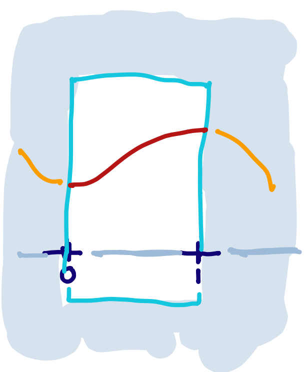
$g(0) = x$
$g(1) = y$ $\quad$ "$g = (x,y,z)$"
$g(2) = z$

...then it would feel natural to define

$$\langle f,g \rangle = (r,s,t)\cdot(x,y,z) = \sum_{i=0}^{2} f(i)\,g(i)$$



f defined on $[0,1]$



g defined on $[0,1]$

$\boxed{\text{Integration is continuous summation}}$

If the interval $[0,1]$ is all that exists,

$$\langle f,g \rangle = \int_0^1 f(x)\,g(x)\,dx$$

is exactly the same idea.

Want to find orthogonal basis for W using $\langle f,g\rangle$

$\boxed{v_1(x) = x-1}$ (belongs to W, $v_1(1) = 1-1 = 0$ ✓)

$$\langle x-1, ax^2+bx+c\rangle$$
$$= \int_0^1 (x-1)(ax^2+bx+c)\,dx$$
$$= \int_0^1 \left[ax^3 + (b-a)x^2 + (c-b)x - c\right] dx$$
$$= a\underbrace{\boxed{\int_0^1 x^3 dx}}_{1/4} + (b-a)\underbrace{\boxed{\int_0^1 x^2 dx}}_{1/3} + (c-b)\underbrace{\boxed{\int_0^1 x^1 dx}}_{1/2} - c\underbrace{\boxed{\int_0^1 x^0 dx}}_{1}$$

(x12) $\sim= \quad 3a + 4(b-a) + 6(c-b) - 12c = 0$

$$-a - 2b - 6c = 0$$
$$a + b + c = 0 \Leftarrow f(1) = 0$$

$$\begin{bmatrix} -1 & -2 & -6 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{(2) \leftarrow (2)+(1)} \begin{bmatrix} -1 & -2 & -6 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix} = 0$$

$5 \quad -1$

or $\begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$ (change sign)

$$\boxed{\begin{aligned} v_1(x) &= x-1 \\ v_2(x) &= 4x^2 - 5x + 1 \end{aligned}}$$

$$\langle x-1, 4x^2-5x+1\rangle = \int_0^1 (x-1)(4x^2-5x+1)\,dx$$

	1	$-5x$	$4x^2$
-1	-1	$5x$	$-4x^2$
x	x	$-5x^2$	$4x^3$

$-1 + 6x - 9x^2 + 4x^3$

	1	-5	4
-1	-1	5	-4
1	1	-5	4

$-1 + 6x - 9x^2 + 4x^3$

$$4x^3 - 9x^2 + 6x - 1$$
$$4/4 - 9/3 + 6/2 - 1$$
$$1 - 3 + 3 - 1 = 0$$

$v_1(x)$ and $v_2(x)$ are orthogonal ☑

$$\langle ax^2+bx+c,\ dx^2+ex+f \rangle$$

$$= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$\square = d \underbrace{\boxed{\int_0^1 x^4 \, dx}}_{1/5}$$

not the same d

$$(\times 60) \quad \sim= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 12 & 15 & 20 \\ 15 & 20 & 30 \\ 20 & 30 & 60 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

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we want $\langle x-1,\ ax^2+bx+c \rangle = 0$

$$\underbrace{\begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 12 & 15 & 20 \\ 15 & 20 & 30 \\ 20 & 30 & 60 \end{bmatrix}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

divide out 5

$① - ② =$
$0 \quad 1 \quad 5$

$$\begin{bmatrix} 1 & 2 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$f(1) = 0$

$$\boxed{\begin{aligned} v_1(x) &= x - 1 \\ v_2(x) &= 4x^2 - 5x + 1 \end{aligned}}$$