Exam 3

Linear Algebra, Dave Bayer, March 8, 2014

Name: solutions **Uni:** ____________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find the determinant of the matrix

$$\begin{bmatrix} 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \\ 3 & 3 & 4 & 9 \end{bmatrix}$$

Subtract multiples of row 2 from others

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix} \quad \text{now row 1} \Longrightarrow \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

expand down col 3

$$+1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

triangular so $\boxed{\det = 5}$

check expand col 1

$$-1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 6 \end{vmatrix} = -1(-1)\begin{vmatrix} 1 & 1 \\ 1 & 6 \end{vmatrix} = 5 \quad ☑$$

[2] Find the inverse of the matrix

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 4 \\ 2 & -1 & 3 \\ 3 & 2 & -6 \end{bmatrix} \Big/ 7 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \checkmark$$

$$\Downarrow$$

$$\begin{array}{c|cccc} & 0 & 3 & 1 & 0 & 3 \\ \hline 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & 1 & 0 & 3 \\ 2 & 0 & 1 & 2 & 0 \end{array}$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 1 & 4 \\ 2 & -1 & 3 \\ 3 & 2 & -6 \end{bmatrix} \Big/ 7}$$

check: $AA^{-1}\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{trial value}} = A \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} \Big/ 7 = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} \Big/ 7$ $\checkmark$

[3] Find w/y where

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{vmatrix} a & 1 & 1 & 0 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix} = a \underbrace{\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}_{\substack{-1(-1)+1\cdot 1 \\ 2}} - b \underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}_{0} + c \underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}}_{0} - d \underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}}_{\substack{1\cdot 1+1\cdot 1 \\ 2}}$$

$$\begin{vmatrix} 0 & 1 & a & 0 \\ 1 & 0 & b & 1 \\ 1 & 1 & c & 1 \\ 2 & 1 & d & 0 \end{vmatrix} = a \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} - b \underbrace{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}}_{2} + c \underbrace{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}}_{2} - d \underbrace{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}}_{0}$$

$$\underbrace{1(-1)+1\cdot(-1)}_{-2}$$

$$\boxed{w/y = \frac{a-d}{c-a-b}}$$

check: $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \\ 5 \end{bmatrix}$

trial values

$$\frac{3-5}{4-3-5} = \frac{-2}{-4} = 1/2 \ \boxed{\checkmark}$$

Cramer's rule

$w = |\,a\,b\,c\,d \ldots| / \mathrm{Det}(A)$

$y = |\ldots a\,b\,c\,d \ldots| / \mathrm{Det}(A)$

$\Rightarrow w/y = |\,a\,b\,c\,d \ldots| \,/\, |\ldots a\,b\,c\,d \ldots|$

expand by minors

[4] Find A^n where A is the matrix

$$\begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$$

Your final answer should be in the form

$$A^n \;=\; r^n\, B + s^n\, C$$

sum = trace = 5
prod = det = −6
$\lambda = -1, 6$

$$\lambda = -1 \quad \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0 \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \end{bmatrix}/7 = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}/7$$

$$\lambda = 6 \quad \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0 \qquad \begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}/7 = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}/7$$

check $6-(-1)=7$ ☑

these matrices swap, one changes sign

$$\boxed{A^n = (-1)^n \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}/7 + 6^n \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}/7}$$

check: I $n=0$ ☑
A $n=1$ ☑

$$\begin{array}{c|c} {}^{-4}\;{}_{18} & {}^{6}\;{}_{36} \\ \hline {}^{2}\;{}_{12} & {}^{-3}\;{}_{24} \end{array}$$ ☑

[5] Find $f(n)$, where $f(n)$ is the determinant of the $n \times n$ matrix in the sequence

$$
\begin{array}{ccccc}
5 & 19 & 65 & & \\
\begin{bmatrix} 5 \end{bmatrix} &
\begin{bmatrix} 5 & 2 \\ 3 & 5 \end{bmatrix} &
\begin{bmatrix} 5 & 2 & 0 \\ 3 & 5 & 2 \\ 0 & 3 & 5 \end{bmatrix} &
\begin{bmatrix} 5 & 2 & 0 & 0 \\ 3 & 5 & 2 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & 3 & 5 \end{bmatrix} &
\begin{bmatrix} 5 & 2 & 0 & 0 & 0 \\ 3 & 5 & 2 & 0 & 0 \\ 0 & 3 & 5 & 2 & 0 \\ 0 & 0 & 3 & 5 & 2 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix}
\end{array}
$$

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Your final answer should be in the form $\quad f(n) = a\,r^n + b\,s^n$

$$\underbrace{\begin{vmatrix} 5 & 2 & & \\ 3 & 5 & 2 & \\ & 3 & 5 & \ddots \\ & & \ddots & \ddots \end{vmatrix}}_{n \times n} = 5 \underbrace{\begin{vmatrix} 5 & 2 & \\ 3 & 5 & \ddots \\ & \ddots & \ddots \end{vmatrix}}_{(n-1)\times(n-1)} - 2 \begin{vmatrix} 3 & 2 & \\ & \underbrace{\begin{matrix} 5 & \ddots \\ \ddots & \ddots \end{matrix}}_{(n-2)\times(n-2)} \end{vmatrix}$$

$$f(n) = 5\,f(n-1) - 6\,f(n-2)$$

n	$f(n)$
0	1
1	5
2	$19 = 5\cdot 5 - 6\cdot 1$
3	$65 = 5\cdot 19 - 6\cdot 5$
$\vdots$	$\vdots$

$$\begin{bmatrix} f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}}_{\text{sum } 5,\ \text{prod } 6 \quad 2,3}^{n} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}^n = 2^n \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}\Big/1 + 3^n \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}\Big/1$$

I ☑
A ☑

6 -6	-2 3
12 -18	-4 9

$$\Rightarrow \boxed{f(n) = -2\cdot 2^n + 3\cdot 3^n}$$

check: $f(3) = -2\cdot 8 + 3\cdot 27$
$= -16 + 81 = 65$ ☑