# Exam 2

Linear Algebra, Dave Bayer, March 6, 2014

**Name:** Solutions **Uni:** ______________

| [1] | [2] | [3] | [4] | [5] | **Total** |
|---|---|---|---|---|---|
|  |  |  |  |  |  |

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find the row space and the column space of the matrix

$$\begin{bmatrix}0&1&2&3&4\\0&2&4&6&8\\0&3&6&9&2\\0&4&8&2&6\end{bmatrix}\begin{array}{l} \\ -2\textcircled{1} = 0\ 0\ 0\ 0\ 0 \\ -3\textcircled{1} = 0\ 0\ 0\ 0\ {-10} \\ -4\textcircled{1} = 0\ 0\ 0\ {-10}\ {-10}\end{array}$$

$$\begin{array}{ccc} -2\textcircled{1} & -3\textcircled{1} & -4\textcircled{1} \\ \| & \| & \| \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \\ 0 & -10 & -10 \end{array}$$

so rank 3

row space basis = rows ①,③,④ $\boxed{\left\{\begin{array}{c}(0,1,2,3,4)\\(0,3,6,9,2)\\(0,4,8,2,6)\end{array}\right\}}$

Note the row space is a subspace of $\mathbb{R}^5$ (5 = # cols) so each vector has $\underline{\underline{5}}$ entries.

col space basis = cols ②,④,⑤ $\boxed{\left\{\begin{array}{c}(1,2,3,4)\\(3,6,9,2)\\(4,8,2,6)\end{array}\right\}}$

Note the col space is a subspace of $\mathbb{R}^4$ (4 = # rows) so each vector has $\underline{\underline{4}}$ entries.

[2] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

can't solve exactly $Ax=b$
Solve instead $A^TAx=A^Tb$
(finds ⊥ projection of b to image(A))

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}}_{\det = 14\cdot 4 - 6\cdot 6 = 56-36=20}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix}\Big/20 = \begin{bmatrix} -2 \\ 28 \end{bmatrix}\Big/20 = \begin{bmatrix} -1/10 \\ 7/5 \end{bmatrix}$$

$$\boxed{y = -\tfrac{1}{10}x + \tfrac{7}{5}}$$

| $x$ | $y$ | $-\frac{1}{10}x+\frac{14}{10}$ | $10\cdot$error |
|---|---|---|---|
| 0 | 1 | 14/10 | 4 |
| 1 | 2 | 13/10 | -7 |
| 2 | 1 | 12/10 | 2 |
| 3 | 1 | 11/10 | 1 |

check:
$(4,-7,2,1)$ should be ⊥ to
image $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ✓$$

[3] Find the 3 × 3 matrix that projects orthogonally onto the plane

$$x + 3y - 2z = 0$$

$\boxed{\text{First method:}}$ Find ⊥ basis, add projections

$v_1 = (3,-1,0)$

$v_2 = (1,3,5)$ ← want ⊥ to $v_1$ and solu to $x+3y-2z=0$

$$\begin{bmatrix}3\\-1\\0\end{bmatrix}\begin{bmatrix}3 & -1 & 0\end{bmatrix}/10 + \begin{bmatrix}1\\3\\5\end{bmatrix}\begin{bmatrix}1 & 3 & 5\end{bmatrix}/35$$

$$\begin{bmatrix}9 & -3 & 0\\-3 & 1 & 0\\0 & 0 & 0\end{bmatrix}/10 + \begin{bmatrix}1 & 3 & 5\\3 & 9 & 15\\5 & 15 & 25\end{bmatrix}/35 = \begin{bmatrix}65 & -15 & 10\\-15 & 25 & 30\\10 & 30 & 50\end{bmatrix}/70$$

$$\boxed{\begin{bmatrix}13 & -3 & 2\\-3 & 5 & 6\\2 & 6 & 10\end{bmatrix}/14}$$

check:

$$\begin{bmatrix}13 & -3 & 2\\-3 & 5 & 6\\2 & 6 & 10\end{bmatrix}/14 \begin{bmatrix}1 & 3 & 2\\3 & -1 & 0\\-2 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 42 & 28\\0 & -14 & 0\\0 & 0 & 14\end{bmatrix}/14$$

normal

basis for plane

$$= \begin{bmatrix}0 & 3 & 2\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix} ✓$$

[3] Find the $3 \times 3$ matrix that projects orthogonally onto the plane

$$x + 3y - 2z = 0$$

second method:

Let A project $\perp$ onto plane, B $\perp$ onto normal $(1,3,-2)$

$$I = A + B \Rightarrow A = I - B$$

$$B = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} / 14 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix} / 14$$

$$A = \begin{bmatrix} 14 & & \\ & 14 & \\ & & 14 \end{bmatrix} / 14 - \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix} / 14 = \boxed{\begin{bmatrix} 13 & -3 & 2 \\ -3 & 5 & 6 \\ 2 & 6 & 10 \end{bmatrix} / 14}$$

[3] Find the $3 \times 3$ matrix that projects orthogonally onto the plane

$$x + 3y - 2z = 0$$

$\boxed{\text{alternate method}}$

Find for basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

Subtract multiple of normal $(1,3,-2)$ so in plane

$$[\underbrace{(1,0,0) - t(1,3,-2)}_{(13,-3,2)/14}] \cdot (1,3,-2) = 0 \Rightarrow 1 - 14t = 0 \quad t = 1/14$$

$$[\underbrace{(0,1,0) - t(1,3,-2)}_{(-3,5,6)/14}] \cdot (1,3,-2) = 0 \Rightarrow 3 - 14t = 0 \quad t = 3/14$$

$$[\underbrace{(0,0,1) - t(1,3,-2)}_{(2,6,10)/14}] \cdot (1,3,-2) = 0 \Rightarrow t = -2/14$$

from pattern $(1,3,-2)$

$$\boxed{\begin{bmatrix} 13 & -3 & 2 \\ -3 & 5 & 6 \\ 2 & 6 & 10 \end{bmatrix} / 14}$$

proj of $(1,0,0)$ etc...

[3] Find the $3 \times 3$ matrix that projects orthogonally onto the plane

$$x + 3y - 2z = 0$$

(Alternate method)

Plane is image of $\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$

Can't solve $\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ but $A^TAx = A^Tb$ version computes $\perp$ projection of $(x, y, z)$ to image.

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 10 & 6 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3x-y \\ 2x+z \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}\Big/ 14 \begin{bmatrix} 3x-y \\ 2x+z \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}\Big/ 14 \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -5 & -6 \\ 2 & 6 & 10 \end{bmatrix}\Big/ 14}$$

$$\boxed{\begin{bmatrix} 13 & -3 & 2 \\ -3 & 5 & 6 \\ 2 & 6 & 10 \end{bmatrix}\Big/ 14}$$

(Several students successfully used this approach. Not the easiest method but exciting that it works!)

[4] Find an orthogonal basis for the subspace $V$ of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,0,0) \qquad (0,1,1,0) \qquad (0,0,1,1) \qquad \underbrace{(1,2,1,0) \qquad (0,1,2,1)}$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

$$\begin{array}{cc} 1100 & 0110 \\ \underline{0110} & \underline{0011} \\ 1210 & 0121 \end{array}$$

dependent vectors

$w - x + y - z = 0$

$(1,-1,1,-1) \cdot (w,x,y,z) = 0$

$\dim V = 3$

$v_1 = 1100 \quad w_1 = 1100$

$v_2 = 0110 \quad w_2 = -1120$

$v_3 = 0011 \quad w_3 = 1-113$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$$

$$= 0110 - \frac{1}{2} 1100$$

$$= -1120 \quad \text{(doubled)}$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$$

$$= 0011 - \frac{0}{2} ww - \frac{2}{6}(-1120)$$

$$= 0033 + 1-1-20$$

$$= 1-113$$

$\perp$ basis for V:

$$\boxed{\begin{array}{c} (1,1,9,0) \\ (-1,1,2,0) \\ (1,-1,1,3) \\ \hline (1,-1,1,-1) \end{array}}$$

extend to $\perp$ basis for $\mathbb{R}^4$

checks: $w_1, w_2, w_3$ in V? ☑ ☑ ☑ $\underbrace{(1,-1,1,-1) \cdot w_i = 0}_{\text{part of same question}}$

All 4 vectors $\perp$ to each other?

$$\begin{tikzcd} 1100 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & -1120 \arrow[d, dash] \arrow[dl, dash] \\ 1-113 \arrow[r, dash] & 1-11-1 \end{tikzcd}$$

☑

[4] Find an orthogonal basis for the subspace $V$ of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,0,0) \qquad (0,1,1,0) \qquad (0,0,1,1) \qquad \underbrace{(1,2,1,0) \qquad (0,1,2,1)}_{\substack{\text{dependent} \\ \Rightarrow \dim V = 3}}$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

$\boxed{\text{Second approach}}$

$(1,1,0,0) \perp (0,0,1,1)$

Need third vector in $V$, $\perp$ to these and normal $(1,-1,1,-1)$

$$(s,-s,t,-t) \cdot \begin{matrix}(1,1,0,0)=0 \\ (0,0,1,1)=0\end{matrix}$$

$$(s,-s,t,-t)\cdot(1,-1,1,-1) = 2s+2t = 0 \Rightarrow s=-t$$

$$(1,-1,-1,1)$$

$$\boxed{\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & -1 \end{array}} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \perp \text{ basis for } V \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{extend to } \perp \text{ basis for } \mathbb{R}^4 \end{array}$$

[5] Let $V$ be the vector space of all polynomials of degree $\leqslant 2$ in the variable $x$ with coefficients in $\mathbb{R}$. Let $W$ be the subspace of polynomials of degree $\leqslant 1$. Find the orthogonal projection of the polynomial $x^2$ onto the subspace $W$, with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

Want $ax+b$ so $x^2-ax-b$ is $\perp$ to $x, 1$

$$\langle x^2-ax-b, 1\rangle = \tfrac{1}{3}x^3-\tfrac{a}{2}x^2-bx\Big|_0^1 = \tfrac{1}{3}-\tfrac{a}{2}-b=0$$

$$\langle x^2-ax-b, x\rangle = \tfrac{1}{4}x^4-\tfrac{a}{3}x^3-\tfrac{b}{2}x^2\Big|_0^1 = \tfrac{1}{4}-\tfrac{a}{3}-\tfrac{b}{2}=0$$

$$\begin{bmatrix}3 & 6\\ 4 & 6\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}1\\ -1/6\end{bmatrix}$$

$$\boxed{x-1/6}$$

check: $\langle x^2-x+1/6, 1\rangle = \tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{6}=0$ ☑

$\langle x^2-x+1/6, x\rangle = 1/4-1/3+1/12=0$ ☑