Exam 1

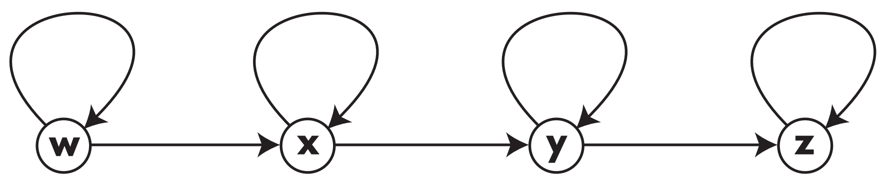
Linear Algebra, Dave Bayer, February 11, 2014

Name: Solutions **Uni:** ____________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Using matrix multiplication, count the number of paths of length six from w to z.



$$\begin{array}{c|cccc} & w & x & y & z \ (\text{in}) \\ \hline w & 1 & 0 & 0 & 0 \\ x & 1 & 1 & 0 & 0 \\ y & 0 & 1 & 1 & 0 \\ z & 0 & 0 & 1 & 1 \\ (\text{out}) & & & & \end{array} = A$$

$$A^2 = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & & & \\ 4 & 1 & & \\ 6 & 4 & 1 & \\ 4 & 6 & 4 & 1 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 1 & & & \\ 6 & 1 & & \\ 15 & 6 & 1 & \\ 20 & 15 & 6 & 1 \end{bmatrix}$$

only need this entry, but I want to see the pattern.

(pascal's triangle)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$\boxed{\text{20 paths}}$

$20 = \binom{6}{3}$ = ways to choose 3 objects from set of 6 objects.

$= \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$ Pascal's triangle.

So, why? Each path is 3 steps ↻ and 3 steps →

$20 = \binom{6}{3}$ ways to choose which steps are → not ↻

[2] Solve the following system of equations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

particular solution: $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, last three cols of matrix.

so, $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

homogeneous solutions: 3 rows are independent, so matrix rank 3, so expect $4-3=1$ dim homog solu.

Find one vector. $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \underbrace{3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{forced by last entry}} \underbrace{-3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{" middle entry}} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

check $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} t}$$

check:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} t \right) = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} t = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} ✓$$

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 4 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{1 \leftarrow 1 - 3\,2} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2 \leftarrow 2 - 2\,3} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{1 \leftarrow \frac{1}{4}\,1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{1 \leftarrow 1 + 3\,2} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{2 \leftarrow 2 + 2\,3} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{1 \leftarrow 4\,1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & 3 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 1 & \\ & & 1 \end{bmatrix}}$$

check:

$$\begin{bmatrix} 1 & 3 & \\ & 1 & \\ & & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\begin{bmatrix} 4 & & \\ & 1 & 2 \\ & & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 4 & 3 & 6 \\ & 1 & 2 \\ & & 1 \end{bmatrix} ✓$$

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

rank 2 = dim 2

expect 2 equs

4 − 2 = 2

① $\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}}_{\text{find this}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$

② $\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \left.\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right\}$ find this

$$\boxed{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}}$$

check:

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \checkmark$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$V \cap W$

$$V: \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$W: \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

equs for V: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

plug in W: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \Leftarrow \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

check $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓