Exam 2

Linear Algebra, Dave Bayer, 10:10 AM, March 12, 2013

Name: ______________________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find a basis for the set of solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 2 & 2 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Extend this basis to a basis for $\mathbb{R}^5$.

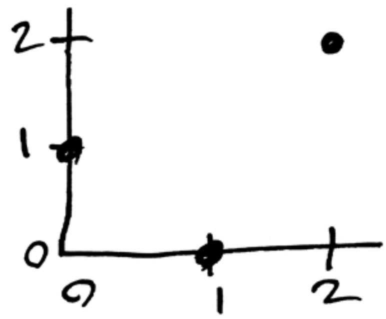
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ -1 & -1 & -1 \\ -4 & -4 & -4 \end{bmatrix}$$

$$\left.\begin{matrix} (2,0,0,-1,-4) \\ (0,2,0,-1,-4) \\ (0,0,2,-1,4) \end{matrix}\right\} \text{basis for solus}$$

$$\left.\begin{matrix} (0,0,0,1,0) \\ (0,0,0,0,1) \end{matrix}\right\} \text{extend to } \mathbb{R}^5$$

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (0, 1), \quad (x_2, y_2) = (1, 0), \quad (x_3, y_3) = (2, 2)$$



$$\begin{array}{cc} x & 1 \end{array} \qquad\qquad y$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

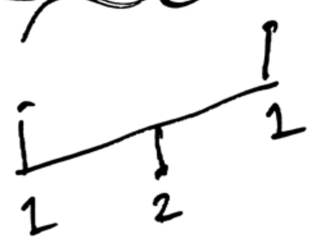
$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} \Big/ 6 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big/ 2 \qquad \boxed{y = \tfrac{1}{2}x + \tfrac{1}{2}}$$

check:

x	y	est	error
0	1	1/2	1
1	0	1	-2
2	2	3/2	1

/2

(springs balance)



[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the subspace

$$x + 2y + z = 0$$

Find the matrix A which represents L in standard coordinates.

Let B project $\perp$ onto line $(1,2,1)$

$A + B = I$

$$\text{proj } (x,y,z) \text{ onto } (1,2,1) = \frac{xyz \cdot 121}{121 \cdot 121}(1,2,1)$$

$$= \frac{x+2y+z}{6}(1,2,1)$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \Big/ 6 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

check: $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \Big/ 6 \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ ✓

$$\text{So } A = I - B = \left(\begin{bmatrix} 6 & & \\ & 6 & \\ & & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \right) \Big/ 6 = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{bmatrix} \Big/ 6$$

$$\boxed{A = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{bmatrix} \Big/ 6}$$

check $L(1,2,1) = 0$ $\perp$ to space

$\left.\begin{array}{l} L(2,-1,0) = (2,-1,0) \\ L(0,-1,2) = (0,-1,2) \end{array}\right\}$ in space

$$\begin{bmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 12 & 0 \\ 0 & -6 & -6 \\ 0 & 0 & 12 \end{bmatrix} \Big/ 6$$ ✓

[4] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,1,1), \quad (1,2,1,2), \quad (2,1,2,1), \quad (2,2,2,2)$$

$(w, x, y, z) \qquad w = y, \quad x = z$

$$\boxed{\begin{array}{c}(1,0,1,0)\\(0,1,0,1)\end{array}}$$

check: mutually $\perp$ ✓

belong to space?

$(1,0,1,0) = (2,1,2,1) - (1,1,1,1)$ ✓

$(0,1,0,1) = (1,2,1,2) - (1,1,1,1)$ ✓

correct number? subspace dim $= 2$ ✓

[5] Let V be the vector space of all polynomials of degree $\leqslant 3$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials satisfying $f(0) = f'(0) = 0$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

$$V = \{ ax^3 + bx^2 + cx + d \}$$

$$f(0) = \quad d = 0$$

$$f'(0) = \quad c = 0$$

$$W = \{ ax^3 + bx^2 \}$$

$$\underbrace{\begin{aligned} v_1 &= x^2 \\ v_2 &= x^3 \end{aligned}}_{\substack{f(0)=f'(0)=0 \\ \checkmark \quad \checkmark}} \qquad \underbrace{\begin{aligned} w_1 &= x^2 \\ w_2 &= 6x^3 - 5x^2 \end{aligned}}_{\substack{f(0)=f'(0)=0 \\ \checkmark \quad \checkmark}}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = v_2 - \frac{1/6}{1/5} w_1 = v_2 - 5/6\, w_1$$

$$= x^3 - 5/6\, x^2$$

scale to $6x^3 - 5x^2$

$$\langle v_2, w_1\rangle = \int_0^1 x^3x^2\,dx = \int_0^1 x^5\,dx = 1/6$$

$$\langle w_1, w_1\rangle = \int_0^1 x^2x^2\,dx = \int_0^1 x^4\,dx = 1/5$$

check:

$$\langle w_1, w_2\rangle = \int_0^1 x^2(6x^3 - 5x^2)\,dx = \int_0^1 6x^5\,dx - \int_0^1 5x^4\,dx$$

$$= 1 - 1 = 0 \quad \checkmark$$