Practice Final A

Linear Algebra, Dave Bayer, April 25, 2012

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	**Total**

Please draw a box around your final answer. Please use each printed sheet (front and back) *only* for that problem, not for any other problem. There are blank sheets at the end of the exam, to give you more room to work. However, your final answer will not be graded unless it appears on the same sheet (front or back) as the printed problem.

[1] Find A^n where A is the matrix

$$\begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$$

$\lambda_1 + \lambda_2 = -1 + -2 = -3$

$\lambda_1 \lambda_2 = (-1)(-2) - 3 \cdot 2 = -4$

$\lambda = -4, 1$

$-4+1 = -3$ ✓

$(-4)1 = -4$ ✓

$\lambda = -4 \quad A + 4I = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0$

$\lambda = 1 \quad A - I = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$A = \underbrace{\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}}_{S \leftarrow E} \underbrace{\begin{bmatrix} -4 & \\ & 1 \end{bmatrix}}_{E \leftarrow E} \underbrace{\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} / 5}_{E \leftarrow S}$

$A = (-4) \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} / 5 + 1 \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} / 5$

$$\boxed{A^n = (-4)^n \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} / 5 + \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} / 5}$$

check $\quad n = 0 \quad \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} / 5 + \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} / 5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^0$ ✓

$n = 1 \quad -4 \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} / 5 + \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} / 5 = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} = A^1$ ✓

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alternatively,

if $A = \begin{bmatrix} -4 & \\ & 1 \end{bmatrix}$ then $(A+4I)/5 = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$

$(I - A)/5 = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$

$(A + 4I)/5 = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} / 5$ ✓

$(I - A)/5 = \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} / 5$ ✓

[2] Find A^n where A is the matrix

$$\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$$

sum $= -2 =$ trace

product $= -8 =$ det

$\lambda = -4, 2$

$$\lambda = -4: \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\lambda = 2: \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & \\ & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} / 2$$

$$A = -4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} / 2 + 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} / 2$$

$$I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} / 2 + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} / 2$$

$$\boxed{A^n = (-4)^n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} / 2 + 2^n \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} / 2}$$

Alternatively,

$$\text{formulas hold in any coordinates} \left\{ \begin{array}{ccc} A & (2-A)/6 & (A+4)/6 \\ \hline \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} / 2 & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} / 2 \\ \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \right.$$

[**3**] Find e^{At} where A is the matrix

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

$$\begin{array}{l} \text{sum} = 4 = \text{trace} \\ \text{prod} = 0 = \det \end{array} \Rightarrow \lambda = 0, 4$$

$$\lambda = 0: \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & \\ & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \Big/ 4$$

$$\lambda = 4: \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \qquad e^{At} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} e^{0t} & \\ & e^{4t} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \Big/ 4$$

$$\boxed{e^{At} = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \Big/ 4 + e^{4t} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \Big/ 4}$$

checks: $t = 0 \Rightarrow I = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \Big/ 4 + \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \Big/ 4$ ✓

$\frac{d}{dt} e^{At} \Big|_{t=0} = A$ ✓

Alternatively,

$$\begin{array}{l|ccc} & A & (4-A)/4 & A/4 \\ \hline \text{same in any coords} \left\{ \right. & \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} & \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \Big/ 4 & \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \Big/ 4 \end{array}$$

✓ ✓

[**4**] Find e^{At} where A is the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\begin{matrix} \text{sum} = 4 = \text{trace} \\ \text{prod} = 4 = \det \end{matrix} \implies \lambda = 2, 2 \qquad A - 2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$v_2 \xmapsto{A-2I} v_1 \xmapsto{A-2I} 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{matrix} (A-2I)v_2 = v_1 \\ (A-2I)v_1 = 0 \end{matrix} \qquad \begin{matrix} Av_2 = v_1 + 2v_2 \\ Av_1 = 2v_1 \end{matrix} \qquad \text{so } \underset{V \leftarrow V}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}$$

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\boxed{e^{At} = e^{2t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}}$$

checks $\quad e^{At}\Big|_{t=0} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ ✓

$$\frac{d}{dt} e^{At}\Big|_{t=0} = A = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \quad ✓$$

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alternatively, $\quad A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ where $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}^2 = 0$

$$e^{At} = e^{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} t} e^{\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} t}$$

$$e^{At} = e^{2t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} t + \underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}^2 t^2/2 + \dots}_{\text{zero from here on}} \right)$$

$$\boxed{e^{At} = e^{2t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}}$$

[5] Express $x^2 + 6xy + y^2$ as a linear combination of squares of orthogonal linear forms.

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

sum = 2 = trace
prod = −8 = det
$\lambda = -2, 4$

$\lambda = -2$: $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$ $\lambda = 4$: $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & \\ & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Big/ 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x-y & x+y \end{bmatrix} \begin{bmatrix} -2 & \\ & 4 \end{bmatrix} \Big/ 2 \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$

$$= \boxed{-(x-y)^2 + 2(x+y)^2}$$

checks: $x-y = (1,-1) \cdot (x,y)$ and $(1,-1) \cdot (1,1) = 0$
$x+y = (1,1) \cdot (x,y)$ forms are orthogonal ✓

$$-(x^2 - 2xy + y^2) + 2(x^2 + 2xy + y^2)$$
$$= (-1+2)x^2 + (2+4)xy + (-1+2)y^2$$
$$= x^2 + 6xy + y^2 \quad ✓$$

[**6**] Convert the differential equation $y'' - 3y' + 2y = 0$ to matrix form, and solve by exponentiating.

$$y_1 = y \qquad y_2 = y' \quad \Rightarrow \quad y_1' = y_2 \qquad y_2' = -2y_1 + 3y_2 \qquad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

sum = 3 = trace
prod = 2 = det

$\lambda = 1, 2$

check: $(\lambda-1)(\lambda-2) = \lambda^2 - 3\lambda + 2 \Leftrightarrow (D^2 - 3D + 2)y = y'' - 3y' + 2y$ ✓

$$\lambda = 1: \quad \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\lambda = 2: \quad \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$y' = Ay \Rightarrow y = e^{At}c$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = e^t \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + e^{2t} \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

(I ✓, A ✓)

$\Downarrow$

so $$\boxed{\begin{bmatrix} y \\ y' \end{bmatrix} = \left(e^t \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + e^{2t} \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

(use eigenvectors to find basis of solutions)

$\boxed{y = e^t, e^{2t} \text{ is basis of solutions}}$ (need to also show work!)

check: $y = e^t \quad y' = e^t \quad y'' = e^t \quad y'' - 3y' + 2y = 0$ ✓
$y = e^{2t} \quad y' = 2e^{2t} \quad y'' = 4e^{2t} \quad y'' - 3y' + 2y = 0$ ✓

[7] Find e^{At} where A is the matrix

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & -2 & 3 \end{bmatrix}$$

We could see $\lambda = 1, 2, 3$ by block triangular structure.

Or,

$$\begin{array}{cccc} 1 & -4 & & \\ 0 & 2 & 0 & \\ & -2 & 3 & 0 \\ & & 2 & 1 \end{array}$$

$$-\det|A - \lambda I| = \lambda^3 - \operatorname{trace}(A)\lambda^2 + \boxed{}\lambda - \det(A) = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$2 + 6 + 3 = 11$

$(1\cdot 2 + 1\cdot 3 + 2\cdot 3 = 11 ✓)$

divides 6
$$\left\{\begin{array}{c|ccccccc} 1 & 1 & -6 & + & 11 & - & 6 & = 0 \\ 2 & 8 & -24 & + & 22 & - & 6 & = 0 \\ 3 & 27 & -54 & + & 33 & - & 6 & = 0 \end{array}\right.$$

so $\lambda = 1, 2, 3$ don't need to check $-1, -2, -3$

$$(\lambda-1)(\lambda-2)(\lambda-3) = \lambda^3 - (1+2+3)\lambda^2 + (1\cdot 2 + 1\cdot 3 + 2\cdot 3)\lambda - 1\cdot 2\cdot 3$$
$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6 \quad ✓$$

$A - 1I$
$$\begin{bmatrix} 0 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$A - 2I$
$$\begin{bmatrix} -1 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0$$

$A - 3I$
$$\begin{bmatrix} -2 & -4 & 2 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

So
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} ✓ \Leftarrow$$

$$\begin{array}{c|cccc} 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{array}$$

✓ $$A = 1\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

✓ I

$$\boxed{e^{At} = e^{t}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}}$$

check: (use above $A-1$, $A-2$, $A-3$)

$$\begin{array}{cccc} A & (A-2)(A-3)/2 & -(A-1)(A-3) & (A-1)(A-2)/2 \\ \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} & \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} & \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} & \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \end{array}$$

✓✓✓

[8] Find e^{At} where A is the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & 4 & 1 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} 1 & 1 & & \\ -2 & 4 & 1 & \\ & -2 & 0 & 2 \\ & & 0 & 1 \end{array}$$

$6 + 2 + 0 = 8$

$-1\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = 4$

$$-\det|A - \lambda I| = \lambda^3 - \text{trace}(A)\lambda^2 + 8\lambda - \det(A) = 0$$

$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -5 & +8 & -4 = 0 \\ 2 & 8 & -20 & +16 & -4 = 0 \end{array}$$

trace $= 1 + 2 + \lambda = 5$ so last λ also 2

check: $(\lambda-1)(\lambda-2)(\lambda-2) = \lambda^3 - (\underbrace{1+2+2}_{5\ ✓})\lambda^2 + (\underbrace{1\cdot2 + 1\cdot2 + 2\cdot2}_{8\ ✓})\lambda - \underbrace{1\cdot2\cdot2}_{4\ ✓}$

$\lambda = 1, 2, 2$

$A - I$

$$\begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0$$

v_1

$A - 2I$

$$\begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & 1 \\ 2 & -2 & -2 \end{bmatrix}$$

$(A - 2I)^2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

v_3

$$v_1 \xmapsto{A-1} 0 \qquad v_3 \xmapsto{A-2I} v_2 \xmapsto{A-2I} 0$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 2 & 1 \\ & & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -2 & 2 & 1 \end{bmatrix} ✓ \Leftarrow$$

$$\begin{array}{c|cccc} & 1 & 0 & 2 & 1 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{array}$$

$$\begin{array}{c|ccc} & -1 & 1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -2 & 2 & 2 \end{array}, \quad \begin{array}{c|ccc} & 2 & -1 & -1 \\ \hline 1 & 2 & -1 & -1 \\ 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array} + \begin{array}{c|ccc} & -2 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 1 \\ -1 & 2 & -2 & -1 \end{array} = \begin{array}{ccc} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & -1 \end{array}, \quad \begin{array}{c|ccc} & -2 & 2 & 1 \\ \hline 1 & -2 & 2 & 1 \\ 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\boxed{e^{At} = e^t\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -2 & 2 & 2 \end{bmatrix} + e^{2t}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix} + te^{2t}\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$e^{At}\big|_{t=0} = I\ ✓ \qquad \frac{d}{dt}e^{At}\Big|_{t=0} = A\ ✓$$