Practice Problems 2

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[1] Let V and W be the subspaces of $\mathbb{R}^2$ spanned by (1,1) and (1,2), respectively. Find vectors $v \in V$ and $w \in W$ so $v + w = (2, -1)$.

$$v = s(1,1)$$
$$w = t(1,2)$$

$$v + w = s(1,1) + t(1,2) = (2,-1)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \frac{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}{1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$\begin{array}{r} \boxed{\begin{array}{l} v = (5,5) \\ + \quad w = (-3,-6) \end{array}} \\ \hline (2,-1) \ \checkmark \end{array}$$

[**2**] Let V and W be the subspaces of $\mathbb{R}^2$ spanned by $(1,-1)$ and $(2,1)$, respectively. Find vectors $v \in V$ and $w \in W$ so $v + w = (1,1)$.

$$V = s(1,-1)$$
$$W = t(2,1)$$
$$V + W = s(1,-1) + t(2,1) = (1,1)$$
$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} / 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} / 3$$
$$\boxed{\begin{aligned} V &= (-1/3,\ 1/3) \\ W &= (4/3,\ 2/3) \end{aligned}}$$
$$(1,\ 1) \checkmark$$

[3] Let V and W be the subspaces of $\mathbb{R}^2$ spanned by (1,1) and (1,4), respectively. Find vectors $v \in V$ and $w \in W$ so $v + w = (2,3)$.

$$s(1,1) + t(1,4) = (2,3)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \Big/ 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Big/ 3$$

$$\boxed{\begin{aligned} v &= (5/3, 5/3) \\ + \quad w &= (1/3, 4/3) \end{aligned}}$$

$$(6,9)/3 = (2,3) \quad \boxed{\checkmark}$$

[4] Let V be the subspace of $\mathbb{R}^3$ consisting of all solutions to the equation $x + y + z = 0$. Let W be the subspace of $\mathbb{R}^3$ spanned by $(1,1,0)$. Find vectors $v \in V$ and $w \in W$ so $v + w = (0,0,1)$.

$$w = t(1,1,0) \in W$$

$$v = (0,0,1) - t(1,1,0) = (-t,-t,1)$$

find t so $v \in V$

$$(1,1,1) \cdot (-t,-t,1) = 1 - 2t = 0$$

$$\text{so } t = 1/2$$

$$\boxed{\begin{aligned} w &= (1/2, 1/2, 0) \\ v &= (-1/2, -1/2, 1) \end{aligned}}$$

$$+ \quad (0, 0, 1) \checkmark$$

check: $(1,1,1) \cdot (-1/2, -1/2, 1) = 0$ ✓

[5] Let V be the subspace of $\mathbb{R}^3$ consisting of all solutions to the equation $x + y - z = 0$. Let W be the subspace of $\mathbb{R}^3$ spanned by (1,0,4). Find vectors $v \in V$ and $w \in W$ so $v + w = (1,1,1)$.

$w = t(1,0,4) \in W$

$v = (1,1,1) - t(1,0,4)$

find t so $v \in V$

$(1,1,-1) \cdot [(1,1,1) - t(1,0,4)] = 1 + 3t = 0$

so $t = -1/3$

$$\boxed{\begin{aligned} w &= (-1,0,-4)/3 \\ + \quad v &= (4,3,7)/3 \end{aligned}}$$

$(3,3,3)/3$ ✓

check: $(1,1,-1) \cdot (4,3,7) = 0$ ✓

[6] Let V be the subspace of $\mathbb{R}^3$ consisting of all solutions to the equation $x+2y+z=0$. Let W be the subspace of $\mathbb{R}^3$ spanned by $(1,1,1)$. Find vectors $v \in V$ and $w \in W$ so $v+w=(1,1,0)$.

$$w = t(1,1,1) \in W$$
$$v = (1,1,0) - t(1,1,1)$$

find t so $v \in V$

$$(1,2,1)\cdot\left[(1,1,0) - t(1,1,1)\right]$$
$$= 3 - 4t = 0 \qquad \text{so } t = 3/4$$

$$\boxed{\begin{aligned} w &= (3,3,3)/4 \\ + \quad v &= (1,1,-3)/4 \end{aligned}}$$
$$(4,4,0)/4 \ \checkmark$$

check: $(1,2,1)\cdot(1,1,-3) = 0$ ✓

[7] Let V be the subspace of $\mathbb{R}^4$ consisting of all solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let W be the orthogonal complement of V. Find vectors $v \in V$ and $w \in W$ so $v + w = (1,0,1,0)$.

Project $(1,0,1,0)$ to W:

$$\frac{(1,0,1,0)\cdot(1,1,0,0)}{(1,1,0,0)\cdot(1,1,0,0)}(1,1,0,0) + \frac{(1,0,1,0)(0,0,1,1)}{(0,0,1,1)(0,0,1,1)}(0,0,1,1)$$

same up to symmetry

$$= \frac{1}{2}(1,1,0,0) + \frac{1}{2}(0,0,1,1) = \boxed{\begin{aligned} &(\tfrac12,\tfrac12,\tfrac12,\tfrac12) = w \\ + &(\tfrac12,-\tfrac12,\tfrac12,-\tfrac12) = v \end{aligned}}$$

$$(1,0,1,0) \ ✓$$

and $v \perp w$ ✓

[**8**] Let V be the subspace of $\mathbb{R}^4$ consisting of all solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let W be the orthogonal complement of V. Find vectors $v \in V$ and $w \in W$ so $v + w = (0,0,1,1)$.

w = nearest point in span of $(1,1,1,1)$, $(0,1,1,1)$ to $(0,0,1,1)$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 4 \end{bmatrix} \Big/ 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Big/ 3$$

So

$$w = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \Big/ 3, \quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \Big/ 3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ✓$$

$$w \cdot v = 0 \quad ✓$$

$$\boxed{\begin{aligned} v &= (0,-2,1,1)/3 \\ w &= (0,2,2,2)/3 \end{aligned}}$$

[9] Let V be the subspace of $\mathbb{R}^4$ consisting of all solutions to the system of equations

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let W be the orthogonal complement of V. Find vectors $v \in V$ and $w \in W$ so $v + w = (1,0,0,0)$.

$\overline{W}$ spanned by rows of matrix so

$w = s(0,1,2,3) + t(3,2,1,0)$ for some s,t

$v = (1,0,0,0) - w$ is in $\overline{V}$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

equivalently solve to find $s,t \Rightarrow v,w$ by

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 4 \\ 4 & 14 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 14 & -4 \\ -4 & 14 \end{bmatrix} \Big/ 180 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ 42 \end{bmatrix} \Big/ 180$$

$14 \cdot 14 - 4 \cdot 4 = 140 + 56 - 16 = 180$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix} \Big/ 30$$

$$\boxed{w = \begin{bmatrix} 0+21 \\ -2+14 \\ -4+7 \\ -6+0 \end{bmatrix} \Big/ 30 = \begin{bmatrix} 21 \\ 12 \\ 3 \\ -6 \end{bmatrix} \Big/ 30 \quad \text{so} \quad v = \begin{bmatrix} +9 \\ -12 \\ -3 \\ 6 \end{bmatrix} \Big/ 30}$$

check: $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} +9 \\ -12 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -12-6+18 \\ +27-24-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

so $v \in \overline{V}$

$(21, 12, 3, -6) \cdot (9, -12, -3, 6) = 189 - 144 - 9 - 36 = 0$ ✓