Exam 2, 11:00-12:15

Linear Algebra, Dave Bayer, March 27, 2012

[1] Let V be the vector space of all polynomials $f(x)$ of degree $\leqslant 2$. Find a basis for the subspace W defined by

$$f'(1) = 0.$$

Extend this to a basis for V.

Usual basis for $V = \{x^2, x, 1\}$

$V = \{ax^2+bx+c\} \leftrightarrow (a,b,c) \in \mathbb{R}^3$

$f(x) = ax^2+bx+c$

$f'(x) = 2ax+b$

$f'(1) = 2a+b = 0$ defines plane in $\mathbb{R}^3$, W

dim=2, expect 2 basis vectors

independent, start in different columns

$$\left\{ \begin{array}{lll} (1,-2,0) \in W & \leftrightarrow & x^2-2x \\ (0,0,1) \in W & \leftrightarrow & 1 \\ \hline (0,1,0) \text{ extends} & \leftrightarrow & x \end{array} \right.$$

$$\boxed{\text{extends to } V \left\{ \begin{array}{c} x^2-2x \\ 1 \\ \hline x \end{array} \right. \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{basis for } W}$$

(using polynomial notation. Vector notation is ambiguous, we use only while computing, not for final answer.)

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

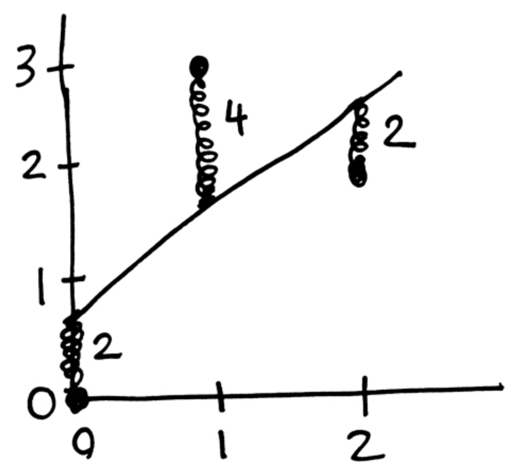
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \Big/ 6$$

$a = 1,\ b = 2/3$

$$\boxed{y = x + 2/3}$$

check

x	y	$x + 2/3$	Δ
0	0	$2/3$	$-2/3$
1	3	$5/3$	$4/3$
2	2	$8/3$	$-2/3$



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[3] Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,1,2), \quad (2,1,1,1).$$

Find an orthogonal basis for the subspace V.

$v_1 = (1,1,1,2)$ $\qquad$ $w_1 = (1,1,1,2)$

$v_2 = (2,1,1,1)$ $\qquad$ $w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w$

$$= (2,1,1,1) - \frac{2111 \cdot 1112}{1112 \cdot 1112}(1,1,1,2)$$

$$= (2,1,1,1) - \frac{6}{7}(1,1,1,2)$$

rescale to $\quad 7(2,1,1,1) - 6(1,1,1,2)$

$$= (8,1,1,-5)$$

$$\boxed{\{(1,1,1,2),\ (8,1,1,-5)\}}$$

check $w_1 \cdot w_2 = 0$ ✓

Need 2 equs on span $\quad (a,b,c,d)$

$b = c$ $\qquad$ $(1,1,1,2)$ ✓✓

$3(b+c) = 2(a+d)$ $\qquad$ $(2,1,1,1)$ ✓✓

$8, \overbrace{1, 1}^{b=c\ ✓}, -5 \qquad 3(1+1) = 2(8-5)$ ✓

[4] Find the determinant of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & a & 0 \\ 1 & b & 0 & 0 & 0 \\ 0 & 0 & 1 & c & 0 \\ 1 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}.$$

Using 3 row swaps, get block diagonal matrix:

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & b \\ 1 & d \end{matrix}} & & \\ & \boxed{\begin{matrix} 1 & a \\ 1 & c \end{matrix}} & \\ & & \boxed{e} \end{bmatrix}$$

$$\boxed{-(d-b)(c-a)e}$$

Check $\quad a=b=1 \quad c=d=2 \quad$ (sample values)

$$\begin{matrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e \end{matrix} \quad \Longrightarrow \quad \begin{matrix} 0 & 0 & \boxed{1} & 1 & 0 \\ \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{e} \end{matrix}$$

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det has one term

again 3 swaps to identity

$$-(d-b)(c-a)e = -(2-1)(2-1)e = -e \; ✓$$

[5] Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,1,1), \quad (1,1,2,2), \quad (1,1,3,3).$$

Find the matrix A which projects orthogonally onto the subspace V.

$$\begin{bmatrix} 1&1&1&1 \\ 1&1&2&2 \\ 1&1&3&3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1&1&1&1 \\ 0&0&1&1 \\ 0&0&2&2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 0&0&0&0 \end{bmatrix}$$

So basis is $(1,1,0,0), (0,0,1,1)$ perp

$$(1,0,0,0) \mapsto \frac{1000 \cdot 1100}{1100 \cdot 1100}(1,1,0,0) + \frac{1000 \cdot 0001}{0011 \cdot 0011}(0,0,1,1)$$

$$= \tfrac{1}{2}(1,1,0,0) + \tfrac{0}{2}(0,0,1,1)$$

$$(0,1,0,0) \mapsto \tfrac{1}{2}(1,1,0,0) + 0/2\,(0,0,1,1)$$

$$(0,0,1,0) \mapsto 0/2\,(1,1,0,0) + 1/2\,(0,0,1,1)$$

$$(0,0,0,1) \mapsto 0/2\,(1,1,0,0) + 1/2\,(0,0,1,1)$$

so

$$\boxed{A = \begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&1 \\ 0&0&1&1 \end{bmatrix} \Big/ 2}$$