Final Exam

Linear Algebra, Dave Bayer, May 10, 2011

Name: Solutions

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	[7] (5 pts)	[8] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x_i & 1 \\ & \\ & \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_i \\ \\ \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \begin{matrix} Ax = b \\ A^TAx = A^Tb \end{matrix}$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix}\Big/14 \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}\Big/14$$

x	y	$ax+b$	Δ
0	1	12/14	-2/14
1	1	17/14	3/14
3	2	27/14	-1/14

$$\boxed{y = \tfrac{5}{14}x + \tfrac{12}{14}}$$

[2] Extend the vector (1,1,1,2) to an orthogonal basis for $\mathbb{R}^4$.

$$\left.\begin{array}{l} v_1 = (1,1,1,2) \\ v_2 = (1,-1,0,0) \\ v_3 = (0,0,2,-1) \end{array}\right\} \text{mutually } \perp$$

Want $v_4 \perp v_1, v_2, v_3$

$\Leftrightarrow$ in kernel of $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

$v_4 = (a, a, b, 2b)$ is $\perp$ to v_2, v_3

$$\left.\begin{aligned} v_1 \cdot v_4 &= (1,1,1,2)\cdot(a,a,b,2b) \\ &= 2a + 5b = 0 \end{aligned}\right\} \text{make } v_4 \perp v_1$$

so $a = 5, b = -2$

$v_4 = (5, 5, -2, -4)$

$$\boxed{\begin{array}{l} v_1 = (1,1,1,2) \\ v_2 = (1,-1,0,0) \\ v_3 = (0,0,2,-1) \\ v_4 = (5,5,-2,-4) \end{array}}$$

Or use Gram-Schmidt

or solve for kernel of A

"

[3] Find the orthogonal projection of the vector (1,0,0,0) onto the subspace of $\mathbb{R}^4$ spanned by the vectors (1,1,1,0) and (0,1,1,1).

(1,1,1,0) and (0,1,1,1) are not $\perp$, make $\perp$

$$(0,1,1,1) - \frac{(0,1,1,1)\cdot(1,1,1,0)}{(1,1,1,0)\cdot(1,1,1,0)}(1,1,1,0)$$

$$= (0,1,1,1) - \frac{2}{3}(1,1,1,0) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 1\right)$$

scale to $(-2,1,1,3)$

$(-2,1,1,3) = -2(1,1,1,0) + 3(0,1,1,1)$ ✓

now $(1,0,0,0) \mapsto$

$$\frac{(1,0,0,0)\cdot(1,1,1,0)}{(1,1,1,0)\cdot(1,1,1,0)}(1,1,1,0) + \frac{(1,0,0,0)\cdot(-2,1,1,3)}{(-2,1,1,3)\cdot(-2,1,1,3)}(-2,1,1,3)$$

$$= \frac{1}{3}(1,1,1,0) + \frac{-2}{15}(-2,1,1,3)$$

$$= [(5,5,5,0) + (4,-2,-2,-6)]/15 = (9,3,3,-6)/15$$

$$= \boxed{(3,1,1,-2)/5}$$

checks: $(3,1,1,-2) = 3(1,1,1,0) + -2(0,1,1,1)$ ✓

in space

$(1,0,0,0) - (3,1,1,-2)/5$

$= (2,-1,-1,2)/5 \perp \begin{matrix}(1,1,1,0)\\(0,1,1,1)\end{matrix}$ ✓

difference $\perp$ to subspace

[4] Find the matrix A which projects $\mathbb{R}^4$ orthogonally onto the subspace spanned by the vectors (1,1,1,1) and (1,1,2,2).

$$(1,1,2,2) - \frac{(1,1,2,2)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1) = (1,1,2,2) - \frac{6}{4}(1,1,1,1)$$
$$= (-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$

scales to $(1,1,-1,-1)$

could have guessed

project (w,x,y,z) onto $(1,1,1,1)$:

$$\frac{(w,x,y,z)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1) = \frac{w+x+y+z}{4}(1,1,1,1)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} / 4 \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

project (w,x,y,z) onto $(1,1,-1,-1)$

$$\frac{(w,x,y,z)\cdot(1,1,-1,-1)}{(1,1,-1,-1)\cdot(1,1,-1,-1)}(1,1,-1,-1) = \frac{w+x-y-z}{4}(1,1,-1,-1)$$

$$= \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} / 4 \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

sum $\boxed{A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} / 2}$

checks

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} / 2$$

$$(1,0,0,0) - (1,1,0,0)/2$$
$$= (1,-1,0,0)/2$$

difference $\perp$ to subspace

✓ same pattern for rest of standard basis.

[5] Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$\det|A - \lambda I| = \lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda + 1)(\lambda - 5) = 0$$

$$\lambda = -1, 5$$

$\lambda = -1$ $\quad A - \lambda I:$ $\quad \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v = (1, -2)$

$\lambda = 5$ $\quad A - \lambda I:$ $\quad \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v = (1, 1)$

checks: $\quad \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ ✓

$\quad \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

$$\boxed{\begin{array}{ll} \lambda = -1, & v = (1, -2) \\ \lambda = 5, & v = (1, 1) \end{array}}$$

[6] Find the matrix exponential e^{At}, for the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$$

$$\det(A-\lambda I) = \lambda^2 - \operatorname{trace}(A)\lambda + \det(A) = 0$$
$$= \lambda^2 - 5\lambda + 0 = 0$$
$$(\lambda - 0)(\lambda - 5) = 0$$
$$\lambda = 0, 5$$

$$\lambda=0 \quad A-0I: \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v = (1,-4)$$
$$\lambda=5 \quad A-5I: \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v = (1,1)$$

$$\begin{array}{c|cc} & 1 & -1 \\ \hline 1 & 1 & -1 \\ -4 & -4 & 4 \end{array} \qquad \begin{array}{c|cc} & 4 & 1 \\ \hline 1 & 4 & 1 \\ 1 & 4 & 1 \end{array}$$

$$\underset{S \leftarrow S}{\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}} = \underset{S \leftarrow E}{\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}} \underset{E \leftarrow E}{\begin{bmatrix} 0 & \\ & 5 \end{bmatrix}} \underset{E \leftarrow S}{\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}} \Big/ 5$$

$$A = 0\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\Big/5 + 5\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\Big/5$$

$$I = 1\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\Big/5 + 1\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\Big/5$$

checks

$$e^{At} = e^{0t}\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\Big/5 + e^{5t}\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\Big/5$$

$$\boxed{e^{At} = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\Big/5 + e^{5t}\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\Big/5}$$

t=0 gives

$$\frac{d}{dt}: \quad Ae^{At} = 5e^{5t}\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\Big/5$$

t=0 gives

[7] Find the matrix exponential e^{At}, for the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^3 - \text{trace}(A)\,\lambda^2 + \Big(\underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}}_{3} + \underbrace{\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}_{0}\Big)\lambda - \det(A)$$

$$= \lambda^3 - 4\lambda^2 + 3\lambda$$

$$= \lambda(\lambda-1)(\lambda-3) \qquad \lambda = 0, 1, 3$$

$\lambda = 0$ $$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 1$ $$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 3$ $$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{S \leftarrow S}{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}} = \underset{S \leftarrow E}{\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}} \underset{E \leftarrow E}{\begin{bmatrix} 0 & & \\ & 1 & \\ & & 3 \end{bmatrix}} \underset{E \leftarrow S}{\begin{bmatrix} 0 & -2 & 2 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}/2}$$

$$\begin{array}{c|cccc} 1 & -1 & 0 & 1 & -1 \\ \hline 2 & -1 & -1 & 2 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 2 & -1 & -1 & 2 & -1 \end{array}$$

$$\begin{array}{c|ccc} & 0 & -2 & 2 \\ \hline 1 & 0 & -2 & 2 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} & 1 & 1 & -1 \\ \hline 2 & 2 & 2 & -2 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 \end{array} \qquad \begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

$$I = 1\begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}/2 + 1\begin{bmatrix} 2 & 2 & -2 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}/2 + 1\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}/2 \quad \checkmark$$

$$A = 0\begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}/2 + 1\begin{bmatrix} 2 & 2 & -2 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}/2 + 3\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}/2 \quad \checkmark$$

$$\boxed{e^{At} = 1\begin{bmatrix} 0 & -2 & +2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}/2 + e^{t}\begin{bmatrix} 2 & 2 & -2 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}/2 + e^{3t}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}/2}$$

[8] Find a formula for A^n, for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\lambda^3 - 6\lambda^2 + \left(\underbrace{\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}}_{2} + \underbrace{\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}}_{6} + \underbrace{\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}}_{4}\right)\lambda - \underbrace{\begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix}}_{8} = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$(\lambda - 2)^3 = 0$$

$$\lambda = 2, 2, 2$$

$$B = A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$B \neq 0$ rank 1, kernel dim $= 2$

$B^2 = 0$

$$\overset{v_2}{(1,0,0)} \xmapsto{B} \overset{v_1}{(0,1,1)} \xmapsto{B} 0$$

So want basis $v_2 \xmapsto{B} v_1 \xmapsto{B} 0$

$v_3 \xmapsto{B} 0$

Need v_3 independent, $\xmapsto{B} 0$

$$\underset{v_3}{(1,1,0)} \xmapsto{B} 0$$

$$\underset{S \leftarrow S}{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}} = \underset{S \leftarrow E}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \underset{E \leftarrow E}{\begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix}} \underset{E \leftarrow E}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}} / 1$$

$$\begin{array}{ccc|ccc} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}$$

$$= \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{c|ccc} & 1 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad ✓$$

$$\boxed{A^n = 2^n \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + n 2^{n-1} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}}$$

checks

$n = 0$ ✓

$$n = 2: \; A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 0 & 4 \\ 4 & -4 & 8 \end{bmatrix} = 4 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad ✓$$