Exam 2

Linear Algebra, Dave Bayer, March 29, 2011

Name: Solutions

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] Find a basis for the rowspace of the following matrix. Extend this basis to a basis for all of $\mathbb{R}^4$.

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 3 & 2 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

rank 2, rows are dependent

$3(2) = (1) + (3)$

$$\begin{array}{l} \text{rowspace} \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \text{extend} \left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

basis because triangular, det = ~~2~~ 3 $\neq 0$

$$\boxed{\begin{array}{l} \text{basis for rowspace: } \{(3\ 2\ 1\ 0), (0,1,2,3)\} \\ \text{extend to } \mathbb{R}^4 : \{\ '',\ '',\ (0,0,1,0), (0,0,0,1)\} \end{array}}$$

[2] Find the determinant of each of the following matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 9 \end{bmatrix} \qquad \begin{bmatrix} a & b & c & d \\ a & b+1 & c & d \\ a & b & c+1 & d \\ a & b & c & d+1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 4 \\ 1 & 2 & 6 & 4 \\ 1 & 2 & 3 & 8 \end{bmatrix}$$

$1 \cdot 3 \cdot (9-6) = 9$

$\boxed{\det = 9}$

$\Downarrow$ ② ← ② − ①, ③ ← ③ − ①, ④ ← ④ − ①

$$\begin{bmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\det = a}$

$a \cdot 1 \cdot 1 \cdot 1 = a$

$\Downarrow$ same

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\boxed{\det = 24}$

$1 \cdot 2 \cdot 3 \cdot 4 = 24$

[3] Find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & c \\ b & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ a & 1 & 0 & c & 0 & 1 & 0 & 0 \\ b & 0 & 1 & d & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \qquad [A \mid I]$$

$$\Downarrow \qquad\qquad \Downarrow =$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c & -a & 1 & 0 & 0 \\ 0 & 0 & 1 & d & -b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \qquad [I \mid A^{-1}]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & & & & 1 & & & 0 \\ & 1 & & & -a & 1 & & -c \\ & & 1 & & -b & & 1 & -d \\ & & & 1 & 0 & & & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & -c \\ -b & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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or stare and fill in, forced moves:

$$\left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & c \\ b & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -a \\ -c \\ 0 \end{bmatrix} \qquad \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & c \\ b & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & -c \\ -b & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

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or formula using minors (not so hard here...)

[4] Let

$$v_1 = (1,0,0), \qquad v_2 = (1,1,0), \qquad v_3 = (0,1,1)$$

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$L(v_1) = v_2, \qquad L(v_2) = v_3, \qquad L(v_3) = v_1,$$

Find the matrix A (in standard coordinates) which represents the linear map L.

$$(**) \quad \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\text{check} \quad \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad ✓$$

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or

$$\underset{S \leftarrow S}{\overset{L}{\begin{bmatrix} A \end{bmatrix}}} = \underset{S \leftarrow V}{\overset{Id}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}} \underbrace{\underset{V \leftarrow V}{\overset{L}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}} \underset{V \leftarrow S}{\overset{Id}{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}}}_{\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad ✓$$

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or transpose first equation (**)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

$$AB = C \overset{T}{\Downarrow} B^T A^T = C^T$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \underset{②\leftarrow②-①}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \underset{③\leftarrow③-②}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & & & 1 & 1 & 0 \\ & 1 & & -1 & 0 & 1 \\ & & 1 & 2 & 0 & -1 \end{array}\right]$$

transpose back

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad ✓$$

[5] Find the ratio x/y for the solution to the matrix equation

$$\begin{bmatrix} a & d & 1 \\ b & e & 1 \\ c & f & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Use Cramer's rule

$$x = \begin{vmatrix} 1 & d & 1 \\ 0 & e & 1 \\ 0 & f & 1 \end{vmatrix} \Big/ \det = \frac{e-f}{\det}$$

$$y = \begin{vmatrix} a & 1 & 1 \\ b & 0 & 1 \\ c & 0 & 1 \end{vmatrix} \Big/ \det = \frac{c-b}{\det}$$

so $\boxed{x/y = \dfrac{e-f}{c-b}}$

No need to compute denominator determinants, they cancel out.

[6] Find the determinant of the following 5×5 matrix. What is the determinant for the $n \times n$ case?

$$\begin{bmatrix} x & x^2 & 0 & 0 & 0 \\ 1 & x & x^2 & 0 & 0 \\ 0 & 1 & x & x^2 & 0 \\ 0 & 0 & 1 & x & x^2 \\ 0 & 0 & 0 & 1 & x \end{bmatrix}$$

A_n is general case

$f(n) = \det(A_n)$

$A_1 = [x] \quad f(1) = x$

$A_2 = \begin{bmatrix} x & x^2 \\ 1 & x \end{bmatrix} \quad f(2) = 0$

$A_3 = \begin{bmatrix} x & x^2 & 0 \\ 1 & x & x^2 \\ 0 & 1 & x \end{bmatrix} \quad f(3) = x \begin{vmatrix} x & x^2 \\ 1 & x \end{vmatrix} - 1 \begin{vmatrix} x^2 & 0 \\ 1 & \boxed{x} \end{vmatrix}$

$= x \cdot 0 - 1 \cdot x^3 = -x^3$

$= x f(2) - \cancel{x} \; x^2 f(1)$

$\boxed{\begin{matrix} + \\ - \\ + \end{matrix}}$

$A_4 = \begin{bmatrix} x & x^2 & 0 & 0 \\ 1 & x & x^2 & 0 \\ 0 & 1 & x & x^2 \\ 0 & 0 & 1 & x \end{bmatrix} \quad f(4) = x \begin{vmatrix} x & x^2 & 0 \\ 1 & x & x^2 \\ 0 & 1 & x \end{vmatrix} - 1 \begin{vmatrix} x^2 & 0 & 0 \\ 1 & x & x^2 \\ 0 & 1 & x \end{vmatrix}$

$= x \begin{vmatrix} x & x^2 & 0 \\ 1 & x & x^2 \\ 0 & 1 & x \end{vmatrix} - x^2 \begin{vmatrix} x & x^2 \\ 1 & x \end{vmatrix}$

$= x f(3) - x^2 f(2)$

$= -x^4 + 0 = -x^4$

n	$f(n)$
0	1
1	x
2	0
3	$-x^3$
4	$-x^4$
5	0
6	x^6
7	x^7
8	0
9	$-x^9$
$\vdots$	$\vdots$

Pattern:
$f(1) = x$
$f(2) = 0$
$f(n) = x f(n-1) - x^2 f(n-2)$

$f(5) = 0$

$f(n) = c_n x^n$ where

c_n cycles (period 6)

1 1 0 -1 -1 0 1 1 0 -1 -1 0 1 1 0 ...