Exam 1

Linear Algebra, Dave Bayer, February 15, 2011

Name: Solutions

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	TOTAL
/	/	/	/	/	/	/

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 2 & 0 & 1 & 5 \end{array}\right] \quad 1 \leftarrow 1 - 2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 7 \\ 2 & 0 & 1 & 5 \end{array}\right] \quad 2 \leftarrow 2 - 3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 5 \end{array}\right] \quad 3 \leftarrow 3 - 2\,1$$

$$\left[\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & 2 \\ & & 1 & 3 \end{array}\right]$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

check: $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$ ✓

[2] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 0 & 1 & 1 & 5 & 0 & 9 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$① \leftarrow ① - ② \qquad \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 3 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 & 1 & 6 & 4 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix} \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 2 & 0 & -2 & -4 \\ \hline 3 & 0 & -3 & -5 \\ \hline 0 & 0 & 1 & 0 \\ \hline 4 & 0 & 0 & -6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{array}\right]$$

free cols

$$\boxed{\begin{bmatrix} u \\ v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -4 \\ -5 \\ 0 \\ -6 \\ 1 \end{bmatrix}}$$

check against original matrix:

$$\checkmark \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} \qquad \checkmark \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \checkmark \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \checkmark \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

[3] Use Gaussian elimination to find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

$$[A \mid I] \Rightarrow [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & & \\ 0 & 1 & 0 & & 1 & \\ 1 & 3 & 0 & & & 1 \end{array}\right] \xRightarrow{\;1 \leftarrow 1 - 2(2)\;}$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xRightarrow{\;3 \leftarrow 3 - 3(2)\;}$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -3 & 1 \end{array}\right] \xRightarrow{\;1 \rightleftarrows 3\;}$$

$$\left[\begin{array}{ccc|ccc} 1 & & & 0 & -3 & 1 \\ & 1 & & 0 & 1 & 0 \\ & & 1 & 1 & -2 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}}$$

check

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

[4] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underset{\xleftarrow{1 \rightleftarrows 3}}{\xrightarrow{1 \rightleftarrows 3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \underset{\xleftarrow{3 \leftarrow 3 + 3\,2}}{\xrightarrow{3 \leftarrow 3 - 3\,2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \underset{\xleftarrow{3 \leftarrow -3}}{\xrightarrow{3 \leftarrow -3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

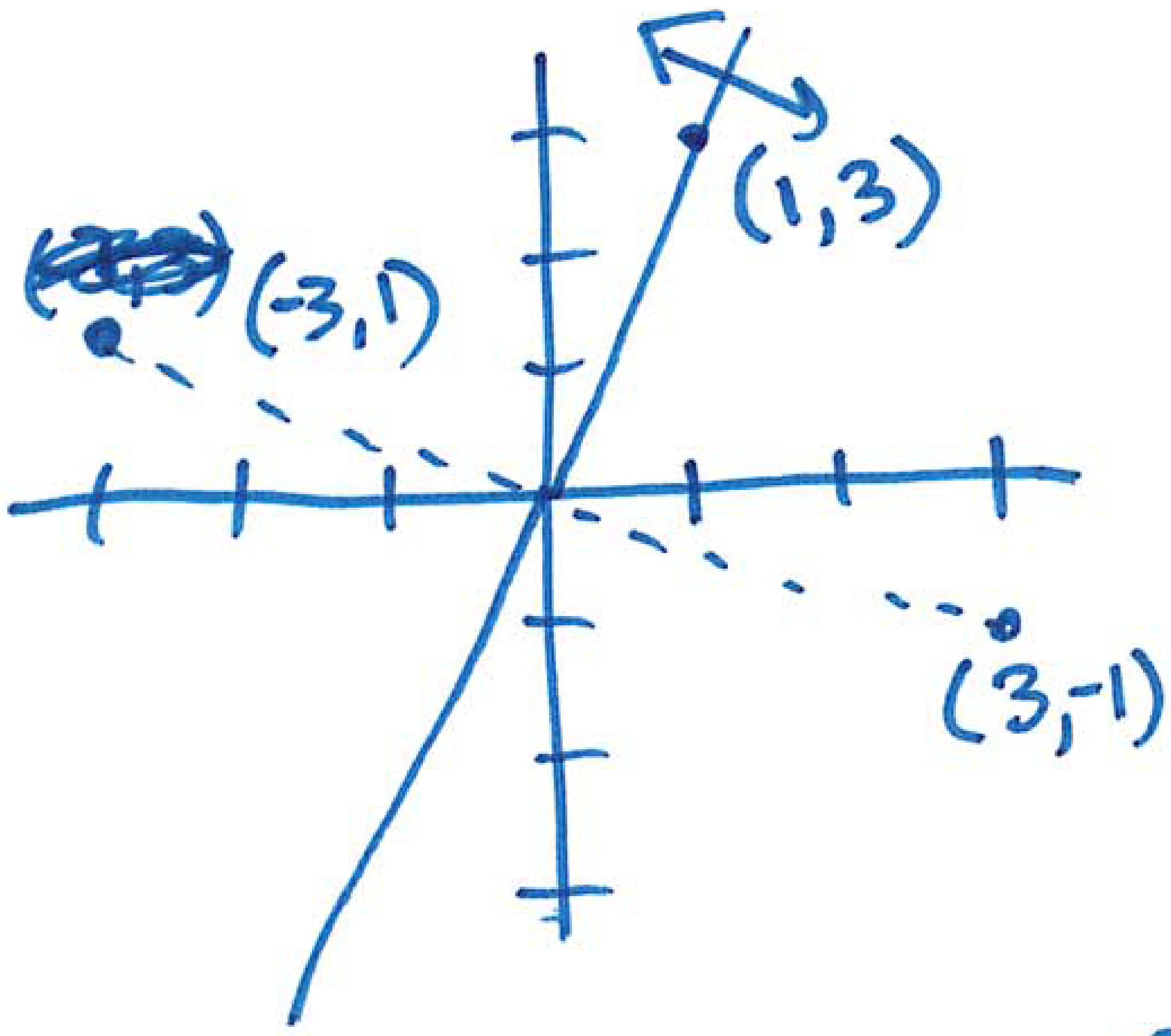
$$\boxed{A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}}}$$

check

$$\begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

✓

[5] Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix which flips the plane $\mathbb{R}^2$ across the line $3x = y$. Find A.



$$A\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

$$A\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix}$$

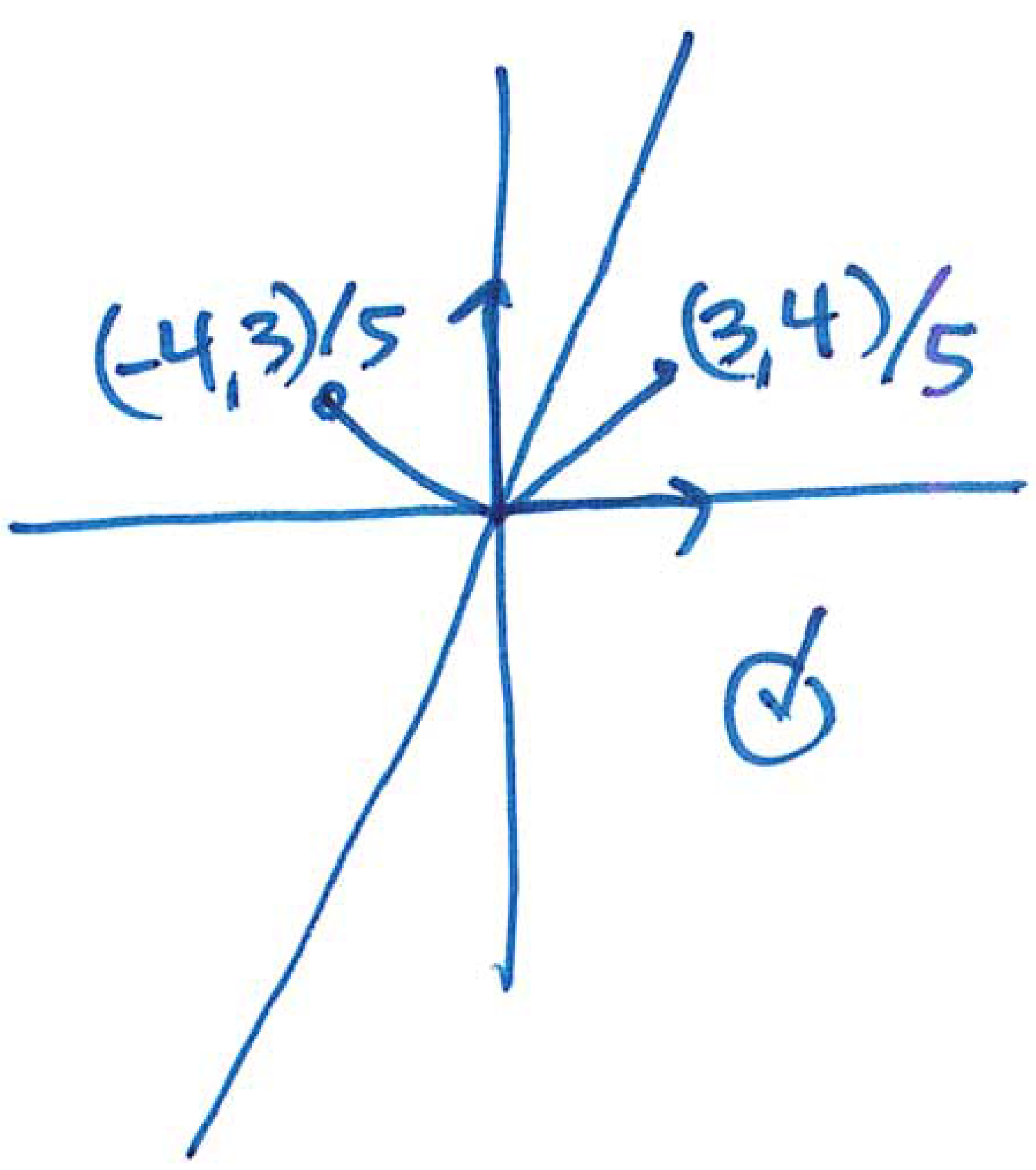
$$A\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix} = \begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix}^{-1} = \begin{bmatrix}-1 & -3\\-3 & 1\end{bmatrix}\Big/{-10} = \begin{bmatrix}1 & 3\\3 & -1\end{bmatrix}\Big/10 \qquad \underbrace{\cdot\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix} = \begin{bmatrix}10 & 0\\0 & 10\end{bmatrix}\Big/10}_{\text{check}} \quad \checkmark$$

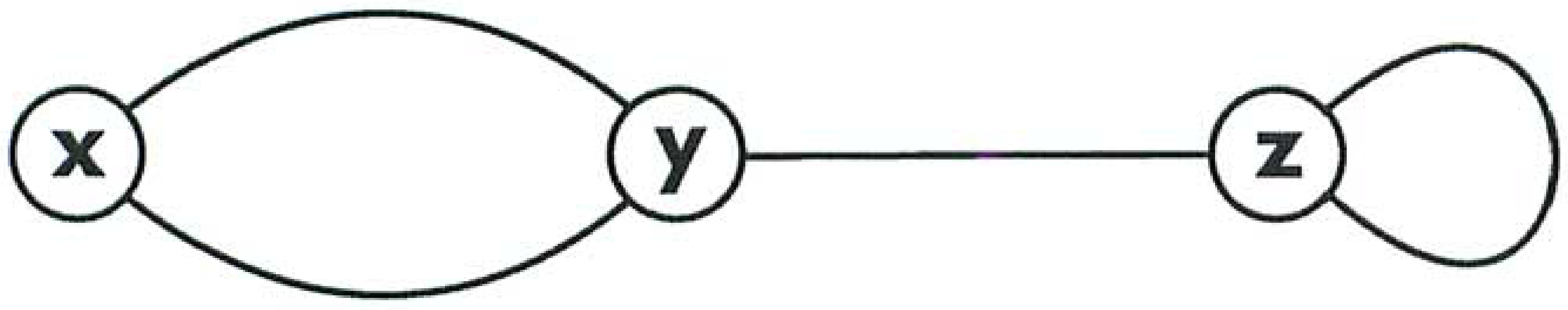
$$A = \begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix}\Big/10 = \begin{bmatrix}-8 & 6\\6 & 8\end{bmatrix}\Big/10 = \boxed{\begin{matrix}\begin{bmatrix}-4 & 3\\3 & 4\end{bmatrix}\Big/5\\ \shortparallel \\ A\end{matrix}}$$

check

$$\begin{bmatrix}-4 & 3\\3 & 4\end{bmatrix}\Big/5 \begin{bmatrix}1 & 3\\3 & -1\end{bmatrix} = \begin{bmatrix}5 & -15\\15 & 5\end{bmatrix}\Big/5 = \begin{bmatrix}1 & -3\\3 & 1\end{bmatrix} \quad \checkmark$$



[6] Using matrix multiplication, count the number of paths of length four from y to itself.



$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 20 & 2 & 12 \\ 2 & 26 & 7 \\ 12 & 7 & 9 \end{bmatrix}$$

$\boxed{\text{26 paths of length 4}}$

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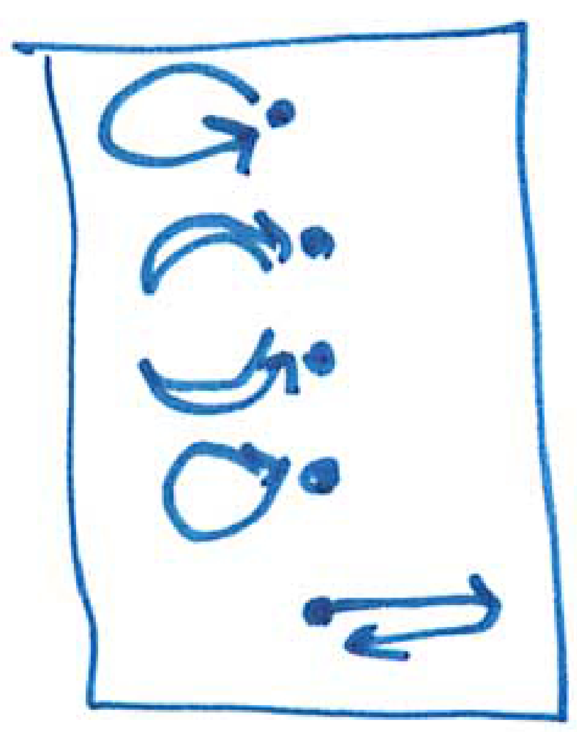
check: $26 = 5^2 + 1$

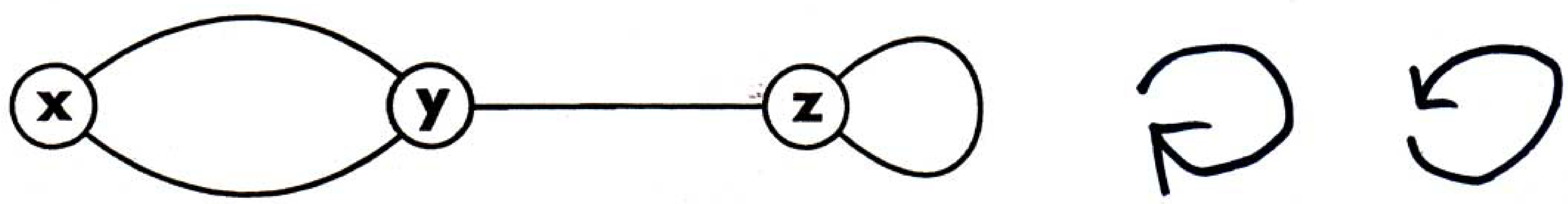
5 of length 2

(do twice)



1 of length 4

[6] Using **matrix multiplication**, count the number of paths of length four from y to itself.



The previous solution counted the loop from z to z <u>once</u>. If we instead count the two directions around this loop as different, we get more paths:

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} \ast & \ast & \ast \\ \ast & 29 & \ast \\ \ast & \ast & \ast \end{bmatrix}$$

$\boxed{\text{29 paths}}$