



Exam 4

Linear Algebra, Dave Bayer, May 10, 2007

Name: _____

Answer Key

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer.
If a problem continues on a new page, clearly state this fact on both the old and the new pages.

Do not use calculators or decimal notation. Please simplify each answer as far as possible.

[1] Find an orthogonal basis for the subspace V of \mathbb{R}^5 spanned by the vectors

$$(1, 0, -1, 0, 1) \quad (0, 1, -1, 0, 0) \quad (0, 0, 1, -1, 0)$$

$$\begin{aligned} v_1 &= (0, 1, -1, 0, 0) \\ v_2 &= (0, 0, 1, -1, 0) \\ v_3 &= (1, 0, -1, 0, 1) \end{aligned}$$

$w_1 = (0, 1, -1, 0, 0)$
 $w_2 = (0, 1, 1, -2, 0)$
 $w_3 = (3, -1, -1, -1, +3)$

$w_1 \cdot w_2 = 0 \quad \checkmark$
 $w_2 \cdot w_3 = 0 \quad \checkmark$
 $w_1 \cdot w_3 = 0 \quad \checkmark$

answer

checks

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (0, 0, 1, -1, 0) - \frac{-1}{2}(0, 1, -1, 0, 0)$$

$$\text{rescale (x2)} = (0, 0, 2, -2, 0) + (0, 1, -1, 0, 0)$$

$$= (0, 1, 1, -2, 0)$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$$

$$= (1, 0, -1, 0, 1) - \frac{1}{2}(0, 1, -1, 0, 0) - \frac{-1}{6}(0, 1, 1, -2, 0)$$

$$\text{rescale (x6)} = (6, 0, -6, 0, 6) - (0, 3, -3, 0, 0) + (0, 1, 1, -2, 0)$$

$$= (6, -2, -2, -2, 6)$$

$$\text{rescale (x2)} = (3, -1, -1, -1, 3)$$

check also: Do w_1, w_2, w_3 belong to V ?

$V = \text{nullspace of}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}}_{\text{inspection}}$$

w_1, w_2, w_3 also in nullspace?
 \checkmark \checkmark \checkmark
 (this check caught a copying error!)

[2] Let V be the vector space of all polynomials $f(x)$ of degree ≤ 3 . Find a basis for the subspace W defined by

$$f(x) = f(-x)$$

Extend this basis to a basis for V .

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d && \text{even polynomials} \\ \Rightarrow f(-x) &= -ax^3 + bx^2 - cx + d \\ f(x) - f(-x) &= 2ax^3 + 2cx = 0 \Rightarrow a=0, c=0 \end{aligned}$$

Subspace W basis: $\{1, x^2\}$

extend to basis for V : $\{1, x^2, x, x^3\}$

answer

[3] Define the inner product of two polynomials f and g by the rule

$$\langle f, g \rangle = \int_{-1}^1 f(x) g(x) dx$$

Using this definition of the inner product, find an orthogonal basis for the vector space of all polynomials of degree ≤ 2 .

$f(x) = a + bx + cx^2$ represent as (a, b, c)

$$M = \int_{-1}^1 \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix} dx = \left[\begin{array}{c|c} x & \frac{1}{2}x^2 & \frac{1}{3}x^3 \\ \hline \frac{1}{2}x^2 & \frac{1}{3}x^3 & \frac{1}{4}x^4 \\ \hline \frac{1}{3}x^3 & \frac{1}{4}x^4 & \frac{1}{5}x^5 \end{array} \right] \Big|_1^1 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} - \begin{bmatrix} -1 & \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \\ -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{bmatrix} \quad \text{rescale by dropping } 2, \text{ now}$$

$$f(x) = (a, b, c) \quad \frac{1}{2} \langle f, g \rangle = [a \ b \ c] \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$\begin{array}{l|l} v_1 = 1 & w_1 = 1 \\ v_2 = x & w_2 = x \\ v_3 = x^2 & w_3 = x^2 - \frac{1}{3} \end{array}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$\boxed{\{1, x, x^2 - \frac{1}{3}\}} \quad \text{answer}$$

$$\langle v_2, w_1 \rangle = [0 \ 1 \ 0] \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

(already \perp . nice...)

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\langle w_1, w_1 \rangle = [1 \ 0 \ 0] \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\langle w_2, w_2 \rangle = [0 \ 1 \ 0] \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}$$

$$= x^2 - \frac{1}{3}$$

$$\langle v_3, w_1 \rangle = [0 \ 0 \ 1] \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}$$

$$\langle v_3, w_2 \rangle = [0 \ 0 \ 1] \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

check

$$\langle w_1, w_2 \rangle = \int_{-1}^1 1 \cdot x dx = 0 \quad \text{OK}$$

$$\langle w_1, w_3 \rangle = \int_{-1}^1 (x^2 - \frac{1}{3}) dx = (\frac{1}{3}x^3 - \frac{1}{3}) \Big|_{-1}^1 = 0 - 0 = 0 \quad \text{OK}$$

$$\langle w_2, w_3 \rangle = \int_{-1}^1 x(x^2 - \frac{1}{3}) dx = \int_{-1}^1 (x^3 - \frac{1}{3}x) dx = (\frac{1}{4}x^4 - \frac{1}{6}x^2) \Big|_{-1}^1 = 0$$

or

$$[0 \ 1 \ 0] \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} = 0$$

$$2\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 0$$

$$\det(A) = 0$$

[4] Find the matrix e^{At} , where $A = \begin{bmatrix} 2 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

$$\lambda^3 - \underbrace{\text{trace}(A)\lambda^2}_{2} + \underbrace{\text{trace}(A^2)\lambda}_{|2 \ 2| + |2 \ -2| + |-1 \ 1|} - \underbrace{\det(A)}_{-2 \ 2 \ 0} = 0$$

$$\lambda^3 - 2\lambda^2 = 0$$

$$(\lambda - 2)(\lambda - 0)(\lambda - 0) = 0$$

$$\lambda = 0, 0, 2$$

$$\lambda = 2 \quad A - 2I = \begin{bmatrix} 0 & 2 & -2 \\ 0 & -3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AV_3 = 2V_3$$

$$\lambda = 0 \quad A - 0I = A = \begin{bmatrix} 2 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{rank 2, only one eigenvector.}$$

Can't find $v_1, v_2 \xrightarrow{A} 0$

find instead $v_2 \xrightarrow{A} v_1 \xrightarrow{A} 0$

So look for v_2 so $A^2v_2 = 0$ but $Av_2 \neq 0$. Then set $v_1 = Av_2$.

$$A^2 = \begin{bmatrix} 4 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \cancel{v_2 \neq 0} \quad v_2 = [1, 0, 1] \quad Av_2 = [0, 1, 1] = v_1 \quad Av_1 = 0$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}}_{\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & -2 \end{bmatrix}} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & -2 \end{bmatrix}} \quad \text{④}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

inverse

$$e^{At} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{ot} & te^{ot} & t^2e^{ot} \\ 0 & e^{ot} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

... continued on next page \Rightarrow

Problem 4 continued...

$$e^{At} = \dots \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$+ t \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$+ e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

work

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

answer

checks: $e^{At} \Big|_{t=0} = I :$ $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left(\frac{d}{dt} e^{At} \right) \Big|_{t=0} = \left(A e^{At} \right) \Big|_{t=0} = A :$$

$$\frac{d}{dt} e^{At} = 0 + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{I_{t=0}} + 2e^{2t} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = A$$
✓

[5] Find a matrix A so $A^2 = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}$.

can find in any words. Are there better words? Diagonalize?

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda-1)(\lambda-4) = 0$$

eigenvalues $\lambda = 1, 4$

λ	$A - \lambda I$	V
1	$\begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
4	$\begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A \quad A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}_{\text{diag}} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$\begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ -4 & 8 \end{bmatrix}$

phew!

Simpler problem: Find D so $D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.
 hmm... $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \sqrt{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{\text{diag}} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = A}$$

$\boxed{\text{answer}}$

check: $\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} \quad \text{OK}$