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Exam 3

Linear Algebra, Dave Bayer, April 17, 2007

Name: Solutions

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

Do not use calculators or decimal notation. Please simplify each answer as far as possible.

[1] By least squares, find the equation of the form $z = ax + by + c$ which best fits the data

$$(x_1, y_1, z_1) = (0,0,0), \quad (x_2, y_2, z_2) = (1,0,0), \quad (x_3, y_3, z_3) = (0,1,0), \quad (x_4, y_4, z_4) = (1,1,1)$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underbrace{a\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} + c\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

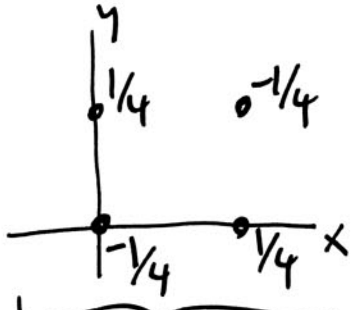
(guess a=b by symmetry)

$$a\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} + c\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so $a = b = \frac{1}{2}$, $c = 1/4$ $\quad \boxed{z = \frac{1}{2}x + \frac{1}{2}y - \frac{1}{4}}$

checks:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \\ -1/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$



plot errors ✓

x,y	actual	fit	error
00	0	-1/4	-1/4
10	0	1/4	1/4
01	0	1/4	1/4
11	1	3/4	-1/4

(no up, down or twisting pull to these errors)

[2] Find e^{At} for the matrix

$$A = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$$

$$\begin{aligned} \lambda_1 + \lambda_2 &= \text{trace}(A) = 4 \\ \lambda_1 \lambda_2 &= \det(A) = 3 \end{aligned} \Rightarrow \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 3 \end{aligned} \quad \left(A \text{ has same trace, det as } \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right)$$

λ	$A-\lambda I$	v
1	$\begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
3	$\begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & \\ & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Big/ 1$$

$$\begin{array}{c|cc} & 2 & 1 \\ \hline 1 & 2 & 1 \\ -1 & -2 & -1 \end{array} \qquad \begin{array}{c|cc} & 1 & 1 \\ \hline -1 & -1 & -1 \\ 2 & 2 & 2 \end{array}$$

$$\begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix} = 1 \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + 3 \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \checkmark$$

$$\left(\text{check: } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \checkmark \right)$$

So $$\boxed{e^{At} = e^t \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + e^{3t} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}$$

same

check: $t=0 \quad e^{A0} = I = e^0 \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + e^{3\cdot 0} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \checkmark$

same

$\frac{d}{dt}$: $\quad Ae^{At} = e^t \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + 3e^{3t} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$

... $t=0 \quad A = 1 \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + 3 \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \checkmark$

[3] Find e^{At} for the matrix

$$A = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\lambda_1+\lambda_2 = \text{trace}(A) = 0$$
$$\lambda_1\lambda_2 = \det(A) = 1$$

$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \frac{-0 \pm\sqrt{0^2-4}}{2} = \pm i$$

(Yes, $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ acts on $\mathbb{R}^2$ just like i acts on $\mathbb{C}$)

λ	$A-\lambda I$	v
$-i$	$\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}$	$\begin{bmatrix}1\\ i\end{bmatrix}$
i	$\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}$	$\begin{bmatrix}i\\ 1\end{bmatrix}$

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\begin{bmatrix}-i & \\ & i\end{bmatrix}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix}\Big/2$$

$$\begin{array}{c|cc} & 1 & -i\\ \hline 1 & 1 & -i\\ i & i & 1\end{array} \qquad \begin{array}{c|cc} & -i & 1\\ \hline i & 1 & i\\ 1 & -i & 1\end{array}$$

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = -i\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}\Big/2 + i\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}\Big/2$$

$$e^{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}t} = e^{-it}\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}\Big/2 + e^{it}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}\Big/2$$

$$\left.\begin{array}{l} e^{it} = \cos t + i\sin t\\ e^{-it} = \cos t - i\sin t\end{array}\right\} \Rightarrow e^{At} = (\cos t - i\sin t)\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}\Big/2$$
$$+ (\cos t + i\sin t)\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}\Big/2$$

$$e^{At} = \cos t\Bigg(\underbrace{\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}\Big/2 + \begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}\Big/2}_{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}\Bigg) + i\sin t\Bigg(\underbrace{-\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}\Big/2 + \begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}\Big/2}_{\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}}\Bigg)$$

$$i\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\boxed{e^{At} = \cos t\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \sin t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}$$

check

$$e^{it} = \cos t\cdot 1 \quad + \sin t\cdot i$$

$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is 1, $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ is i, same pattern. ✓

[4] Find A^n for the matrix

$$A = \begin{bmatrix} 2 & -4 \\ 1 & 6 \end{bmatrix}$$

$$\left.\begin{array}{l} \lambda_1 + \lambda_2 = \text{trace}(A) = 8 \\ \lambda_1 \lambda_2 = \det(A) = 16 \end{array}\right\} \quad \lambda_1 = \lambda_2 = 4, \text{ repeated root}$$

$$B = A - 4I = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

rank one, can't find $v_1, v_2 \xrightarrow{B} 0$

instead look for

$$v_2 \xrightarrow{B} v_1 \xrightarrow{B} 0$$

$$\boxed{A = B + 4I}$$

Try any v_2, e.g.

$$\underset{v_2}{(1,0)} \xrightarrow{B} \underset{v_1}{(-2,1)} \xrightarrow{B} 0$$

$$\boxed{\begin{array}{l|l} Bv_2 = v_1 & Av_2 = v_1 + 4v_2 \\ Bv_1 = 0 & Av_1 = \quad 4v_1 \end{array}}$$

so

$$\begin{bmatrix} 2 & -4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & +1 \\ +1 & +2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

same

$$\left(\begin{array}{c} \begin{bmatrix} 2 & -4 \\ 1 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark \end{array} \right) \quad = 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

same

answer

$$\text{so} \quad \boxed{A^n = 4^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + n4^{n-1} \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}}$$

check:

$$A^2 = \begin{bmatrix} 0 & -32 \\ 8 & 32 \end{bmatrix} = 4^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \cdot 4 \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} + \begin{bmatrix} -16 & -32 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} 0 & -32 \\ 8 & 32 \end{bmatrix}$$

$\checkmark$

[5] The quadratic form

$$2xy \;=\; \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \frac{1}{2}\left((x+y)^2 - (x-y)^2\right)$$

can be expressed as shown as a linear combination of squares of linear forms. Do the same for the quadratic form

$$x^2 + 2xy + 2yz + z^2$$

$\boxed{\text{Solution 1}}$ The problem does not specify that the squares be orthogonal, so there are many solutions:

$$x^2 + 2xy + \qquad + 2yz + z^2$$

$$= \underbrace{x^2 + 2xy + \boxed{y^2}}\; \underbrace{+\, \boxed{y^2} + 2yz + z^2}\; \boxed{-2y^2} \qquad \text{(add \& subtract } 2y^2\text{)}$$

$$= \boxed{(x+y)^2 \;+\; (y+z)^2 \;-\; 2y^2} \qquad \text{(for example)}$$

5.1

$\boxed{\text{Solution 2}}$

$$x^2 + 2xy + 2yz + z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\begin{bmatrix} x+y \\ x+z \\ y+z \end{bmatrix}}$$

$$\begin{aligned} &= x(x+y) && = x^2 + xy \\ &\;\; + y(x+z) && \;\; + xy + yz \\ &\;\; + z(y+z) && \;\; + yz + z^2 \end{aligned}$$

so diagonalize $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

$$\lambda^3 - \underbrace{\mathrm{trace}(A)}_{\substack{1+0+1 \\ =2}}\lambda^2 + \underbrace{\mathrm{trace}(\Lambda^2 A)}_{}\lambda - \underbrace{\det(A)}_{} = 0$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \qquad \begin{matrix} -1 & +1 & -1 \end{matrix} \qquad = -1$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -2$$

$$\lambda^3 - (2)\lambda^2 + (-1)\lambda - (-2) = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

λ	$\lambda^3 - 2\lambda^2 - \lambda + 2 = ?$
1	$1 - 2 - 1 + 2 = 0$
-1	$-1 - 2 + 1 + 2 = 0$
2	$8 - 8 - 2 + 2 = 0$
-2	$-8 - 8 + 2 + 2 =$ [illegible]

so $\lambda = 1, -1, 2$

$\Rightarrow$ see next page

$\boxed{\text{problem 5, continued}}$

check: Does $D = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 2 \end{bmatrix}$ have same characteristic poly?

$$1 + (-1) + 2 = 2 = \operatorname{trace}(A) \ ✓$$
$$1(-1) + 1 \cdot 2 + (-1)2 = -1 = \operatorname{trace}(\Lambda^2 A) \ ✓$$
$$1(-1)2 = -2 = \det(A) \ ✓$$

λ	$A - \lambda I$	v
1	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
-1	$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
2	$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

A symmetric, $v \perp$ ✓

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c|cccc} 1 & 0 & -1 & 1 & 0 \\ \hline 1 & -2 & 1 & 1 & -2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 \end{array} \qquad C^{-1} = \begin{bmatrix} -3 & 0 & 3 \\ -1 & 2 & -1 \\ -2 & -2 & -2 \end{bmatrix} \Big/ -6$$

or, use orthogonality: rows of C^T are $\perp$ to cols of C

$$C^T C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

so $$\underbrace{\begin{bmatrix} 3 & & \\ & 1 & \\ & & 2 \end{bmatrix} \Big/ 6}_{} \, C^T C = \begin{bmatrix} 3 & & \\ & 1 & \\ & & 2 \end{bmatrix} \Big/ 6 \begin{bmatrix} 2 & & \\ & 6 & \\ & & 3 \end{bmatrix} = I$$

$$C^{-1} = \begin{bmatrix} 3 & & \\ & 1 & \\ & & 2 \end{bmatrix} \Big/ 6 \; C^T = \begin{bmatrix} 3 & & \\ & 1 & \\ & & 2 \end{bmatrix} \Big/ 6 \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 & -3 \\ 1 & -2 & 1 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 6}_{\text{same answer as before}}$$

$\Rightarrow$ see next page

$\boxed{\text{problem 5, continued}}$

$$\left\{\begin{array}{l} S = \text{original coords} \\ T = \text{eigenvector coords} \end{array}\right\}$$

$$\underset{S \leftarrow S}{\overset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}} = \underset{S \leftarrow T}{\overset{C}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}}} \underset{T \leftarrow T}{\overset{D}{\begin{bmatrix} 1 & & \\ & -1 & \\ & & 2 \end{bmatrix}}} \underset{T \leftarrow S}{\overset{C^{-1}}{\begin{bmatrix} 3 & 0 & -3 \\ 1 & -2 & 1 \\ 2 & 2 & 2 \end{bmatrix}}} \Big/ 6$$

so

$$[x\ y\ z]\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{[x\ y\ z]\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 2 \end{bmatrix}\underbrace{\begin{bmatrix} 3 & 0 & -3 \\ 1 & -2 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\Big/ 6}$$

$$= [(x-z)\ (x-2y+z)\ (x+y+z)]\begin{bmatrix} 1 & & \\ & -1 & \\ & & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}(x-z) \\ \frac{1}{6}(x-2y+z) \\ \frac{1}{3}(x+y+z) \end{bmatrix}$$

$$= \boxed{\frac{1}{2}(x-z)^2 - \frac{1}{6}(x-2y+z)^2 + \frac{2}{3}(x+y+z)^2} \text{ answer}$$

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check:

$$\frac{1}{2}\left(\begin{array}{c|ccc} & 1 & 0 & -1 \\ \hline 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right) - \frac{1}{6}\left(\begin{array}{c|ccc} & 1 & -2 & 1 \\ \hline 1 & 1 & -2 & 1 \\ -2 & -2 & 4 & -2 \\ 1 & 1 & -2 & 1 \end{array}\right) + \frac{2}{3}\left(\begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) = A$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

spot checks:

	$x^2+2xy+2yz+z^2$	$\frac{1}{2}u^2+\frac{1}{6}v^2+\frac{2}{3}w^2$
$x=y=z=1$:	6	6
$x=1, y=z=0$:	1	1
$x=0, y=z=1$:	$\frac{1}{2}-\frac{1}{6}+\frac{8}{3}=3$	3

✓