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Exam 2

Linear Algebra, Dave Bayer, March 20, 2007

Name: answer key

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

Do not use calculators or decimal notation.

[1] Use Cramer's rule to solve for z in the system of equations

$$\begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{z = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{vmatrix} = \frac{a^2-a}{a^3-2a} = \frac{a-1}{a^2-2}}$$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{vmatrix} = +a\begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = a(a-1) = a^2-a$$

$$\begin{vmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{vmatrix} = +a\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & a \end{vmatrix} = a(a^2-1) - a = a^3-2a$$

check:

$$\begin{vmatrix} a & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & a \end{vmatrix} = +a\begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & a \end{vmatrix} = a(a-1) - a = a^2-2a, \text{ so } y = \frac{a^2-2a}{a^3-2a} = \frac{a-2}{a^2-a}$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = \underbrace{-\begin{vmatrix} 1 & 1 & a \\ 1 & a & 1 \\ 1 & 1 & 0 \end{vmatrix}}_{\text{swap rows}} = \underbrace{+\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{vmatrix}}_{\text{swap cols}} = a^2-a, \text{ so } x = \frac{a^2-a}{a^3-2a} = \frac{a-1}{a^2-2}$$

$$\begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix} \begin{bmatrix} a-1 \\ a-2 \\ a-1 \end{bmatrix} \Big/ a^2-2 = \begin{bmatrix} a^2-2 \\ a^2-2 \\ a^2-2 \end{bmatrix} \Big/ a^2-2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad ✓$$

[2] Find a basis for the subspace V of $\mathbb{R}^4$ defined by the following system of equations. Extend this basis to a basis for all of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] - \boxed{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & \underbrace{1 \quad 1}_{\text{free cols}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \}\text{ forced} \\ \\ \}\text{ identity block} \end{matrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so basis for $V = \left\{ \begin{matrix} (0,-1,1,0) \\ (1,-1,0,1) \end{matrix} \right\}$

extend to basis for $\mathbb{R}^4$:

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \}V \\ \\ \end{matrix}$$

$\Leftarrow$ choose from $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$

(go with new pivot columns to insure full rank.)

basis extending V to $\mathbb{R}^4 = \left\{ \begin{matrix} (0,-1,1,0) \\ (1,-1,0,1) \\ (0,0,1,0) \\ (0,0,0,1) \end{matrix} \right\}$

[3] Find a 3×3 matrix A such that

$$A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}, \qquad A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\3\\3\end{bmatrix}$$

collect columns:

$$A\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}1&2&0\\1&2&3\\0&2&3\end{bmatrix} \quad \text{so} \quad A = \begin{bmatrix}1&2&0\\1&2&3\\0&2&3\end{bmatrix}\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}^{-1}$$

inverse:

$$\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}^{-1} = \begin{array}{c|cccc|}1&1&0&1&1\\ \hline 1&1&1&1&1\\0&1&1&0&1\\1&1&0&1&1\\1&1&1&1&1\\ \hline\end{array} = \begin{bmatrix}0&-1&1\\-1&1&-1\\1&-1&0\end{bmatrix}\Big/(-1) = \begin{bmatrix}0&1&-1\\1&-1&1\\-1&1&0\end{bmatrix} \checkmark$$

$$\text{so} \quad A = \begin{bmatrix}1&2&0\\1&2&3\\0&2&3\end{bmatrix}\begin{bmatrix}0&1&-1\\1&-1&1\\-1&1&0\end{bmatrix} = \begin{bmatrix}2&-1&1\\-1&2&1\\-1&1&2\end{bmatrix}$$

check:

$$\begin{bmatrix}2&-1&1\\-1&2&1\\-1&1&2\end{bmatrix}\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}1&2&0\\1&2&3\\0&2&3\end{bmatrix} \checkmark$$

[4] Find the characteristic polynomial, and a system of eigenvalues and eigenvectors, for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$$

$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$

$\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0$

$\lambda = +3, -2$

$\lambda_1 + \lambda_2 = 1 = \text{trace}(A)$

$\lambda_1 \lambda_2 = -6 = \det(A)$

$\lambda_1 = 3$

$\lambda_2 = -2$

λ	$A - \lambda I$	v
3	$\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
-2	$\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

So $\boxed{\begin{array}{l} \text{char poly} = \lambda^2 - \lambda - 6 \\ \text{eigenvalues, vectors:} \quad \lambda = 3 \quad v = (1,2) \\ \qquad\qquad\qquad\qquad\quad \lambda = -2 \quad v = (1,-3) \end{array}}$

checks:

$\begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ ✓

$\lambda^2 - \lambda - 6 \begin{cases} (3)^2 - (3) - 6 = 0 \checkmark & \lambda = 3 \\ (-2)^2 - (-2) - 6 = 0 \checkmark & \lambda = -2 \end{cases}$

[5] For each of the following matrices, find the determinant. What is the general pattern?

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{f(2)} \quad \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{f(3)}, \quad \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{f(4)}, \quad \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{f(5)}$$

$$f(4) = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = +1 \underbrace{\begin{vmatrix} 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}}_{f(3)} -1 \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= +f(3) + \underbrace{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}_{f(2)}$$

so $f(4) = f(3) + f(2)$

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{matrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{matrix} = 3$$

$-(-1\ \ -1)$

$+(\cancel{3},0,0)$

$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} + 1 \underbrace{\begin{vmatrix} 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}}_{\text{above, } 3} = 2 + 3 = 5$$

(be different)

pattern:

$f(3) = 3$

$f(4) = 5$

$f(5) = 8$

$\vdots$

$f(n) = f(n-1) + f(n-2)$